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On the Systems of Finite Weights on the Algebra of Bounded Operators and Corresponding Translation Invariant Measures

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Abstract—We describe the class of translation invariant measures on the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ and some of its subalgebras. In order to achieve this we apply two steps. First we show that a total minimal system of finite weights on the operator algebra defines a family of rectangles in this algebra through construction of operator intervals. The second step is construction of a translation invariant measure on some subalgebras of algebra $\mathcal{B}(\mathcal{H})$ by the family of rectangles. The operator intervals in the Jordan algebra $\mathcal{B}(\mathcal{H})^{\mathrm{sa}}$ is investigated. We also obtain some new operator inequalities.

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1. INTRODUCTION

According to the A. Weil theorem (see [1]), there does not exist a Lebesgue measure on an infinite-dimensional Banach space $\mathcal{X}$. So it is impossible to build a non-zero σ-additive σ-finite locally finite shift-invariant measure defined on all Borel subsets of an infinite-dimensional Banach space $\mathcal{X}$. Therefore we turn to the following definition of a translation invariant measure on Banach space. An additive function λ on some ring $\mathcal{R}$ of subsets of Banach space $\mathcal{X}$ is called a translation invariant measure on the space X if for any set $A \in \mathcal{R}$ and any vector $\mathbf{a} \in \mathcal{X}$) the conditions $A + \mathbf{a} \in \mathcal{R}$, $\lambda(A + \mathbf{a}) = \lambda(A)$ hold.

Let $\mathcal{A}$ be a C^*-algebra of bounded linear operators acting on a Hilbert space $\mathcal{H}$. The translation invariant measure on the algebra $\mathcal{A}$ is constructed by the following scheme. At the first step we choose a family of the finite weights on algebra $\mathcal{A}$ and construct the corresponding family of operator intervals in algebra $\mathcal{A}$ (see [2, 3]). Hence the family of measurable rectangles in algebra $\mathcal{A}$ is defined by the family of the operator intervals. According to the approach of work [4] the additive function of a set is defined on the family of measurable rectangles by means of the product of the operator intervals lengths of the measurable rectangle. At the second step we extend the family of measurable rectangles onto the ring of measurable subsets of the algebra $\mathcal{A}$ applying constructions of [4]. It is shown that an additive function of a set on the family of measurable rectangles is uniquely extendable to the translation invariant measure λ on the ring of measurable sets of algebra $\mathcal{A}$. We study the properties of countable additivity, σ-finiteness, local finiteness of the constructed measure λ. Moreover, the invariance of this measure with respect to some transformations of algebra $\mathcal{A}$ is considered (in particular, with respect to unitary equivalence, see Remark 3.3). In Chapter 4 we investigate the operator intervals in the Jordan algebra $\mathcal{B}(\mathcal{H})^{\mathrm{sa}}$ and obtain some new operator inequalities.

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2. NOTATION AND DEFINITIONS

A complex Banach $*$-algebra $\mathcal{A}$ such that $||x^*x|| = ||x||^2$ for all $x \in \mathcal{A}$ is called a *C^*-algebra*. An element $x \in \mathcal{A}$ is called a *projection*, if $x = x^2 = x^*$. The symbols $\mathcal{A}^{\rm sa}$ and $\mathcal{A}^+$ denote the subsets of Hermitian and positive elements of C^*-algebra $\mathcal{A}$, respectively.

Let $\mathcal{H}$ be a separable Hilbert space. Let $\mathcal{B}(\mathcal{H})$ be an algebra of all bounded linear operators on the space $\mathcal{H}$. Any C^*-algebra can be realized as the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (Gelfand–Naimark; see [5], Theorem 3.4.1). Let $\mathcal{B}(\mathcal{H})^{\rm sa}$ be a Jordan algebra of all bounded self-adjoint operators on the space $\mathcal{H}$. Let $|\mathbf{T}| = \sqrt{\mathbf{T}^*\mathbf{T}} \in \mathcal{B}(\mathcal{H})^+$ for any $\mathbf{T} \in \mathcal{B}(\mathcal{H})$. If $\mathbf{T} \in \mathcal{B}(\mathcal{H})^{\rm sa}$ then the operators $\mathbf{T}_+ = (|\mathbf{T}| + \mathbf{T})/2$ and $\mathbf{T}_- = (|\mathbf{T}| - \mathbf{T})/2$ belong to the set $\mathcal{B}(\mathcal{H})^+$ and meet the equalities $\mathbf{T} = \mathbf{T}_+ - \mathbf{T}_-$ and $\mathbf{T}_+\mathbf{T}_- = \theta$. Let $\mathbf{I}$ be the identity operator in the space $\mathcal{H}$. For any operator $\mathbf{X} \in \mathcal{B}(\mathcal{H})^+$ define the following two sets: $I_\mathbf{X} = \{\mathbf{Y} \in \mathcal{B}(\mathcal{H})^{\rm sa} : -\mathbf{X} \le \mathbf{Y} \le \mathbf{X}\}$ and $M_\mathbf{X} = \{\mathbf{Y} \in \mathcal{B}(\mathcal{H}) : |\mathbf{Y}| \le \mathbf{X}\}$.

