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On Operators all of Which Powers have the same Trace

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Abstract

We introduce the class $K_{\mathcal{A},\phi} = \{A \in \mathcal{A} : \phi(A^k) = \phi(A)$ for all $k \in \mathbb{N}\}$ for a linear functional ϕ on an algebra $\mathcal{A}$ and consider the properties of this class. Also we prove the "0–1 number lemma": if a set $\{z_k\}_{k=1}^n \subset \mathbb{C}$ is such that $z_1+\ldots+z_n = z_1^2+\ldots+z_n^2 = \cdots = z_1^{n+1}+\ldots+z_n^{n+1}$, then $z_k \in \{0, 1\}$, for all $k = 1, 2, \ldots, n$. This lemma helps us to show that $\{\phi(A) : A \in K_{\mathcal{A},\phi}\} = \{0, 1, \ldots, n\}$ and $\det(A) \in \{0, 1\}$ for $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$ and $\phi = \mathrm{tr}$, the canonical trace. We have $A = P + Z$ where P is a projection and Z is a nilpotent for any $A \in K_{\mathcal{A},\phi}$. Assume that for a trace class operator A there exists a constant $C \in \mathbb{C}$ such that $\mathrm{tr}(A^k) = C$ for all $k \in \mathbb{N}$. Then $C \in \mathbb{N}\bigcup\{0\}$ and the spectrum $\sigma(A)$ is a subset of $\{0, 1\}$. Finally we give the description of all the elements of the class $K_{\mathcal{A},\phi}$ for $\mathbb{M}_2(\mathbb{C})$.

Keywords Hilbert space · Normed algebra · Idempotent · C^*-algebra · W^*-algebra · Linear functional · State · Tracial functional · Trace class operator · Vandermonde matrix · Spectrum · Determinant

1 Introduction

Let $\mathcal{A}$ be an algebra. A number of authors consider various properties of the idempotent ($A^2 = A$) set $\mathcal{A}^{\mathrm{id}} \subset \mathcal{A}$ (see, e.g., [1, 7, 9]) or traces of powers of matrices (see, e.g., [16]). Note that any tripotent ($A^3 = A$) of $\mathcal{A}$ is a difference of two idempotents [2], so tripotents inherit many properties of idempotents [3]. Here for any linear functional on $\mathcal{A}$ we introduce the class that contains $\mathcal{A}^{\mathrm{id}}$, namely the class $K_{\mathcal{A},\phi} = \{a \in \mathcal{A} : \phi(A^k) = \phi(A)$ for all $k \in \mathbb{N}\}$ and consider its properties (Propositions 1–4, Corollaries 1 and 2). We prove the "0–1 number lemma": if a set $\{z_k\}_{k=1}^n \subset \mathbb{C}$ is such that $z_1 + \ldots + z_n = z_1^2 + \ldots + z_n^2 = \cdots = z_1^{n+1} + \ldots + z_n^{n+1}$, then $z_k \in \{0, 1\}$, for all $k = 1, 2, \ldots, n$ (Lemma 2). This lemma helps us to show that for $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$ and $\phi = \mathrm{tr}$ we have $\{\phi(A) : A \in K_{\mathcal{A},\phi}\} = \{0, 1, \ldots, n\}$ and $\det(A) \in \{0, 1\}$ (Theorem 2). So for $A \in K_{\mathcal{A},\phi}$ we obtain $\det(\exp(\exp(A))) = e^{n+(e-1)\mathrm{tr}(A)} \in \{e^n, e^{n-1+e}, e^{n-2+2e}, \ldots, e^{ne}\}$ (Corollary 7). Similar relations hold true for

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the determinants of the matrices $\exp(\sinh(A))$, $\exp(\cosh(A))$, $\exp(\sin(A))$, $\exp(\cos(A))$. We have $A = P + Z$ where P is a projection and Z is a nilpotent. Moreover, if $A \in K_{\mathcal{A},\phi}$ is invertible then $A^{-1} \in K_{\mathcal{A},\phi}$ (Corollary 3). If a matrix A lies in $K_{\mathcal{A},\phi}$ then $A^n - A^k$ is a commutator for all $n, k \in \mathbb{N}$ (Corollary 6). Assume that for a trace class operator A there exists a constant $C \in \mathbb{C}$ such that $\forall k \in \mathbb{N}$ $\mathrm{tr}(A^k) = C$. Then $C \in \mathbb{N} \bigcup \{0\}$ and the spectrum $\sigma(A)$ is a subset of $\{0, 1\}$ (Theorem 3). The canonical trace and trace class operators play an important part in quantum mechanics. Finally we give the complete description of all the elements of class $K_{\mathcal{A},\phi}$ for $\mathbb{M}_2(\mathbb{C})$ (Example 5).

2 Definitions and Notation

Let $\mathcal{A}$ be an algebra, $\mathcal{A}^{\mathrm{id}}$ be the idempotent ($A^2 = A$) subset of $\mathcal{A}$. Let $\mathcal{A}^{(n)} = \{A \in \mathcal{A} : A^n = A\}$ be the set of all n-potent elements in $\mathcal{A}$, thus $\mathcal{A}^{\mathrm{id}} \subset \mathcal{A}^{(n)}$ for all $n \in \mathbb{N}, n \geq 3$. The algebra $\mathcal{A}$ is *normed* if $\mathcal{A}$ admits a norm $\|\cdot\|$ such that $\|XY\| \leq \|X\|\|Y\|$ for all $X, Y \in \mathcal{A}$. The algebra $\mathcal{A}$ is *unital* (i.e. possesses a unit) if there exists an element $(0 \neq)I \in \mathcal{A}$ such that $IX = XI = X$ for all $X \in \mathcal{A}$. An element X of a unital normed *-algebra is *unitary* if $X^*X = XX^* = I$; an *isometry* if $X^*X = I$; a *partial isometry* if $XX^*X = X$; a *projection* if $X = X^* = X^2$.

