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Trace and Differences of Idempotents in C^*-Algebras

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Abstract—Let φ be a trace on a unital C^*-algebra $\mathcal{A}$, let $\mathfrak{M}_\varphi$ be the ideal of definition of the trace φ, and let $P, Q \in \mathcal{A}$ be idempotents such that $QP = P$. If $Q \in \mathfrak{M}_\varphi$, then $P \in \mathfrak{M}_\varphi$ and $0 \le \varphi(P) \le \varphi(Q)$. If $Q - P \in \mathfrak{M}_\varphi$, then $\varphi(Q - P) \in \mathbb{R}^+$. Let $A, B \in \mathcal{A}$ be tripotents. If $AB = B$ and $A \in \mathfrak{M}_\varphi$, then $B \in \mathfrak{M}_\varphi$ and $0 \le \varphi(B^2) \le \varphi(A^2) < +\infty$.
Let $\mathcal{A}$ be a von Neumann algebra. Then

$$\varphi(|PQ - QP|) \le \min\{\varphi(P), \varphi(Q), \varphi(|P - Q|)\}$$

for all projections $P, Q \in \mathcal{A}$. The following conditions are equivalent for a positive normal functional φ on a von Neumann algebra $\mathcal{A}$:
(i) φ is a trace;
(ii) $\varphi(Q - P) \in \mathbb{R}^+$ for all idempotents $P, Q \in \mathcal{A}$ with $QP = P$;
(iii) $\varphi(|PQ - QP|) \le \min\{\varphi(P), \varphi(Q)\}$ for all projections $P, Q \in \mathcal{A}$;
(iv) $\varphi(PQ + QP) \le \varphi(PQP + QPQ)$ for all projections $P, Q \in \mathcal{A}$.

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1. INTRODUCTION

A bounded linear operator A on a Hilbert space $\mathcal{H}$ is called a *tripotent* if $A = A^3$, an *idempotent* if $A = A^2$, and a *projection* if $A = A^2 = A^*$.

Let P and Q be idempotents on $\mathcal{H}$. Various properties of the difference $P - Q$ (invertibility, Fredholm property, trace-class property, positivity, etc.) were studied in [1]–[7]. Every tripotent is the difference $P - Q$ of some idempotents P and Q with $PQ = QP = 0$ [8, Proposition 1]. Hence tripotents inherit some properties of idempotents [9]. Let φ be a trace on a unital C^*-algebra $\mathcal{A}$, let $\mathfrak{M}_\varphi$ be the ideal of definition of the trace φ, and let $P, Q \in \mathcal{A}$ be idempotents. If $P - Q \in \mathfrak{M}_\varphi$, then $\varphi(P - Q) \in \mathbb{R}$ [10, Theorem 3]; this is a C^*-analog of the well-known assertion that if P and Q are idempotents in $\mathcal{H}$ and $P - Q$ belongs to the ideal $\mathfrak{S}_1$ of trace-class operators, then $\mathrm{tr}(P - Q) \in \mathbb{Z}$, where tr stands for the canonical trace [6].

The results obtained in this paper are as follows. Let $P, Q \in \mathcal{A}$ be idempotents such that $QP = P$. If $Q \in \mathfrak{M}_\varphi$, then $P \in \mathfrak{M}_\varphi$ and $0 \le \varphi(P) \le \varphi(Q)$. If $Q - P \in \mathfrak{M}_\varphi$, then $\varphi(Q - P) \in \mathbb{R}^+$ (Theorem 1). Let $A, B \in \mathcal{A}$ be tripotents. If $AB = B$ and $A \in \mathfrak{M}_\varphi$, then $B \in \mathfrak{M}_\varphi$ and $0 \le \varphi(B^2) \le \varphi(A^2) < +\infty$ (Corollary 1). Let $\mathcal{A}$ be a von Neumann algebra. Then

$$\varphi(|PQ - QP|) \le \min\{\varphi(P), \varphi(Q), \varphi(|P - Q|)\}$$

for all projections $P, Q \in \mathcal{A}$ (Corollary 2). In Theorem 3, it is proved that the following conditions are equivalent for a positive normal functional φ on a von Neumann algebra $\mathcal{A}$:

(i) φ is a trace;

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(ii) $\varphi(Q-P)\in\mathbb{R}^+$ for all idempotents $P,Q\in\mathcal{A}$ with $QP=P$;

(iii) $\varphi(|PQ-QP|)\le\min\{\varphi(P),\varphi(Q)\}$ for all projections $P,Q\in\mathcal{A}$;

(iv) $\varphi(PQ+QP)\le\varphi(PQP+QPQ)$ for all projections $P,Q\in\mathcal{A}$.

2. DEFINITIONS AND NOTATION

A *C^*-algebra* is a complex Banach $*$-algebra $\mathcal{A}$ such that $\|A^*A\|=\|A\|^2$ for all $A\in\mathcal{A}$. An element $A\in\mathcal{A}$ is called a *tripotent* if $A=A^3$ and an *idempotent* if $A=A^2$. A self-adjoint idempotent is called a *projection*. For a C^*-algebra $\mathcal{A}$, by $\mathcal{A}^{\rm tri}$, $\mathcal{A}^{\rm id}$, $\mathcal{A}^{\rm pr}$, and $\mathcal{A}^+$ we denote the subsets of tripotents, idempotents, projections, and positive elements of $\mathcal{A}$, respectively. If $A\in\mathcal{A}$, then $|A|=\sqrt{A^*A}\in\mathcal{A}^+$. If I is the unit of an algebra $\mathcal{A}$ and $P\in\mathcal{A}^{\rm id}$, then $P^\perp=I-P\in\mathcal{A}^{\rm id}$.

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$, and let $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all bounded linear operators on $\mathcal{H}$. If $P,Q\in\mathcal{B}(\mathcal{H})^{\rm pr}$, then the projection $P\wedge Q$ is defined by the rule

$$(P\wedge Q)\mathcal{H}=P\mathcal{H}\cap Q\mathcal{H}$$

and $P\vee Q=(P^\perp\wedge Q^\perp)^\perp$ is the projection onto $\overline{\mathrm{lin}(P\mathcal{H}\cup Q\mathcal{H})}$. The *commutant* of a set $\mathcal{X}\subset\mathcal{B}(\mathcal{H})$ is defined as the set

$$\mathcal{X}'=\{Y\in\mathcal{B}(\mathcal{H}):XY=YX\text{ for all }X\in\mathcal{X}\}.$$

By a *von Neumann algebra* acting on a Hilbert space $\mathcal{H}$ we mean a $*$-subalgebra $\mathcal{A}$ of the algebra $\mathcal{B}(\mathcal{H})$ for which $\mathcal{A}=\mathcal{A}''$. Every C^*-algebra can be regarded as a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (this was proved by Gel'fand and Naimark; see [11, Theorem 3.4.1]).

By a *trace* on a C^*-algebra $\mathcal{A}$ we mean a mapping $\varphi\colon\mathcal{A}^+\to[0,+\infty]$ such that

$$\varphi(X+Y)=\varphi(X)+\varphi(Y),\quad \varphi(\lambda X)=\lambda\varphi(X)\qquad\text{for all}\quad X,Y\in\mathcal{A}^+,\quad \lambda\ge 0$$

(here $0\cdot(+\infty)\equiv 0$), and

$$\varphi(Z^*Z)=\varphi(ZZ^*)\qquad\text{for all}\quad Z\in\mathcal{A}.$$

A trace φ is said to be *faithful* if $\varphi(X)=0\Rightarrow X=0$ for $X\in\mathcal{A}^+$. For a trace φ, we write

$$\mathfrak{M}_\varphi^+=\{X\in\mathcal{A}^+:\varphi(X)<+\infty\},\quad \mathfrak{M}_\varphi=\mathrm{lin}_{\mathbb{C}}\,\mathfrak{M}_\varphi^+.$$