An additive homogeneous map $\omega : \mathcal{A}^+ \to [0, +\infty)$ is called a *finite weight* on a C^*-algebra $\mathcal{A}$. A system of finite weights $\{\omega_a, a \in \Omega\}$ on an algebra $\mathcal{A}$ is called *complete* if an operator $\mathbf{A} \in \mathcal{A}$ subject to the conditions $\omega_a(\mathbf{A}) = 0 \forall a \in \Omega$, must be the zero operator: $\mathbf{A} = \theta$.

A complete system of finite weights $\{\omega_a, a \in \Omega\}$ is called *minimal*, if for any proper subset $\Omega_1 \subset \Omega$ the system $\{\omega_a, a \in \Omega_1\}$ is not complete on an algebra $\mathcal{A}$.

A system of finite weights $\{\omega_a, a \in \Omega\}$ on an operator algebra $\mathcal{A}$ is called *independent* if for any element $a \in \Omega$ $\mathcal{A}$ contains the operator $\mathbf{A}$ such that $\omega_a(\mathbf{A}) \ne 0$ and $\omega_b(\mathbf{A}) = 0$ $\forall b \in \Omega \backslash \{a\}$. For example, if a finite weight ω_1 is faithful (see [6]) then the one-element system $\{\omega_1\}$ is a complete minimal independent system of weights on the algebra $\mathcal{B}(\mathcal{H})$.

Let $\{e_k\}$ be a countable system of unit vectors which is dense in the unit sphere of the space $\mathcal{H}$. If $\omega_{e_k}(\mathbf{A}) = (\mathbf{A}e_k, e_k)_H$ then the system of finite weights $\{\omega_{e_k}, k \in \mathbb{N}\}$ is complete on the algebra $\mathcal{B}(\mathcal{H})$ but is not a minimal system.

Let $\{e_k\}$ be an orthonormal basis in the space $\mathcal{H}$. Then the system $\{\omega_{e_k}\}$ is not complete on the algebra $\mathcal{B}(\mathcal{H})$ since there exist the nontrivial operators $\mathbf{A} \in B(\mathcal{H})$ such that $(e_k, \mathbf{A}e_k) = 0 \, \forall k \in \mathbb{N}$. But the system $\{\omega_{e_k}\}$ is complete and minimal on the commutative subalgebra $\mathcal{A}$ of operators sharing the common orthonormal basis of eigenvectors $\{e_k\}$.

3. TRANSLATION INVARIANT MEASURES

Lemma 3.1. *Any finite independent system of finite weights $\Sigma = \{\omega_1, \dots, \omega_m\}$ on a Jordan algebra $\mathcal{B}(\mathcal{H})^{sa}$ defines the translation invariant measure on the algebra $\mathcal{B}(\mathcal{H})^{sa}$.*

Proof. In fact, the system Σ defines the continuous surjective map $\Phi_\Sigma : \mathcal{B}(\mathcal{H})^{\rm sa} \to \mathbb{R}^m$. Let λ_m be the Lebesgue measure on the Euclidean space $\mathbb{R}^m$. Let μ_Σ be the preimage of the measure λ_m of the map Φ_Σ: $\mu_\Sigma(A_\Pi) = \lambda_m(\Pi)$ for any $\Pi \in \mathcal{B}(\mathbb{R}^m)$ where $A_\Pi = \Phi_\Sigma^{-1}(\Pi)$.

The preimage of the disjoint sets Δ_1 and Δ_2 of the σ-ring $\mathcal{K}(\mathbb{R}^m)$ of bounded Borel sets consists of the disjoint sets $\Phi_\Sigma^{-1}(\Delta)$ and $\Phi_\Sigma^{-1}(\Delta_2)$. The preimage of the σ-ring $\mathcal{K}(\mathbb{R}^m)$ of the map Φ_Σ is the σ-ring $\mathcal{R}_\Sigma = \Phi_\Sigma^{-1}(\mathcal{K}(\mathbb{R}^m))$ of subsets of the space $\mathcal{B}(\mathcal{H})^{\rm sa}$. The σ-additivity of the function μ_Σ on the σ-ring of sets $\Phi_\Sigma^{-1}(\mathcal{K}(\mathbb{R}^m))$ is the consequence of the σ-additivity of the function λ_m on the σ-ring of bounded Borel sets $\mathcal{K}(\mathbb{R}^m)$. □