C^*-algebra is a complex Banach $*$-algebra such that $\|X^*X\| = \|X\|^2$ for all $X \in \mathcal{A}$. By the Gel'fand-Naimark theorem any C^*-algebra can be realized as a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. A linear functional ϕ on a C^*-algebra $\mathcal{A}$ is said to be *positive* if $\phi(\mathcal{A}^+) \subset \mathbb{R}^+$, here $\mathcal{A}^+$ is the cone of positive elements in $\mathcal{A}$. A positive functional ϕ on a C^*-algebra $\mathcal{A}$ is said to be *tracial* if $\phi(X^*X) = \phi(XX^*)$ for all $X \in \mathcal{A}$; *faithful*, if $\phi(X) > 0$ for all $X \in \mathcal{A}^+$, $A \neq 0$. A *state* of a C^*-algebra $\mathcal{A}$ (or a state on $\mathcal{A}$) is any positive functional of norm 1. A *W^*-algebra* is a C^*-algebra $\mathcal{A}$ with a predual Banach space $\mathcal{A}_* : \mathcal{A} \simeq (\mathcal{A}_*)^*$.

3 Main Results

Let ϕ, ψ be linear functionals on an algebra $\mathcal{A}$. Consider the class

$$K_{\mathcal{A},\phi} = \{A \in \mathcal{A} : \phi(A^k) = \phi(A) \text{ for all } k \in \mathbb{N}\}.$$

Clearly, $\mathcal{A}^{\mathrm{id}} \subset K_{\mathcal{A},\phi} = K_{\mathcal{A},\lambda\phi}$ ($\lambda \in \mathbb{C} \setminus \{0\}$) and $K_{\mathcal{A},\phi} \bigcap K_{\mathcal{A},\psi} \subset K_{\mathcal{A},\phi+\psi}$. If $A, B \in K_{\mathcal{A},\phi}$ and $AB = BA = 0$ then $A + B \in K_{\mathcal{A},\phi}$. Put $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$ and let $\phi = \mathrm{tr}$ be the canonical trace. If $A \in \mathcal{A}$ is nilpotent then $A \in K_{\mathcal{A},\phi}$.

Example 1 Let S be a set with $\mathrm{card}S \geq 2$, $\mathcal{A} = l^\infty(S)$, then the set of all bounded complex-valued functions on S is a unital Banach algebra where the operations are defined pointwise:

$$(f + g)(x) = f(x) + g(x), \quad (fg)(x) = f(x)g(x), \quad (\lambda f)(x) = \lambda f(x),$$

and the norm is the sup-norm $\|f\|_\infty = \sup_{x \in S} |f(x)|$. Also $\mathcal{A}^{\mathrm{id}} = \{0, \chi_A, A \subset S\}$. For fixed $y \in S$ define the continuous linear functional

$$\phi(f) = f(y), \quad f \in \mathcal{A}.$$

We have $K_{\mathcal{A},\phi} = \{f \in \mathcal{A} : f(y) \in \{0, 1\}\} \neq \mathcal{A}^{\mathrm{id}}$.

Example 2 Let a normed algebra $\mathcal{B}$ have a subalgebra isomorphic to algebra $\mathcal{A} = l^\infty(S)$ for some set S with card$S \geq 2$. Then $\mathcal{B}$ admits a continuous linear functional ψ such that $K_{\mathcal{B},\psi} \neq \mathcal{B}^{\mathrm{id}}$. Indeed, we define a continuous linear functional ϕ on $\mathcal{A}$ as in Example 1 and consider its extension (by Hahn–Banach Theorem) ψ on $\mathcal{B}$.

Example 3 Let $\mathcal{A} = \mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. Let $\xi, \eta \in \mathcal{H}$, $\|\xi\| = \|\eta\| = 1$, $\xi \perp \eta$ and

$$\phi(T) = \langle T\xi, \xi \rangle, \quad T \in \mathcal{A}.$$

We have $K_{\mathcal{A},\phi} \neq \mathcal{A}^{\mathrm{id}}$ (the operator $T = \langle \cdot, \xi \rangle \xi + 2\langle \cdot, \eta \rangle \eta$ lies in $K_{\mathcal{A},\phi} \setminus \mathcal{A}^{\mathrm{id}}$).

Proposition 1 *Let ϕ be a linear functional on an algebra $\mathcal{A}$ and $A \in K_{\mathcal{A},\phi}$. Then*

(i) $\lambda A^n + (1-\lambda)A^m \in K_{\mathcal{A},\phi}$ *for all* $\lambda \in [0, 1]$ *and* $n, m \in \mathbb{N}$.
Moreover if $\mathcal{A}$ is unital and $\phi(A) = \phi(I)$ then
(ii) $I - A \in K_{\mathcal{A},\phi}$ *and* $\phi((I - A)^n) = 0$ *for all* $n \in \mathbb{N}$;
(iii) $(2I - A)^n \in K_{\mathcal{A},\phi}$ *for all* $n \in \mathbb{N}$.