The restriction $\varphi|_{\mathfrak{M}_\varphi^+}$ can be well extended by linearity to a functional on $\mathfrak{M}_\varphi$; we denote this extension by the same letter φ. This extension enables one to identify the finite traces (such that $\varphi(X)<+\infty$ for all $X\in\mathcal{A}^+$) with positive functionals on $\mathcal{A}$. A positive functional on a von Neumann algebra $\mathcal{A}$ is said to be *normal* if

$$A_i\nearrow A,\quad A_i,A\in\mathcal{A}^+,\qquad\Longrightarrow\qquad \varphi(A)=\sup_i\varphi(A_i).$$

3. TRACE AND DIFFERENCES OF IDEMPOTENTS IN C^*-ALGEBRAS

Let $\mathcal{A}$ be a C^*-algebra. For $P,Q\in\mathcal{A}^{\rm id}$, we write $P\preceq Q$ if $QP=P$. Obviously, $0\preceq P\preceq I$ for all $P\in\mathcal{A}^{\rm id}$. The relation $\preceq$ is reflexive and transitive on $\mathcal{A}^{\rm id}$. For any $P,Q\in\mathcal{A}^{\rm id}$, we have the equivalencies

$$P\preceq Q\quad\Longleftrightarrow\quad Q^{*\perp}\preceq P^{*\perp}\quad\Longleftrightarrow\quad P\mathcal{A}\subset Q\mathcal{A};$$

therefore, $P\mathcal{A}=Q\mathcal{A}\Leftrightarrow P\preceq Q$ and $Q\preceq P$ [12, p. 686]. If $P\preceq Q$ and $Q\preceq P$, then, for $T=I-P+Q$, we have $T^{-1}=I+P-Q$ and $T^{-1}PT=Q$, i.e., P and Q are similar. According to [13, Theorem 1.3], for arbitrary $P,Q\in\mathcal{A}^{\rm id}$, there are unique decompositions

$$P=\widetilde{P}+Z,\qquad Q=\widetilde{Q}+T,$$

where $\widetilde{P},\widetilde{Q}\in\mathcal{A}^{\rm pr}$ and $Z,T\in\mathcal{A}$ are nilpotents with $Z^2=T^2=0$ and

$$Z\widetilde{P}=T\widetilde{Q}=0,\qquad \widetilde{P}Z=Z,\qquad \widetilde{Q}T=T.\tag{1}$$

Lemma 1. *Let $\mathcal{A}$ be a C^*-algebra and $P, Q \in \mathcal{A}^{\mathrm{id}}$. The following equivalencies hold*:

(i) $P \preceq Q \Leftrightarrow \widetilde{P} \le \widetilde{Q}$, $T\widetilde{P} = 0$, *and* $QZ = Z$;

(ii) $P - Q \in \mathcal{A}^{\mathrm{tri}} \Leftrightarrow PQP = QPQ$ *and, moreover,* $R = PQP \in \mathcal{A}^{\mathrm{id}}$, $R \preceq P$, *and* $R \preceq Q$;

(iii) $P - Q \in \mathcal{A}^{\mathrm{id}} \Leftrightarrow PQP = Q$ *and* $PQ = QP = Q \preceq P$.

Proof. (i), "$\Rightarrow$". We have

$$\widetilde{Q}\widetilde{P} + \widetilde{Q}Z + T\widetilde{P} + TZ = \widetilde{P} + Z. \tag{2}$$

Multiplying both sides of (2) on the left by the projection $\widetilde{P}$ and applying (1), we obtain

$$\widetilde{Q}\widetilde{P} + T\widetilde{P} = \widetilde{P}. \tag{3}$$

Multiplying both sides of (2) on the left by the projection $\widetilde{Q}$ and applying (1), we obtain $\widetilde{Q}\widetilde{P} + T\widetilde{P} = \widetilde{Q}\widetilde{P}$, whence $T\widetilde{P} = 0$, and (3) yields $\widetilde{Q}\widetilde{P} = \widetilde{P}$, i.e., $\widetilde{P} \le \widetilde{Q}$. Now it follows from (2) that $QZ = Z$.