Example 3.2. Let $\mathcal{A}$ be the Jordan ring of the self-adjoint trace-class operators. Let ω the trace on the space of trace-class operators. Let $\mathcal{R}_\omega$ be a σ-ring of subsets $\Phi_\omega^{-1}(\mathcal{K}(\mathbb{R}))$. Then the function $I_\omega : \mathcal{R}_\omega \to \mathbb{R}$ such that $I_\omega(\{\mathbf{X} \in \mathcal{A} : \ \omega(\mathbf{X}) \in [a, b)\}) = b - a \ \forall a, b : -\infty < a < b < +\infty$ is a translation invariant countable additive measure on the Jordan ring $\mathcal{A}$ defined on the σ-ring of subsets $\mathcal{R}_\omega$.

We say that a family of subsets $\mathcal{S}$ of the set X splits points of the set X if $(X, \tau_\mathcal{S})$ is the Hausdorff topological space, where $\tau_\mathcal{S}$ is the topology on the set X generated by the family $\mathcal{S}$.

Lemma 3.1 and Example 3.2 describe the class of translation invariant countable additive measures on the spaces of operators acting in a Hilbert space $\mathcal{H}$. Any measure of this class is generated by a Lebesgue measure on a finite dimensional Euclidean space. But any measure of this class is not invariant with respect to a unitary mapping of the space $\mathcal{H}$. Moreover, for any $m \in \mathbb{N}$ the σ-ring $\mathcal{R}_\Sigma^{-1}(\mathcal{K}(\mathbb{R}^m))$ does not split points of Jordan algebra $\mathcal{B}(\mathcal{H})^{sa}$ since the mapping Φ_Σ has the nontrivial kernel.

Remark 3.3. If a countable system of finite weights $\mathbf{\Sigma}$ on a C^*-algebra $\mathcal{A}$ is complete then the topology on the algebra $\mathcal{A}$ generated by the family Σ is a Hausdorff topology.

Lemma 3.4. *Let $\mathcal{A}$ be a C^*-algebra of bounded linear operators acting on a Hilbert space $\mathcal{H}$. A countable system of finite weights $\mathbf{\Sigma}$ on the C^*-algebra $\mathcal{A}$ is complete if and only if the mapping $\Phi_\Sigma:\ \mathcal{A} \to \ell_\infty$ is injective. Here the mapping Φ_Σ is defined by the equality $\Phi_\Sigma(\mathbf{A}) = \{\omega_k(\mathbf{A}\}, \mathbf{A} \in \mathcal{A}$.*

In fact, if a system of finite weights $\mathbf{\Sigma}$ is complete (is not complete) on the C^*-algebra $\mathcal{A}$, then the mapping $\Phi_\Sigma:\ \mathcal{A} \to \ell_\infty$ has the trivial (respectively, nontrivial) kernel. □

Let Σ be a complete system of finite weights on a C^*-algebra of operators $\mathcal{A}$. We investigate the existence of a translation invariant measure defined on the ring of subsets of a C^*-algebra $\mathcal{A}$ generated by the system of weights Σ. Moreover, we study measures on the C^*-algebra $\mathcal{A}$ invariant with respect to the unitary equivalence transformations. The unitary equivalence transformations are parametrized by the group $\mathcal{U}(\mathcal{H})$ of unitary operators in the space $\mathcal{H}$ by the rule $\mathbf{A} \to \mathbf{U}^{-1}\mathbf{A}\mathbf{U},\ \mathbf{A} \in \mathcal{A}$, where $\mathbf{U} \in \mathcal{U}(\mathcal{H})$.

Let $\{e_k\}$ be an orthonormal basis in the space $\mathcal{H}$. Let $\mathbf{Q}_{\{e_k\}} \equiv \{\mathbf{P}_k, k \in \mathbb{N}\}$ be a countable systems of the orthogonal projections onto one-dimensional subspaces spanned by the vectors of this basis. Let $\Sigma(\mathbf{Q}_{\{e_k\}}) = \{\omega_{e_k}, k \in \mathbb{N}\}$ be a system of weights on the algebra $\mathcal{B}(\mathcal{H})$ defined by the relation

$$\omega_{e_k}(\mathbf{A}) = \mathrm{Tr}(\mathbf{A}\mathbf{P}_k) = (e_k, \mathbf{A}e_k)_\mathcal{H}, \mathbf{A} \in \mathcal{B}(\mathcal{H}), k \in \mathbb{N}. \quad (1)$$

The symbol $\mathcal{P}$ denotes the set of all the one-dimensional orthogonal projection systems on the space $\mathcal{H}$ each element of which is generated by an orthonormal basis. Any system of projections $\mathbf{Q} \in \mathcal{P}$ defines the system of weights $\Sigma(\mathbf{Q})$ according to equality (1).