Proof We apply the Newton binomial formula.

(i). We have $(\lambda A^n + (1-\lambda)A^m)^k = \lambda^k A^{nk} + \lambda^{k-1}(1-\lambda)kA^{n(k-1)+m} + \ldots + (1-\lambda)^k A^{nk}$, so

$$\begin{aligned}\phi((\lambda A^n + (1-\lambda)A^m)^k) &= (\lambda^k + \lambda^{k-1}(1-\lambda)k + \ldots + (1-\lambda)^k)\phi(A) \\ &= (\lambda + (1-\lambda))^k \phi(A) = \phi(A)\end{aligned}$$

for all $\lambda \in [0, 1]$ and $k \in \mathbb{N}$.

(ii). We have $(I - A)^n = I - nA + \frac{n(n-1)}{2!}A^2 - \ldots + (-1)^n A^n$, so

$$\phi((I - A)^n) = \left(1 - n + \frac{n(n-1)}{2!} - \ldots + (-1)^n\right)\phi(I) = (1-1)^n \phi(I) = 0.$$

(iii). We have $(2I - A)^n = 2^n I - 2^{n-1}nA + 2^{n-2}\frac{n(n-1)}{2!}A^2 - \ldots + (-1)^n A^n$, hence

$$\begin{aligned}\phi((2I - A)^n) &= \left(2^n - 2^{n-1}n + 2^{n-2}\frac{n(n-1)}{2!} - \ldots + (-1)^n\right)\phi(I) \\ &= (2-1)^n \phi(I) = \phi(I)\end{aligned}$$

for all $n \in \mathbb{N}$. The statement is proved. □

Proposition 2 *Let ϕ be a continuous functional on a normed algebra $\mathcal{A}$. Then $K_{\mathcal{A},\phi}$ is closed in $\mathcal{A}$, and if $A \in K_{\mathcal{A},\phi}$ with $\phi(A) \neq 0$ then $\|A\| \geq 1$.*

Proof If $\{A_n\}_{n=1}^\infty \subset \mathcal{A}$ and $A_n \xrightarrow{\|\cdot\|} A \in \mathcal{A}$ as $n \to \infty$ then $A_n^k \xrightarrow{\|\cdot\|} A^k$ as $n \to \infty$ for all $k \in \mathbb{N}$. If $A \in \mathcal{A}$ and $\|A\| < 1$ then $0 \leq \|A^n\| \leq \|A\|^n \to 0$ as $n \to \infty$. Since the functional ϕ is continuous $\phi(A^n) \to 0$ as $n \to \infty$, and $A \notin K_{\mathcal{A},\phi}$. The statement is proved. □

Corollary 1 *If ϕ is a tracial functional on a unital C^*-algebra $\mathcal{A}$ then $K_{\mathcal{A},\phi}$ is a unitarily invariant closed subset of $\mathcal{A}$.*

Proof A tracial functional ϕ is positive, so ϕ is automatically continuous [15, Ch. 1, Proposition 9.12]. □

Let ϕ be a tracial functional on a unital C^*-algebra $\mathcal{A}$, $U \in \mathcal{A}$ be an isometry. If $A \in K_{\mathcal{A},\phi}$ then $UAU^* \in K_{\mathcal{A},\phi}$. This follows from the relations $\phi(UAU^*) = \phi(AU^*U) = \phi(A)$ and $(UAU^*)^k = UA^kU^*$ for all $k \in \mathbb{N}$.

Proposition 3 *Let ϕ be a Hermitian functional on a Banach $*$-algebra $\mathcal{A}$ and $A \in \mathcal{A}$. Then the following conditions are equivalent:*

(i) $A \in K_{\mathcal{A},\phi}$;
(ii) $A^* \in K_{\mathcal{A},\phi}$.

Proof (i) $\Rightarrow$ (ii). We have $X^{*k} = X^{k*}$ and $\phi(X^*) = \overline{\phi(X)}$ for all $X \in \mathcal{A}$ and $k \in \mathbb{N}$ (the bar denotes complex conjugation).
(ii) $\Rightarrow$ (i). The assertion follows from the one proved above, since $(X^*)^* = X$ for all $X \in \mathcal{A}$. □

If ϕ is a tracial normal state on a W^*-algebra $\mathcal{A}$ and $A \in \mathcal{A}$ then the following conditions are equivalent: (i) $\phi(A) = 0$; (ii) $\|I + zA\|_1 \geq 1$ for all $z \in \mathbb{C}$ ([5, Theorem 4.8]); here and in what follows $\|X\|_1 = \phi(\sqrt{X^*X})$ for all $X \in \mathcal{A}$.

Proposition 4 *For a tracial normal state ϕ on a W^*-algebra $\mathcal{A}$ and $A \in \mathcal{A}$ the following conditions are equivalent:*

(i) $A \in K_{\mathcal{A},\phi}$;
(ii) $\|I + z(A^k - A^m)\|_1 \geq 1$ *for all* $z \in \mathbb{C}$ *and* $k, m \in \mathbb{N}$.