The implication (i), "$\Leftarrow$" follows from the equations $\widetilde{Q}\widetilde{P} = \widetilde{P}$, (3), $QZ = Z$, and (2).

(ii) Multiplying both sides of the equation $PQP = QPQ$ on the left and right by P, we obtain $PQP = (PQP)^2$.

(iii), "$\Rightarrow$". It follows from the equation $P - Q = (P - Q)^2$ that

$$2Q = PQ + QP. \tag{4}$$

Multiplying (4) on the left (right) by P, we obtain $PQ = PQP$ (respectively, $QP = PQP$). Therefore, $PQ = QP = PQP$, and it follows from (4) that

$$Q = PQP. \tag{5}$$

We have $PQ = QP = Q \preceq P$.

(iii), "$\Leftarrow$" Multiplying (5) on the left (right) by P, we obtain $PQ = PQP$ (respectively, $QP = PQP$). Now, (5) implies

$$(P - Q)^2 = P + Q - 2PQP = P - Q,$$

i.e., $P - Q \in \mathcal{A}^{\mathrm{id}}$. This completes the proof of the lemma. □

Examples. Let $\mathcal{A}$ be a C^*-algebra.

(1) Let $P \in \mathcal{A}^{\mathrm{id}}$, and let $P = \widetilde{P} + Z$ be the decomposition described above. We set $P_\lambda = \widetilde{P} + \lambda Z$ for $\lambda \in \mathbb{C}$. Then $P_\lambda \in \mathcal{A}^{\mathrm{id}}$ and $P_\lambda \preceq P_\mu$ for all $\lambda, \mu \in \mathbb{C}$.

(2) Let $U \in \mathcal{A}$ be a unitary element, and let $P \in \mathcal{A}^{\mathrm{pr}}$ be such that $UPU^* \preceq P$. Multiplying both sides of the equation $PUPU^* = UPU^*$ on the right by U, we obtain $PUP = UP$. If one of the conditions

(a) $\mathcal{A}$ is finite;

(b) $\varphi(P) < +\infty$ for some faithful trace on $\mathcal{A}$ holds, then $UP = PU$; see Theorem 4.1 of [14].

(3) Let $P_k, Q \in \mathcal{A}^{\mathrm{id}}$ be such that $P_k \preceq Q$ for all $k = 1, \dots, n$. If $\{\lambda_k\}_{k=1}^n \subset \mathbb{C}$ and $\sum_{k=1}^n \lambda_k P_k \in \mathcal{A}^{\mathrm{id}}$, then $\sum_{k=1}^n \lambda_k P_k \preceq Q$.

(4) Let $P, Q \in \mathcal{A}^{\mathrm{id}}$ be such that $R = PQ \in \mathcal{A}^{\mathrm{id}}$. Then $R \preceq P$.

(5) Let $P, Q \in \mathcal{A}^{\mathrm{id}}$ be such that $PQP = P$. Then $R = PQ \in \mathcal{A}^{\mathrm{id}}$ and $R \preceq P$.

Proposition 1. *Let $\mathcal{A}$ be a von Neumann algebra, and let $Q \in \mathcal{A}^{\mathrm{id}}$ and $P_k \in \mathcal{A}^{\mathrm{pr}}$ be such that $P_k \preceq Q$, $k = 1, 2$. Then $P_1 \wedge P_2 \preceq Q$ and $P_1 \vee P_2 \preceq Q$.*