Example 3.5. Let $\mathcal{A}_2$ be a commutative ring of Hilbert–Schmidt self-adjoint operators. This ring is the linear space endowed with the Hilbert–Schmidt norm and is isomorphic to the Hilbert space ℓ_2. Therefore the Hilbert space $\mathcal{A}_2$ possesses the orthonormal basis consisting of one-dimensional orthogonal projections $\mathbf{P}_k, k \in \mathbb{N}$ in the ring $\mathcal{A}_2$. Let $\{I_k(a,b), k \in \mathbb{N},\ -\infty < a \le b < +\infty\}$ be a system of operator intervals on the ring $\mathcal{A}_2$ such that $I_k(a,b) = \{\mathbf{A} \in \mathcal{A}_2:\ a\mathbf{P}_k \le \mathbf{A} \le b\mathbf{P}_k\}$. If $a, b \in \ell_\infty$ then the set $\Pi_{a,b} = \bigcap\limits_{k=1}^{\infty} I_k(a_k, b_k)$ is called an operator rectangle in the ring of operators $\mathcal{A}_2$. Let $\mathcal{R}$ be a minimal ring defined by the family of rectangles $\{\Pi_{a,b}, a, b \in \ell_\infty\}$.

Remark 3.6. The ring of subsets $\mathcal{R}$ of the operator ring $\mathcal{A}_2$ splits points of the set $\mathcal{A}_2$.

Theorem 3.7. *Let $\mathcal{A}_2$ be a commutative ring of self-adjoint Hilbert–Schmidt operators on the space $\mathcal{H}$. Then we have the translation invariant locally finite σ-finite completely finite additive measure μ on the ring of operators $\mathcal{A}_2$ such that the measure μ is defined on the ring of subsets $\mathcal{R}$ and the values of measure μ on any nonempty operator rectangle $\bigcap\limits_{k=1}^{\infty} I_k(a_k, b_k)$ are equal to the product $\prod\limits_{k=1}^{\infty} (b_k - a_k)$ of the lengthes of the operator segments.*

Proof. Let us consider the ring of operators $\mathcal{A}_2$ as the Hilbert space endowed with the scalar product $(\mathbf{A}, \mathbf{B})_{\mathcal{A}_2} = \mathrm{Tr}(\mathbf{A}^*\mathbf{B}), \mathbf{A}, \mathbf{B} \in \mathcal{A}_2$. Then we have the orthonormal basis of one-dimensional projections $\mathbf{P}_k, k \in \mathbb{N}$ in this Hilbert space. Let $\{I_k(a,b), k \in \mathbb{N},\ -\infty < a \le b < +\infty\}$ be the system of operator intervals in the ring of operators $\mathcal{A}_2$ such that $I_k(a,b) = \{\mathbf{A} \in \mathcal{A}:\ a\mathbf{P}_k \le \mathbf{A} \le b\mathbf{P}_k\}$. The system of one-dimensional orthogonal projections $\mathbf{Q} = \{\mathbf{P}_k, k \in \mathbb{N}\}$ defines the system of weights $\Sigma(\mathbf{Q})$ according to equality (1). Since the system $\Sigma(\mathbf{Q})$ is complete and minimal the mapping $\Phi_{\Sigma(\mathbf{Q})}: \mathcal{A} \to \ell_\infty$: $\Phi_{\Sigma(\mathbf{Q})}(\mathbf{A}) = \{a_k\}$ is injective for all $\mathbf{A} \in \mathcal{A}$ (where $a_k = \mathrm{Tr}(\mathbf{A}\mathbf{P}_k), k \in \mathbb{N}$). Moreover, the mapping $\Phi_{\Sigma(\mathbf{Q})}$ is an isometric isomorphism of the ring $\mathcal{A}_2$ with the Hilbert–Schmidt norm onto the space ℓ_2.

Let us apply the construction of shift invariant measures on the space ℓ_2 described in [4, 7]. A set $\Pi \subset \ell_2$ is called rectangle if there exist two elements $a, b \in \ell_\infty$ such that $\Pi = \{x \in \ell_2:\ x_j \in [a_j, b_j), j \in \mathbb{N}\}$. In this case the rectangle Π is denoted by $\Pi_{a,b}$.