Item (ii) of Proposition 1 yields

Corollary 2 *Let ϕ be a tracial normal state on a W^*-algebra $\mathcal{A}$ and $A \in K_{\mathcal{A},\phi}$. If $\phi(A) = 1$ then $\|I + z(I - A)^k\|_1 \geq 1$ for all $k \in \mathbb{N}$ and $z \in \mathbb{C}$.*

Lemma 1 *Consider $\alpha_k \in \mathbb{R}^+ \setminus \{0\}$, $k = 1, \ldots, n$. If $\omega_k \in \mathbb{S}^1$, the unit circle in $\mathbb{C}$, $k = 1, \ldots, n$ and $\sum_{k=1}^{n} \alpha_k \omega_k = \sum_{k=1}^{n} \alpha_k$ then $\omega_k = 1$ for all $k \in \{1, \ldots, n\}$.*

Proof Consider the equality $\mathrm{Re}[\sum_{k=1}^{n} \alpha_k \omega_k] = \mathrm{Re}[\sum_{k=1}^{n} \alpha_k]$. Since for any k $\mathrm{Re}[\omega_k] = \cos(\nu_k)$ for some $\nu_k \in [0, 2\pi)$ we have $\sum_{k=1}^{n} \alpha_k \cos(\nu_k) = \sum_{k=1}^{n} \alpha_k$. Note that positivity of α_k implies that for each $k \in \{1, \ldots, n\}$ $\alpha_k \cos(\nu_k) \leq \alpha_k$ and equality holds only for $\nu_k = 0$. Hence for each $k \in \{1, \ldots, n\}$ we have $\nu_k = 0$. □

Theorem 1 *Let ϕ be a faithful tracial functional on a C^*-algebra $\mathcal{A}$ and $n \in \mathbb{N}$, $n \geq 3$. Then $(\mathcal{A}^{(n)} \setminus \mathcal{A}^{id}) \cap K_{\mathcal{A},\phi} = \varnothing$.*

Proof If $n \geq 3$ and $A \in \mathcal{A}^{(n)} \setminus \mathcal{A}^{\text{id}}$, then

$$A = \sum_{k=1}^{n-1} \omega_k Q_k, \tag{1}$$

where $\{\omega_k\}_{k=1}^{n-1}$ are $(n-1)$-th degree roots of unity, $\{Q_k\}_{k=1}^{n-1} \subset \mathcal{A}^{\text{id}}$ (for $n = 3$ see [2, Proposition 1]; in the general case it is an unpublished result of Adel Abyzov) and

$$\text{card}\{1 \leq k \leq n-1 : \ Q_k \neq 0, \ \omega_k \neq 1\} \geq 1. \tag{2}$$

Equalities $A^{2n-2} = A^n A^{n-2} = AA^{n-2} = A^{n-1}$ yield $A^{n-1} \in \mathcal{A}^{\text{id}}$. In fact,

$$A^{n-1} = \sum_{k=1}^{n-1} Q_k. \tag{3}$$

By [4, Theorem 4.6] we have $\phi(Q) > 0$ for all $Q \in \mathcal{A}^{\text{id}} \setminus \{0\}$. Via (1), (3) the equality $\phi(A) = \phi(A^{n-1})$ is equivalent to the relation

$$\sum_{k=1}^{n-1} \omega_k \phi(Q_k) = \sum_{k=1}^{n-1} \phi(Q_k),$$

which is false by (2) and Lemma 1. □

Example 4 The condition, that ϕ is faithful, cannot be omitted in Theorem 1. Consider an abelian W^*-algebra $\mathcal{A} = l^\infty$ of all bounded complex sequences and the tracial functional defined by relation

$$\phi(x) = x_1 \text{ for } x = \{x_n\}_{n=1}^{\infty} \in l^\infty.$$

Then the tripotent $x = \{x_n\}_{n=1}^{\infty}$ with $x_1 = 1$ and $x_n = -1$ for all $n \geq 2$ lies in $K_{\mathcal{A},\phi}$ (cf. Example 1).

Lemma 2 *Consider a set* $\{z_k\}_{k=1}^{n} \subset \mathbb{C}$ *such that*

$$z_1 + \ldots + z_n = z_1^2 + \ldots + z_n^2 = \cdots = z_1^{n+1} + \ldots + z_n^{n+1}.$$

Then $z_k \in \{0, 1\}$, *for all* $k = 1, 2, \ldots, n$.

Proof The proof is by induction on n.

The induction base is $n = 1$. Then $z_1 = z_1^2$ implies that $z_1 \in \{0, 1\}$.

The induction assumption. Let the assertion hold for $n \leq N - 1$.

The induction step. We prove the statement for $n = N$.

Consider

$$z_1 + \ldots + z_N = z_1^2 + \ldots + z_N^2 = \cdots = z_1^{N+1} + \ldots + z_N^{N+1} = z.$$

Then we have the matrix equation

$$\begin{pmatrix} z_1 & z_2 & \cdots & z_N & -z \\ z_1^2 & z_2^2 & \cdots & z_N^2 & -z \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ z_1^{N+1} & z_2^{N+1} & \cdots & z_N^{N+1} & -z \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4}$$

Denote the matrix by M and the vector of units by $\vec{v}_0$. Rewrite (4) in operator form as follows: $M\vec{v}_0 = \vec{0}$.

Then $\ker M \ni \vec{v}_0 \neq \vec{0}$, and, consequently, $\det(M) = 0$. The matrix M determinant is a multiple of the relative Vandermonde matrix determinant and equals $\det(M) = (-z) \prod\limits_{1\le k<j\le N} (z_k - z_j) \prod\limits_{j=1}^{N} (z_j - 1) \prod\limits_{j=1}^{N} z_j$.