Proof. Since $P_1 \wedge P_2 \preceq P_1$ and $P_1 \preceq Q$, it follows that $P_1 \wedge P_2 \preceq Q$ by transitivity of the relation $\preceq$. Let $T = U|T|$ be the polar representation, and let $R = \mathrm{rp}(|T|)$ be the rank projection of the operator $|T|$. Then $R \in \mathcal{A}^{\mathrm{pr}}$ and, by assertion (i) of Lemma 1, we have $|T|P_k = U^*TP_k = 0$, and thus $RP_k = 0$, $k = 1, 2$. Hence $R \le P_k^{\perp}$, $k = 1, 2$, and, by the definition of the greatest lower bound,

$$R \le P_1^{\perp} \wedge P_2^{\perp} = (P_1 \vee P_2)^{\perp};$$

i.e., $R \cdot P_1 \vee P_2 = 0$. Therefore,

$$0 = |T| \cdot P_1 \vee P_2 = U \cdot |T| \cdot P_1 \vee P_2 = T \cdot P_1 \vee P_2$$

and $P_1 \vee P_2 \preceq Q$ by assertion (i) of Lemma 1. This proves the proposition. $\square$

Proposition 2. *Let $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$, let $a > 0$ be a number, and let $i \in \mathbb{C}$ be such that $i^2 = -1$. Then the following conditions are equivalent*:

i) $PQ = QP$;

ii) $PQP \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$;

iii) $PQ + QP \le PQP + QPQ$;

iv) $i(PQ - QP) \le a(PQ + QP)$.

Proof. (iii) ⇒ (ii) Since $PQP \in \mathcal{B}(\mathcal{H})^+$ and $\|PQP\| \le 1$, we have $(PQP)^2 \le PQP$. Multiplying both sides of the inequality $PQ + QP \le PQP + QPQ$ on the left and right by P, we obtain $PQP \le (PQP)^2$.

(ii) ⇒ (i) By the von Neumann theorem (see [15], the solution of Problem 96), we have

$$PQP = (PQP)^n \to P \wedge Q \qquad \text{as} \quad n \to \infty$$

in the strong operator topology. Therefore, $PQP \le Q$, and, by Proposition 2.1 of [16], we have $PQ = QP$.

(iv) ⇒ (i) In the direct decomposition $\mathcal{H} = P\mathcal{H} \oplus P^{\perp}\mathcal{H}$, we have

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix},$$

where $A \in \mathcal{B}(P\mathcal{H})^+$, $C \in \mathcal{B}(P^{\perp}\mathcal{H})^+$, and $B\colon P^{\perp}\mathcal{H} \to P\mathcal{H}$. Then (iii) is equivalent to the inequality

$$\begin{pmatrix} 2aA & (a-i)B \\ (a+i)B^* & 0 \end{pmatrix} \ge 0.$$

By Theorem 1.1 of [17] (or by Theorem 1.3 of [18]), the 2×2 operator matrix

$$\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$$

is nonnegative if and only if $X \ge 0$, $Z \ge 0$, and $Y = \sqrt{X}\,T\sqrt{Z}$ for some $T \in \mathcal{B}(\mathcal{H})$ with $\|T\| \le 1$. Therefore, $B = 0$ and $PQ = QP$. This proves the proposition. $\square$

Example. The condition $Q = Q^*$ is essential for the equivalence (i)⇔(ii) of Proposition 2. For

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_2(\mathbb{C})^{\mathrm{pr}}, \qquad Q = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_2(\mathbb{C})^{\mathrm{id}},$$

we have $PQP = P$; however, $PQ = Q \ne P = QP$.

Proposition 3. *For any $P, Q \in \mathcal{B}(\mathcal{H})^{\rm pr}$, the following inequality holds*:

$$|PQ - QP| \le |P - Q| \le P \vee Q - P \wedge Q.$$

Proof. Since $0 \le (P-Q)^4 \le (P-Q)^2$, it follows that

$$|PQ - QP|^2 = (P-Q)^2 - (P-Q)^4 \le (P-Q)^2.$$

By the operator monotonicity of the function $f(t) = \sqrt{t}\,(t \ge 0)$, this implies $|PQ - QP| \le |P - Q|$. The inequality $|P - Q| \le P \vee Q - P \wedge Q$ was established in part (ii) of Theorem 2 in [9]. This completes the proof of the proposition. □