The rectangle $\Pi_{a,b}$ is called absolutely measurable if either $\Pi_{a,b} = \varnothing$, or the following condition holds $\sum\limits_{j=1}^{\infty} \max\{\ln(b_j - a_j), 0\} < +\infty$. The function $\lambda_0 :\ \mathcal{K} \to [0, +\infty)$ is defined on the family $\mathcal{K}$ of absolutely measurable rectangles of the space ℓ_2 by the following conditions: $\lambda_0(\Pi_{a,b}) = \exp(\sum\limits_{j=1}^{\infty} \ln(b_j - a_j))$ for any nonempty absolutely measurable rectangle and $\lambda_0(\varnothing) = 0$. Then the set function λ_0 is additive and translation invariant on the collection of sets $\mathcal{K}$ (see [8], Lemma 1). The set function $\lambda_0 :\ \mathcal{K} \to [0, +\infty)$ admits the unique extension onto the translation and rotation invariant finite additive complete measure λ on the space ℓ_2 such that the measure λ is defined on the minimal ring Λ containing the family of absolutely measurable rectangles $\mathcal{K}$ and its image under the actions of the group of orthogonal mappings of the space ℓ_2 (see [4], Corollary 4). The measure λ is not countable additive (see [4], Theorem 1). The measure λ is σ-finite since the space ℓ_2 has the covering by the countable system of rectangles $\{\Pi^{m,N}, m, N \in \mathbb{N}\}$ such that $\lambda(\Pi^{m,N}) = 0\ \forall m, N \in \mathbb{N}$. Here for any $m, N \in \mathbb{N}$ the rectangle $\Pi^{m,N}$ has the edges $[a_k, b_k) = [-N, N)\ \forall k = 1, \dots, m$, and $[a_k, b_k) = [-\frac{1}{3}, \frac{1}{3})\ \forall k > m$.

Let $\mu_{\mathbf{Q}} = \lambda \circ \Phi^{-1}_{\Sigma(\mathbf{Q})}$ be the preimage of measure λ under the mapping $\Phi_{\Sigma(\mathbf{Q})}$. The preimage $\Phi^{-1}_{\Sigma(\mathbf{Q})}(\Lambda)$ of the ring Λ of the subsets of the space ℓ_2 is the ring $\mathcal{R}_{\mathbf{Q}}$ of the subsets of the ring $\mathcal{A}_2$ of operators such that $\mu_{\mathbf{Q}}(B) = \lambda(\Phi_{\Sigma(\mathbf{Q})}(B))$ for any $B \in \mathcal{R}_{\mathbf{Q}}$. The measure $\mu_{\mathbf{Q}}$ meets properties of the measure λ. □

Remark 3.8. Let $\mathcal{U}(\mathcal{H})$ be a group of unitary operators on the space $\mathcal{H}$. The measure $\mu_{\mathbf{Q}}$ is invariant with respect to the subgroup $\mathcal{U}_{\mathcal{A}_2}(\mathcal{H})$ of transformations $\mathbf{T}_{\mathbf{U}}, \mathbf{U} \in \mathcal{U}(\mathcal{H})$ of the ring $\mathcal{A}_2$ by unitary operators such that $\mathcal{A}_2 = \{\mathbf{U}^{-1}\mathbf{A}\mathbf{U}, \mathbf{A} \in \mathcal{A}_2\}$. Here $\mathbf{T}_{\mathbf{U}}(\mathbf{A}) = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}, \mathbf{A} \in \mathcal{A}_2$ for any unitary operator $\mathbf{U} \in \mathcal{U}(\mathcal{H})$. Invariantnes of the measure $\mu_{\mathbf{Q}}$ means that the equality $\mu_{\mathbf{Q}}(B) = \mu_{\mathbf{Q}}(\mathbf{U}^{-1}B\mathbf{U})$ holds for any set $B \in \mathcal{R}_{\mathbf{Q}}$ and any $\mathbf{U} \in \mathcal{U}_{\mathcal{A}_2}(\mathcal{H})$. This properties are the consequence of the rotation invariance of the measure λ. In particular, the measure $\mu_{\mathbf{Q}}$ is invariant with respect to the rearrangements of the system $\mathbf{Q}$ elements. Thus the measure $\mu_{\Sigma(\mathbf{Q})} = \lambda \circ \Phi^{-1}_{\Sigma(\mathbf{Q})}$ is a finite additive translation invariant complete σ-finite and locally finite measure on the operator ring $\mathcal{A}_2$ and this measure does not depend on the choice of a basis of one-dimensional orthogonal projections $\{\mathbf{P}_k\}$.

Remark 3.9. There exist infinitely many different translation invariant locally finite σ-finite complete finite additive measures on the operator ring $\mathcal{A}_2$. Firstly, a bijection $\mathcal{A}_2 \to \ell_2$ can be defined applying different complete minimal systems of weights. Secondly, the choice of a translation invariant measure on the space ℓ_2 with the other properties listed above is not unique (see [4, 7, 8]).