Consider all the possible partial cases.

Case 1. $\exists j \in \{1, \ldots, N\}, z_j \in \{0, 1\}$. Without loss of generality assume that $j = N$. Then

$$z_1+\ldots+z_{N-1} = z_1^2+\ldots+z_{N-1}^2 = \cdots = z_1^N+\ldots+z_{N-1}^N = z_1^{N+1}+\ldots+z_{N-1}^{N+1} = z-\{1 \text{ or } 0\},$$

and by the induction hypothesis we have $z_j \in \{0, 1\}$, $j = 1, 2, \ldots, N-1$.

Case 2. $z \neq 0, \exists k, j \in \{1, 2, , \ldots, N\}, z_k = z_j \neq 0$ or 1. □

Statement 1 rank $M =$ (the number of different z_j) $+1$, $\forall \vec{v} \in \ker M$, $\sum\limits_{j=1}^{N} v_j = 0$.

Proof Denote the number of different z_j by l and renumber $z_1, \ldots, z_N$ so that $\forall j, k$, $1 \le k < j \le l$, $z_j \neq z_k$.

Since the matrix M contains the order $l+1$ minor

$$N_1 = \begin{pmatrix} z_1 & z_2 & \cdots & z_l & -z \\ z_1^2 & z_2^2 & \cdots & z_l^2 & -z \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ z_1^{l+1} & z_2^{l+1} & \cdots & z_l^{l+1} & -z \end{pmatrix},$$

and, moreover, (again with the help of the Vandermoinde determinant)

$$\det(N_1) = (-z) \prod_{1\le k<j\le l} (z_k - z_j) \prod_{j=1}^{l} (z_j - 1) \prod_{j=1}^{l} z_j \neq 0,$$

so we have rank $M \ge l+1$, $\dim\ker M \le (N+1) - (l+1) = N - l$.

Rearrange the first N columns of M as follows:

$$M' = \begin{pmatrix} z_1 \ \cdots \ z_1 & z_2 \ \cdots \ z_2 & \cdots & z_l \ \cdots \ z_l & -z \\ z_1^2 \ \cdots \ z_1^2 & z_2^2 \ \cdots \ z_2^2 & \cdots & z_l^2 \ \cdots \ z_l^2 & -z \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underbrace{z_1^{N+1} \ \cdots \ z_1^{N+1}}_{k_1} & \underbrace{z_2^{N+1} \ \cdots \ z_2^{N+1}}_{k_2} & \cdots & \underbrace{z_l^{N+1} \ \cdots \ z_l^{N+1}}_{k_l} & -z \end{pmatrix}$$

Then

$$\begin{aligned} \ker M' \supseteq \ & \text{span}[(1, -1, 0, \ldots, 0), (1, 0, -1, 0, \ldots, 0), \ldots, (1, 0, \ldots, 0, \overset{k_1}{-1}, 0, \ldots, 0), \\ & (0, \ldots, 0, \overset{k_1+1}{1}, -1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \overset{k_1+1}{1}, 0, \ldots, 0, \overset{k_1+k_2}{-1}, 0, \ldots, 0), \\ & \cdots, \\ & (0, \ldots, 0, \overset{k_1+k_2+\ldots+k_{l-1}+1}{1}, -1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \overset{k_1+k_2+\ldots+k_{l-1}+1}{1}, \\ & 0, \ldots, 0, \overset{k_1+k_2+\ldots+k_l}{-1}, 0)], \end{aligned} \qquad (5)$$

and $\dim\ker M = \dim\ker M' \geq k_1 - 1 + k_2 - 1 + \ldots + k_l - 1 = N - l$. Hence $\dim\ker M = N - l$, rank $M = l + 1$ and formula (5) is an equality.

Now by relation (5) $\sum_{j=1}^{N} e_j = 0$ for any base vector $\vec{e}$ of $\ker M'$. Then $\forall \vec{v} \in \ker M$ we also have $\sum_{j=1}^{N} v_j = 0$. □

Statement 1 implies that $\vec{v}_0 \notin \ker M$, and we obtain a contradiction with the assumption, so we have no $z_j = z_k \neq 0$ or 1 for $z \neq 0$.

Case 3. $z = 0, \exists k, j \in \{1, 2, , \ldots, N\}, z_k = z_j \neq 0$ or 1.

Statement 2 rank $M =$ (the number of different z_j), $\forall \vec{v} \in \ker M$, $\sum_{j=1}^{N} v_j = 0$.

Proof The proof is similar to that of Statement 1. Again denote the number of different z_j by l and renumber $z_1, \ldots, z_N$ so that $\forall j, k, 1 \leq k < j \leq l, z_j \neq z_k$.