Theorem 1. *Let φ be a trace on a unital C^*-algebra $\mathcal{A}$, and let $P, Q \in \mathcal{A}^{\rm id}$ be such that $P \preceq Q$.*

i) *If $Q \in \mathfrak{M}_\varphi$, then $P \in \mathfrak{M}_\varphi$ and $0 \le \varphi(P) \le \varphi(Q)$.*

ii) *If $Q - P \in \mathfrak{M}_\varphi$, then $\varphi(Q-P) \in \mathbb{R}^+$.*

Proof. Let $P = \widetilde{P} + Z$ and $Q = \widetilde{Q} + T$ be the decompositions described above, where $\widetilde{P}, \widetilde{Q} \in \mathcal{A}^{\rm pr}$. Then $Z, T \in \mathcal{A}$ are nilpotent with $Z^2 = T^2 = 0$, and relations (1) hold.

(i) By Corollary 6 of [10] (see also Theorem 3.6 of [19]), we have $Q \in \mathfrak{M}_\varphi \Leftrightarrow \widetilde{Q} \in \mathfrak{M}_\varphi$; in this case, $\varphi(Q) = \varphi(\widetilde{Q})$. Similarly, $P \in \mathfrak{M}_\varphi \Leftrightarrow \widetilde{P} \in \mathfrak{M}_\varphi$; in this case, $\varphi(P) = \varphi(\widetilde{P})$. Since $\widetilde{P} \le \widetilde{Q}$ by Lemma 1 (i), we have $\widetilde{P} \in \mathfrak{M}_\varphi$ (recall that $\mathfrak{M}_\varphi$ is an ideal in $\mathcal{A}$) and, therefore, $P \in \mathfrak{M}_\varphi$ and

$$\varphi(P) = \varphi(\widetilde{P}) \le \varphi(\widetilde{Q}) = \varphi(Q).$$

(ii) At the end of the proof of Theorem 3 in [10], it was established for arbitrary $P, Q \in \mathcal{A}^{\rm id}$ with $Q - P \in \mathfrak{M}_\varphi$ that $\widetilde{Q} - \widetilde{P} \in \mathfrak{M}_\varphi$ and

$$\varphi(Q - P) = \varphi(\widetilde{Q} - \widetilde{P}) \in \mathbb{R}.$$

If $P \preceq Q$, then $\widetilde{Q} - \widetilde{P} \in \mathcal{A}^{\rm pr}$ by part (i) of Lemma 1 and, therefore, $\varphi(\widetilde{Q} - \widetilde{P}) \in \mathbb{R}^+$. This completes the proof of the theorem. □

Corollary 1. *Let φ be a trace on a unital C^*-algebra $\mathcal{A}$, and let $A, B \in \mathcal{A}^{\rm tri}$. If $AB = B$ and $A \in \mathfrak{M}_\varphi$, then $B \in \mathfrak{M}_\varphi$ and $0 \le \varphi(B^2) \le \varphi(A^2) < +\infty$.*

Proof. Let $A = P_1 - Q_1$ and $B = P_2 - Q_2$ be representations from [8, Proposition 1], i.e., $P_k, Q_k \in \mathcal{A}^{\rm id}$ and $P_k Q_k = Q_k P_k = 0$ for $k = 1, 2$. It can readily be seen that the elements $A^2 = P_1 + Q_1$ and $B^2 = P_2 + Q_2$ belong to $\mathcal{A}^{\rm id}$. Multiplying both sides of the equation

$$(P_1 - Q_1)(P_2 - Q_2) = P_2 - Q_2 \qquad (6)$$

on the right by the idempotent P_2, we obtain $(P_1 - Q_1)P_2 = P_2$. Multiplying both sides of this equation on the left by the idempotent Q_1, we obtain $Q_1 P_2 = 0$. Thus, $P_1 P_2 = P_2$.