Theorem 3.10. *Let $\mathcal{B}(\mathcal{H})^{sa}$ be a Jordan algebra of bounded self-adjoint operators on the space $\mathcal{H}$. Then any countable system of finite weights $\Sigma = \{\omega_k\} \in \mathcal{P}$ on the algebra $\mathcal{B}(\mathcal{H})^{sa}$ defines the translation invariant complete finite additive measure on the algebra $\mathcal{B}(\mathcal{H})^{sa}$.*

Proof. In fact, since we have $\Sigma \in \mathcal{P}$, the system Σ is an independent system of weights on the algebra $\mathcal{B}(\mathcal{H})^{sa}$. Therefore it defines the surjective linear mapping $\Phi_\Sigma : \mathcal{B}(\mathcal{H})^{\rm sa} \to \ell_\infty$, acting by the rule $\Phi_\Sigma(\mathbf{A}) = \{a_k\}\ \forall \mathbf{A} \in \mathcal{B}(\mathcal{H})^{\rm sa}$, where $a_k = \omega_k(\mathbf{A}), k \in \mathbb{N}$. If $\mathcal{N} = \mathrm{Ker}(\Phi_\Sigma)$ and $\mathcal{B}(\mathcal{H})^{\rm sa}_0 = \mathcal{B}(\mathcal{H})^{\rm sa}/\mathcal{N}$ then the mapping $\Phi_\Sigma :\ \mathcal{B}(\mathcal{H})^{\rm sa}_0 \to \ell_\infty$ is a bijection. Now we apply the construction of a shift-invariant measure on the Banach space ℓ_∞ given in papers [4, 7, 9]. A rectangle in the space ℓ_∞ is defined as the set of type $\Pi_{a,b}, a, b \in \ell_\infty :\ a_j \le b_j \forall j \in \mathbb{N}\}$ defined by the conditions $\Pi_{a,b} = \{x \in \ell_\infty :\ x_j \in [a_j, b_j), j \in \mathbb{N}\}$. The rectangle is called absolutely measurable if $\sum\limits_{j=1}^{\infty} \max\{\ln(b_j - a_j), 0\} < +\infty$. The set function $\lambda_0 :\ \mathcal{K} \to [0, +\infty)$ is defined by the equalities $\lambda_0(\Pi_{a,b}) = \exp(\sum\limits_{j=1}^{\infty} \ln(b_j - a_j))$ on the family of absolutely measurable rectangles $\mathcal{K}$ in the space ℓ_∞. The set function λ_0 is translation invariant and additive on the family of sets $\mathcal{K}$. It admits the extension to the translation invariant finite additive measure λ on the space ℓ_∞ such that the measure λ is defined on the minimal ring Λ containing the family of a sets $\mathcal{K}$ (see [4, 7, 9]).

The preimage $\Phi^{-1}_\Sigma(\Lambda)$ of the ring Λ of subsets of the space ℓ_∞ is the ring $\mathcal{R}_\Sigma$ of subsets of the algebra $\mathcal{B}(\mathcal{H})^{\rm sa}$. Let $\mu_\Sigma(B) = \lambda(\Phi_\Sigma(B))$ for any $B \in \mathcal{R}_\Sigma$. Therefore $\mu_\Sigma = \lambda \circ \Phi^{-1}_\Sigma$ is the image of the measure λ under the mapping Φ_Σ. The measure μ_Σ inherits certain properties of the measure λ, e.g. translation invariance and completeness. □

4. ON OPERATOR INTERVALS

Next, we study operator intervals in the Jordan algebra $\mathcal{B}(\mathcal{H})^{\mathrm{sa}}$. It is motivated by consideration of the image of the Stieltjes transform over all operator-valued measures generating a given sequence of Stieltjes–Hermite moments (see [10]). Operator intervals naturally appear in the noncommutative integration theory (see [11–13]).

Theorem 4.1. *For every operator $\mathbf{B} \in \mathcal{B}(\mathcal{H})^+$ we have $I_\mathbf{B} \supset M_\mathbf{B} \cap \mathcal{B}(\mathcal{H})^{sa}$. There exists a matrix $\mathbf{B} \in \mathbb{M}(\mathbb{C})_2^+$ such that $I_\mathbf{B} \neq M_\mathbf{B} \cap \mathbb{M}(\mathbb{C})_2^{sa}$.*