The matrix M then contains the order l minor

$$N_2 = \begin{pmatrix} z_1 & z_2 & \cdots & z_l \\ z_1^2 & z_2^2 & \cdots & z_l^2 \\ \cdots & \cdots & \cdots & \cdots \\ z_1^{l+1} & z_2^{l+1} & \cdots & z_l^{l+1} \end{pmatrix}$$

of determinant

$$\det(N_2) = \prod_{1 \leq k < j \leq l} (z_k - z_j) \prod_{j=1}^{l} z_j \neq 0.$$

Moreover,

$$M' = \begin{pmatrix} z_1 \ \ldots \ z_1 & z_2 \ \ldots \ z_2 & \cdots & z_l \ \ldots \ z_l & 0 \\ z_1^2 \ \ldots \ z_1^2 & z_2^2 \ \ldots \ z_2^2 & \cdots & z_l^2 \ \ldots \ z_l^2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underbrace{z_1^{N+1} \ \ldots \ z_1^{N+1}}_{k_1} & \underbrace{z_2^{N+1} \ \ldots \ z_2^{N+1}}_{k_2} & \cdots & \underbrace{z_l^{N+1} \ \ldots \ z_l^{N+1}}_{k_l} & 0 \end{pmatrix}$$

and

$$\ker M' = \text{span}\,[(\text{base of } \ker M' \text{ from the proof of Statement 1}), (0, \ldots, 0, 1)] \qquad (6)$$

Again formula (6) implies that we have $\sum_{j=1}^{N} e_j = 0$ for any $\vec{e}$ of the space $\ker M'$ base. So $\forall \vec{v} \in \ker M$, $\sum_{j=1}^{N} v_j = 0$.

Statement 2 then tells us once more that $\vec{v}_0 \notin \ker M$. Again we obtain a contradiction with the assumption, so we also have no $z_j = z_k \neq 0$ or 1 for $z = 0$.

Case 4. $z = 0$ and all the numbers z_j, $j = 1, 2, \ldots, N$, are different. Then $\ker M = \text{span}[(0, \ldots, 0, 1)]$ and again $\vec{v}_0 \notin \ker M$. This completes the proof of the lemma.

□

Remark 1 The assertion of Lemma 2 can be generalized to the case of $\sum_{k=1}^{n} z_k^{l+st} = z$ for fixed $l, s \in \mathbb{N}$ and all $t \in \mathbb{N}$. The proof is the same.

Theorem 2 *Consider* $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$ *and* $\phi = \mathrm{tr}$. *Then* $\det(A) \in \{0, 1\}$ *for* $A \in K_{\mathcal{A},\phi}$ *and* $\{\phi(A) : A \in K_{\mathcal{A},\phi}\} = \{0, 1, \ldots, n\}$.

Proof Recall that $\mathcal{A}^{\mathrm{id}} \subset K_{\mathcal{A},\phi}$ and $\{\phi(A) : A \in \mathcal{A}^{\mathrm{id}}\} = \{0, 1, \ldots, n\}$. Any matrix $A \in \mathbb{M}_n(\mathbb{C})$ is unitarily similar to some upper triangular matrix B by the Schur decomposition $A = U^*BU$ ([10, Theorem 2.3.1]). Then $A^k = U^*B^kU$ for all $k \in \mathbb{N}$. So we may assume that the matrix $A \in K_{\mathcal{A},\phi}$ is an upper triangular one and rewrite the condition $\mathrm{tr}(A) = \mathrm{tr}(A^2) = \cdots = \mathrm{tr}(A^k) = \cdots$ as follows:

$$\sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} a_{ii}^2 = \cdots = \sum_{i=1}^{n} a_{ii}^k = \cdots .$$

By Lemma 2 we have $a_{ii} \in \{0, 1\}$ for all $i = 1, \ldots, n$. Thus $\det(A) \in \{0, 1\}$. This completes the proof. □

Note that each unitriangular matrix and, in particular, any Heisenberg group matrix belongs to the class $K_{\mathcal{A},\phi}$.

Corollary 3 *Consider* $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$ *and* $\phi = \mathrm{tr}$. *If* $A \in K_{\mathcal{A},\phi}$ *then* $A = P + Z$ *where* P *is a projection and* Z *is a nilpotent. Moreover, if* $A \in K_{\mathcal{A},\phi}$ *is invertible then* $A^{-1} \in K_{\mathcal{A},\phi}$.

Recall that every idempotent A from C^*-algebra $\mathcal{A}$ is a sum $A = P + Z$ where P is a projection and Z is a nilpotent with $Z^2 = 0$, $PZ = Z$ and $ZP = 0$ [11, Theorem 1.3]. A similar representation holds also for (unbounded) τ-measurable idempotents affiliated with a semifinite von Neumann algebra [6, Theorem 2.23]. Corollary 3 generalizes a well-known result (see [8, Theorem 8] or [14, item (j) of Theorem 2]): $\mathrm{tr}(A) = \mathrm{tr}(A^2) = \cdots = \mathrm{tr}(A^n) = 0$ implies that A is nilpotent.

Corollary 4 *If in conditions of Theorem 2 a matrix* $A \in K_{\mathcal{A},\phi}$ *is unitary then* $A = I$.

Proof An upper triangular matrix $B = UAU^* = [b_{ij}]_{i,j=1}^n$ is unitary as the product of unitary matrices. Since $b_{ii} \in \{0, 1\}$ for all $i = 1, 2, \ldots, n$ we have $B = I$, hence $A = I$. □

Corollary 5 *If in conditions of Theorem 2 a matrix* $A \in K_{\mathcal{A},\phi}$ *is nondegenerate and* $Z \neq 0$ *then* $\|A^n\| \to \infty$ *as* $n \to \infty$.

Proof Consider the upper triangular representation of A as in the proof of Theorem 2. Then the first nonzero nondiagonal element a_{ij} in the right and lower corner of A turns into na_{ij} for A^n. □

Corollary 6 *If in conditions of Theorem 2 a matrix* $A \in K_{\mathcal{A},\phi}$ *then* $A^n - A^k$ *is a commutator for all* $n, k \in \mathbb{N}$.