Multiplying both sides of (6) on the right by the idempotent Q_2, we obtain $(P_1 - Q_1)Q_2 = Q_2$. Multiplying both sides of this equation on the left by the idempotent Q_1, we obtain $Q_1 Q_2 = 0$. Thus, $P_1 Q_2 = Q_2$ and $A^2 B^2 = B^2$. The application of Theorem 1 (i) completes the proof. □

Theorem 2. *Let φ be a trace on a von Neumann algebra $\mathcal{A}$. Then*

$$\varphi(|PQ - QP|) \le \min\{\varphi(P), \varphi(Q)\}$$

for all $P, Q \in \mathcal{A}^{\rm pr}$.

Proof. For the unitary operator $S = 2Q - I$, we have $S^2 = I$ and $PQ - QP = (1/2)(PS - SP)$. If $A, U \in \mathcal{A}$ and U is an isometry (i.e., $U^*U = I$), then $|UA| = \sqrt{A^*U^*UA} = \sqrt{A^*A} = |A|$. Obviously, $SPS \in \mathcal{A}^{\mathrm{pr}}$. Now

$$|PQ - QP| = \frac{1}{2}|PS - SP| = \frac{1}{2}|S(PS - SP)| = \frac{1}{2}|SPS - P| \le \frac{1}{2}((SPS) \vee P - (SPS) \wedge P) \quad (7)$$

by Proposition 3. Recall that $\varphi(SPS) = \varphi(P)$ and

$$\varphi(R \vee T) \le \varphi(R) + \varphi(T) \quad \text{for all} \quad R, T \in \mathcal{A}^{\mathrm{pr}} \quad (8)$$

by the von Neumann–Murray equivalence $R \vee T - T \sim R - R \wedge T$; see [20, Chap. 5, Proposition 1.6]. Now it follows from (7) and (8) and from the monotonicity of the trace φ on the cone $\mathcal{A}^+$ that

$$\varphi(|PQ - QP|) \le \frac{1}{2}\varphi((SPS) \vee P) \le \frac{1}{2}(\varphi(SPS) + \varphi(P)) = \varphi(P).$$

Similarly, $\varphi(|PQ - QP|) \le \varphi(Q)$. This proves the proposition. □

Theorem 2 and Proposition 3 imply the following corollary.

Corollary 2. *Let φ be a trace on a von Neumann algebra $\mathcal{A}$. Then*

$$\varphi(|PQ - QP|) \le \min\{\varphi(P), \varphi(Q), \varphi(|P - Q|)\} \qquad \textit{for all} \quad P, Q \in \mathcal{A}^{\mathrm{pr}}.$$

Theorem 3. *For any positive normal functional φ on a von Neumann algebra $\mathcal{A}$, the following conditions are equivalent*:

(i) *φ is a trace*;

(ii) *$\varphi(Q - P) \in \mathbb{R}^+$ for all $P, Q \in \mathcal{A}^{\mathrm{id}}$ with $P \preceq Q$*;

(iii) *$\varphi(|PQ - QP|) \le \min\{\varphi(P), \varphi(Q)\}$ for all $P, Q \in \mathcal{A}^{\mathrm{pr}}$*;

(iv) *$\varphi(PQ + QP) \le \varphi(PQP + QPQ)$ for all $P, Q \in \mathcal{A}^{\mathrm{pr}}$.*

Proof. The implications (i) ⇒ (ii) and (i) ⇒ (iii) were established in Theorem 1 (ii) and in Theorem 2, respectively. The implication (i) ⇒ (iv) follows from the fact that

$$\varphi(PQ + QP) = \varphi(P^2Q) + \varphi(Q^2P) = \varphi(PQP) + \varphi(QPQ) = \varphi(PQP + QPQ)$$

for all $P, Q \in \mathcal{A}^{\mathrm{pr}}$.