Proof. For $\mathbf{A} \in M_\mathbf{B} \cap \mathcal{B}(\mathcal{H})^{\mathrm{sa}}$ we have $\mathbf{A} \le |\mathbf{A}| \le \mathbf{B}$ and $-\mathbf{A} \le |\mathbf{A}| \le \mathbf{B}$, therefore $-\mathbf{B} \le \mathbf{A} \le \mathbf{B}$, i. e. $\mathbf{A} \in I_\mathbf{B}$. Assume that $I_\mathbf{B} = M_\mathbf{B} \cap \mathbb{M}(\mathbb{C})_2^{\mathrm{sa}}$ for all $\mathbf{B} \in \mathbb{M}(\mathbb{C})_2^+$. Since $-|\mathbf{X}| \le \mathbf{X} \le |\mathbf{X}|$ and $-|\mathbf{Y}| \le \mathbf{Y} \le |\mathbf{Y}|$ for all $\mathbf{X}, \mathbf{Y} \in \mathbb{M}(\mathbb{C})_2^{\mathrm{sa}}$, we have $-|\mathbf{X}| - |\mathbf{Y}| \le \mathbf{X} + \mathbf{Y} \le |\mathbf{X}| + |\mathbf{Y}|$. By the assumption we obtain the inequality $|\mathbf{X} + \mathbf{Y}| \le |\mathbf{X}| + |\mathbf{Y}|$ for all $\mathbf{X}, \mathbf{Y} \in \mathbb{M}(\mathbb{C})_2^{\mathrm{sa}}$, but it does not hold for

$$\mathbf{X} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Y} = 2^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

see [14]. Theorem is proved. □

Proposition 4.2. *Consider operators $\mathbf{X} \in \mathcal{B}(\mathcal{H})^+$ and $\mathbf{Y} \in I_\mathbf{X}$. Then $||\mathbf{Y}|| \le ||\mathbf{X}||$ and if $\mathbf{Y}$ is invertible then $\mathbf{X}$ is also invertible.*

Proof. By [15, Theorem 1] there exists a unitary operator $\mathbf{U} \in \mathcal{B}(\mathcal{H})^{\mathrm{sa}}$ such that $|\mathbf{Y}| \le (\mathbf{X} + \mathbf{UXU})/2$. Invertibility of $\mathbf{X}$ for invertible $\mathbf{Y}$ was established in [16, Corollary 2]. □

Proposition 4.3. *Consider operators $\mathbf{X}, \mathbf{Y} \in \mathcal{B}(\mathcal{H})^+$ and $a = \max\{||\mathbf{X}||, ||\mathbf{Y}||\}$. Then $\mathbf{X} - \mathbf{Y} \in I_{\mathbf{X}+\mathbf{Y}}$ and $\mathbf{X} - \mathbf{Y} \in I_{2a\mathbf{I}-\mathbf{X}-\mathbf{Y}}$.*

Proof. We have $-(\mathbf{X} + \mathbf{Y}) \le \mathbf{X} - \mathbf{Y} \le \mathbf{X} + \mathbf{Y}$, $\mathbf{X} \le a\mathbf{I}$, and $\mathbf{Y} \le a\mathbf{I}$. Then

$$\mathbf{X} + \mathbf{Y} - 2a\mathbf{I} \le \mathbf{X} - \mathbf{Y} = (a\mathbf{I} - \mathbf{Y}) - (a\mathbf{I} - \mathbf{X}) \le (a\mathbf{I} - \mathbf{Y}) + (a\mathbf{I} - \mathbf{X}).$$

□

Proposition 4.4. *Consider operators $\mathbf{X}, \mathbf{Y} \in \mathcal{B}(\mathcal{H})$. We have*

(i) $\mathbf{X}^*\mathbf{Y} + \mathbf{Y}^*\mathbf{X} \in I_{\mathbf{X}^*\mathbf{X}+\mathbf{Y}^*\mathbf{Y}}$;

(ii) $\mathbf{XY} + \mathbf{Y}^*\mathbf{X}^* \in I_{\mathbf{X}|\mathbf{Y}^*|^2\mathbf{X}^*+\mathbf{I}}$;

(iii) if $\mathbf{X} \in \mathcal{B}(\mathcal{H})^+$ then $\mathbf{XY}^* + \mathbf{YX} \in I_{\mathbf{X}+\mathbf{YXY}^*}$;

(iv) if $\mathbf{X} \in \mathcal{B}(\mathcal{H})^+$ and $\mathbf{Y} \in \mathcal{B}(\mathcal{H})^{\mathrm{sa}}$ then $\mathbf{Y} \in I_{|\mathbf{Y}-\mathbf{X}|+\mathbf{X}}$.

Proof. (i) Since $(\mathbf{X} \pm \mathbf{Y})^*(\mathbf{X} \pm \mathbf{Y}) \ge \theta$, the relation

$$-(\mathbf{X}^*\mathbf{X} + \mathbf{Y}^*\mathbf{Y}) \le \mathbf{X}^*\mathbf{Y} + \mathbf{Y}^*\mathbf{X} \le \mathbf{X}^*\mathbf{X} + \mathbf{Y}^*\mathbf{Y}$$

holds.