Proof For $X \in \mathbb{M}_n(\mathbb{C})$ we have $\mathrm{tr}(X) = 0$ if and only if X is a commutator. □

Theorem 3 *Assume that for a trace class operator* A *there exists a constant* $C \in \mathbb{C}$ *such that* $\mathrm{tr}(A^k) = C$ *for all* $k \in \mathbb{N}$. *Then* $C \in \mathbb{N}\bigcup\{0\}$ *and the spectrum* $\sigma(A)$ *is a subset of* $\{0, 1\}$.

Proof Assume that the series $\sum_{j=1}^{\infty} z_j$, $z_j \in \mathbb{C}$ absolutely converges and the infinite series

$$\sum_{j=1}^{\infty} z_j^k, \tag{7}$$

converge to the same number C for all $k \in \mathbb{N}$. Assume that $z_j \neq 0$ or 1. Since all the series of (7) converge absolutely we can consider the modified series with mutually different numbers z_{j_l}: $\sum_{l=1}^{\infty} m_{j_l} z_{j_l}^k$, here m_{j_l} is the number of summands equal to z_{j_l}. If the number of nonzero summands is finite then Lemma 2 yields the result. Assume then that there exist infinitely many mutually different nonzero elements $(z_{j_l})_{l=1}^{\infty}$.

Then the infinite Vandermonde operator $V(1, z_{j_1}, z_{j_2}, \ldots) = \begin{pmatrix} 1 & z_{j_1} & z_{j_2} & \ldots \\ 1 & z_{j_1}^2 & z_{j_2}^2 & \ldots \\ 1 & z_{j_1}^3 & z_{j_2}^3 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$ on the weighted space L_2 with the norm $\|(v_1, v_2, \ldots)\| = \sum_k \frac{|v_k|^2}{2^{m_{j_k} k}}$ is nondegenerate as LU product of two nondegenerate operators [13, Theorem 2.1]

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & \ldots \\ 1 & 1 + z_{j_1} & 1 & 0 & \ldots \\ 1 & 1 + z_{j_1} + z_{j_1}^2 & 1 + z_{j_1} + z_{j_2} & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & z_{j_1} & z_{j_2} & z_{j_3} & \ldots \\ 0 & z_{j_1}(z_{j_1} - 1) & z_{j_2}(z_{j_2} - 1) & z_{j_3}(z_{j_3} - 1) & \ldots \\ 0 & 0 & z_{j_2}(z_{j_2} - 1)(z_{j_2} - z_{j_1}) & z_{j_3}(z_{j_3} - 1)(z_{j_3} - z_{j_1}) & \ldots \\ 0 & 0 & 0 & z_{j_3}(z_{j_3} - 1)(z_{j_3} - z_{j_1})(z_{j_3} - z_{j_2}) & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

Its kernel should be zero, but it contains the vector $(-C, m_{j_1}, m_{j_2}, \ldots)$. A contradiction. So $z_j = 0$ or 1 and $C \in \mathbb{N}$.

This fact and Lidskii theorem combined allow us to state that for any trace class operator A such that $\mathrm{tr}(A^k) = C$ for all $k \in \mathbb{N}$ we have the spectrum $\sigma(A) \subset \{0, 1\}$ and $C \in \mathbb{N}$. $\square$

Remark 2 Similar statement holds for real-valued functions from the algebra $L^{\infty}(\mathbb{R}, d\mu) \cap L^1(\mathbb{R}, d\mu)$, where μ is the linear Lebesgue measure. Consider the bounded measurable function $f : \mathbb{R} \to \mathbb{R}$, such that $\int_{-\infty}^{+\infty} f^k(x)dx = C$. We restrict ourselves only to even degrees $2k$ and assume that $C \geq 0$ (otherwise put $\tilde{f} = f^2$). If $f > 1$ on some set $S \subset \mathbb{R}$ of positive measure then $\int_{-\infty}^{+\infty} f^k(x)dx \geq \int_S f^k(x)dx \to +\infty$. Hence $f(x) \in [0, 1]$. Again if $0 < f <$

1 on some set T of positive measure then $\int\limits_{-\infty}^{+\infty} f^k(x)dx = \int\limits_T f^k(x)dx + \int\limits_{\mathbb{R}\setminus T} f^k(x)dx < \int\limits_T f^{k-1}(x)dx + \int\limits_{\mathbb{R}\setminus T} f^{k-1}(x)dx = \int\limits_{-\infty}^{+\infty} f^{k-1}(x)dx$ and we have a contradiction with the assumption. So f equals either 1 or 0 and possesses a support of measure C.

The statement nevertheless is false for complex functions. Consider the function $f(x) = 1 + \lambda e^{ix}$, $x \in [0, 2\pi]$, $\lambda \in \mathbb{C}$. Then $\int\limits_0^{2\pi} f^k(x)dx = 2\pi$ for any $k \in \mathbb{N}$.