(ii) We have $(\mathbf{XY} \pm \mathbf{I})(\mathbf{XY} \pm \mathbf{I})^* \ge \theta$.

(iii) We have $(\sqrt{\mathbf{X}} \pm \mathbf{Y}\sqrt{\mathbf{X}})(\sqrt{\mathbf{X}} \pm \mathbf{Y}\sqrt{\mathbf{X}})^* \ge \theta$.

(iv) We have $\mathbf{Y} \le |\mathbf{Y} - \mathbf{X}| + \mathbf{X}$ and, since $\mathbf{X} \ge \theta$, we obtain $-|\mathbf{Y} - \mathbf{X}| - \mathbf{X} \le -|\mathbf{Y} - \mathbf{X}| + \mathbf{X} \le \mathbf{Y}$. □

There exists $\mathbf{B} \in \mathbb{M}(\mathbb{C})_2^+$ such that $M_\mathbf{B}$ is not convex [2, Theorem 3].

Proposition 4.5. *If an operator $\mathbf{B} \in \mathcal{B}(\mathcal{H})^+$ is a projection then the set $M_\mathbf{B}$ is convex.*

Proof. Consider operators $\mathbf{X}, \mathbf{Y} \in M_B$. Then $|\mathbf{X}|^2 \le |\mathbf{X}|$ and $|\mathbf{Y}|^2 \le |\mathbf{Y}|$. We obtain

$$\begin{aligned} |\lambda\mathbf{X} + (1-\lambda)\mathbf{Y}| &= \sqrt{\lambda^2\mathbf{X}^*\mathbf{X} + (1-\lambda)^2\mathbf{Y}^*\mathbf{Y} + \lambda(1-\lambda)(\mathbf{X}^*\mathbf{Y} + \mathbf{Y}^*\mathbf{X})} \\ &\le \sqrt{\lambda\mathbf{X}^*\mathbf{X} + (1-\lambda)\mathbf{Y}^*\mathbf{Y}} \le \sqrt{\lambda\mathbf{B} + (1-\lambda)\mathbf{B}} = \mathbf{B} \end{aligned}$$

for an arbitrary number $\lambda \in [0,1]$ by operator monotonocity of the function $f(t) = \sqrt{t}$ on $\mathbb{R}^+$ and by inequality $(\mathbf{X} \pm \mathbf{Y})^*(\mathbf{X} \pm \mathbf{Y}) \ge 0$. □

Remark 4.6. For $\mathbf{X} \in \mathcal{B}(\mathcal{H})^+$ the set $I_\mathbf{X}$ is convex. We have $I_\mathbf{X}^+ = I_\mathbf{X} \cap \mathcal{B}(\mathcal{H})^+ = \{\mathbf{Y} \in \mathcal{B}(\mathcal{H})^{\mathrm{sa}} : \theta \le \mathbf{Y} \le \mathbf{X}\}$ and the shift $\mathbf{X} + I_\mathbf{X} = I_{2\mathbf{X}}^+$. Denote by $\mathbf{P}$ the orthogonal projection onto the subspace

$\overline{\mathrm{Ran}\sqrt{2\mathbf{X}}}$. We have $\mathbf{T} \in \mathrm{Ext}I^+_{2\mathbf{X}}$ if and only if $\mathbf{T} = \sqrt{2\mathbf{X}}\mathbf{Q}\sqrt{2\mathbf{X}}$, where $\mathbf{Q}$ is a projection from $I^+_{\mathbf{P}}$, see [17, Theorem]. Since the set of extreme points of an arbitrary convex set with affine mapping goes into a set of extreme points of the image, we have

$$\mathrm{Ext}I_{\mathbf{X}} = -\mathbf{X} + \mathrm{Ext}I^+_{2\mathbf{X}}.$$

The set $\mathrm{Ext}I_{\mathbf{X}}$ is compact in the weak operator topology (that is, in the w-topology), since it is w-closed and lies in the ball of radius $||X||$ centered at θ [18, Proposition 2.4.2]. By the Krein–Milman theorem, $I_{\mathbf{X}}$ coincides with the w-closure of a convex hull sets of its extreme points.

The operator $\mathbf{X} \in \mathcal{B}(\mathcal{H})^+$ is compact if and only if $I_{\mathbf{X}}$ is $||\cdot||$-compact, and if these conditions are met, $I_{\mathbf{X}}$ coincides with $||\cdot||$-closure of a convex hull sets of its extreme points [2].

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