Let $\mathcal{A}$ be a unital Banach algebra. Let us define for any $A \in \mathcal{A}$ the following power series convergent in $\mathcal{A}$:

$$\begin{aligned}
\exp(A) &= I + A + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!} + \cdots, \\
\sinh(A) &= A + \frac{A^3}{3!} + \cdots + \frac{A^{2n+1}}{(2n+1)!} + \cdots, \\
\cosh(A) &= I + \frac{A^2}{2!} + \cdots + \frac{A^{2n}}{(2n)!} + \cdots, \\
\sin(A) &= A - \frac{A^3}{3!} + \frac{A^5}{5!} - \cdots + (-1)^n \frac{A^{2n+1}}{(2n+1)!} + \cdots, \\
\cos(A) &= I - \frac{A^2}{2!} + \cdots + (-1)^n \frac{A^{2n}}{(2n)!} + \cdots.
\end{aligned}$$

Proposition 5 *Let ϕ be a continuous linear functional on a unital Banach algebra $\mathcal{A}$ and $A \in K_{\mathcal{A},\phi}$, $e = \exp(1) = 2.718281828\ldots$. Then*

(i) $\phi(\exp(A)) = \phi(I) + (e-1)\phi(A)$;
(ii) $\phi(\sinh(A)) = \sinh(1) \cdot \phi(A)$;
(iii) $\phi(\cosh(A)) = \phi(I) + (\cosh(1) - 1)\phi(A)$;
(iv) $\phi(\sin(A)) = \sin(1) \cdot \phi(A)$;
(v) $\phi(\cos(A)) = \phi(I) + (\cos(1) - 1)\phi(A)$.

Corollary 7 *Consider $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$ and let $\phi = \mathrm{tr}$ be a canonical trace. If $A \in K_{\mathcal{A},\phi}$ then we have the relation* $\det(\exp(\exp(A))) = e^{n+(e-1)\mathrm{tr}(A)} \in \{e^n, e^{n-1+e}, e^{n-2+2e}, \ldots, e^{ne}\}$.

Proof The assertion follows from item (i) of Proposition 5 and the equality $\det(\exp(X)) = e^{\mathrm{tr(X)}}$ for all $X \in \mathbb{M}_n(\mathbb{C})$, see formula (1) of Section 4.16 in [12]. Theorem 1 tells us that $\mathrm{tr}(A) \in \{0, 1, \ldots, n\}$, so $\det(\exp(\exp(A))) \in \{e^n, e^{n-1+e}, e^{n-2+2e}, \ldots, e^{ne}\}$. Similarly we have

$$\begin{aligned}
\det(\exp(\sinh(A))) &= e^{\sinh(1)\mathrm{tr}(A)} \in \{1, e^{\sinh(1)}, e^{2\sinh(1)}, \ldots, e^{n\sinh(1)}\}, \\
\det(\exp(\cosh(A))) &= e^{n+(\cosh(1)-1)\mathrm{tr}(A)} \in \{e^n, e^{n-1+\cosh(1)}, e^{n-2+2\cosh(1)}, \ldots, e^{n\cosh(1)}\}, \\
\det(\exp(\sin(A))) &= e^{\sin(1)\mathrm{tr}(A)} \in \{1, e^{\sin(1)}, e^{2\sin(1)}, \ldots, e^{n\sin(1)}\}, \\
\det(\exp(\cos(A))) &= e^{n+(\cos(1)-1)\mathrm{tr}(A)} \in \{e^n, e^{n-1+\cos(1)}, e^{n-2+2\cos(1)}, \ldots, e^{n\cos(1)}\}.
\end{aligned}$$

This completes the proof. □

Example 5 Consider $\mathcal{A} = \mathbb{M}_2(\mathbb{C})$, $\phi = \mathrm{tr}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{\mathcal{A},\phi}$. By Theorem 2 we have $\mathrm{tr}(A) \in \{0, 1, 2\}$ and $\det(A) \in \{0, 1\}$. Since

$$\mathrm{tr}(A) = \mathrm{tr}(A^2) \Leftrightarrow (a+d)^2 - (a+d) - 2\det(A) = 0, \tag{8}$$

$\det(A)$ vanishes for $\mathrm{tr}(A) \in \{0, 1\}$.

Case 1. $a + d = 0$. Then $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, $a, b, c \in \mathbb{C}$ and $bc = -a^2$. Hence $A^n = 0$ for all $n = 2, 3, \ldots$.

Case 2. $a + d = 1$. Then $A = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$, $a, b, c \in \mathbb{C}$ and $bc = a - a^2$. We have $A \in \mathcal{A}^{\mathrm{id}}$.

Case 3. $a+d = 2$. Then $\det(A) = 1$ due to (8). So $A = \begin{pmatrix} a & b \\ -(a-1)^2 b^{-1} & 2-a \end{pmatrix}$, $a, b \in \mathbb{C},\ b \neq 0$. We can easily prove by induction the formula

$$A^n = \begin{pmatrix} na - n + 1 & nb \\ -n(a-1)^2 b^{-1} & -na + n + 1 \end{pmatrix},\ n \in \mathbb{N}. \tag{9}$$

Thus $A \in K_{\mathcal{A},\phi}$ and $A^* \in K_{\mathcal{A},\phi}$ by Proposition 3 (for $X = [x_{ij}]_{i,j=1}^2$ we have $X^* = [\overline{x_{ji}}]_{i,j=1}^2$). The matrix inverse to the matrix A^n of (9) then equals

$$A^{-n} = \begin{pmatrix} -na + n + 1 & -nb \\ n(a-1)^2 b^{-1} & na - n + 1 \end{pmatrix},\ a, b \in \mathbb{C},\ b \neq 0,$$

hence we also have $A^{-1} \in K_{\mathcal{A},\phi}$.

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