Below it is proved that, similarly to the situation in several other like cases (see [21] or [22]), the proof of the converse implications for an arbitrary von Neumann algebra reduces to the case of the algebra $\mathbb{M}_2(\mathbb{C})$. As is well known [21], a positive normal functional φ on $\mathcal{A}$ is a trace if and only if $\varphi(A) = \varphi(B)$ for all $A, B \in \mathcal{M}^{\mathrm{pr}}$ such that $AB = 0$ and $A \sim B$ (see also [22, Lemma 2]). Let a $*$-algebra $\mathcal{B}$ be generated in a reduced algebra $(A + B)\mathcal{A}(A + B)$ by a partial isometry $V \in \mathcal{A}$ realizing the equivalence of A and B. Then $\mathcal{B}$ is $*$-isomorphic to $\mathbb{M}_2(\mathbb{C})$, and the relation in (ii) and the inequality in (iii) remain valid for operators in $\mathcal{B}$ and for the restriction $\varphi|\mathcal{B}$ of the functional. We claim that this restriction is a trace functional on $\mathcal{B}$ and, therefore, $\varphi(A) = \varphi(B)$.

As is well known, every linear functional φ on $\mathbb{M}_2(\mathbb{C})$ can be represented in the form $\varphi(\,\cdot\,) = \mathrm{tr}(S_\varphi\,\cdot\,)$. The matrix $S_\varphi \in \mathbb{M}_2(\mathbb{C})$ is called the density matrix for φ. Without loss of generality, we may assume that

$$S_\varphi = \mathrm{diag}\left(\frac{1}{2} - s, \frac{1}{2} + s\right), \qquad 0 \le s \le \frac{1}{2}\,.$$

Thus, $\varphi(X)$ is equal to the quantity $(1/2 - s)x_{11} + (1/2 + s)x_{22}$ for $X = [x_{ij}]_{i,j=1}^2$ in $\mathbb{M}_2(\mathbb{C})$.

Let us prove the implication (ii)⇒(i). We set

$$Q_a = \begin{pmatrix} a & 1-a \\ a & 1-a \end{pmatrix} \qquad \text{for all} \quad a \in \mathbb{C},$$

and let $P = Q_{1/2}$. Then $Q_a \in \mathbb{M}_2(\mathbb{C})^{\mathrm{id}}$ and $Q_a P = P$ for all $a \in \mathbb{C}$. We have

$$\varphi(Q_a - P) = \varphi(Q_a) - \varphi(P) = \mathrm{tr}(S_\varphi Q_a) - \mathrm{tr}(S_\varphi P) = \frac{1}{2} + 2as + s - \frac{1}{2} = (2a+1)s \geq 0$$

for all $a \in \mathbb{R}$ only if $s = 0$.

(iii) ⇒ (i). We set $f(t) = \sqrt{1/4 - t^2}$ and

$$P = \begin{pmatrix} \frac{1}{2} + t & f(t) \\ f(t) & \frac{1}{2} - t \end{pmatrix}, \qquad Q = \begin{pmatrix} \frac{1}{2} - t & f(t) \\ f(t) & \frac{1}{2} + t \end{pmatrix}$$

for $0 \leq t \leq 1/2$. Then $P, Q \in \mathbb{M}_2(\mathbb{C})^{\mathrm{pr}}$ and $|PQ - QP|^2 = 4t^2(1-4t^2)I$ for all $0 \leq t \leq 1/2$. By the uniqueness of the square root of a positive matrix, we obtain $|PQ - QP| = 2t\sqrt{1-4t^2}\, I$ for all $0 \leq t \leq 1/2$. For $t = 1/2\sqrt{2}$, we have

$$\varphi(|PQ - QP|) = \mathrm{tr}(S_\varphi |PQ - QP|) = \frac{1}{2}, \qquad \varphi(P) = \mathrm{tr}(S_\varphi P) = \frac{1}{2} - \frac{s}{\sqrt{2}}.$$

Therefore, the inequality $\varphi(|PQ - QP|) \leq \varphi(P)$ holds only if $s = 0$.

(iv) ⇒ (i). We set

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Then $P, Q \in \mathbb{M}_2(\mathbb{C})^{\mathrm{pr}}$ and

$$\varphi(PQ + QP) = \mathrm{tr}(S_\varphi(PQ + QP)) = \frac{1}{2} + s,$$
$$\varphi(PQP + QPQ) = \mathrm{tr}(S_\varphi(PQP + QPQ)) = \frac{1}{2} + \frac{s}{2}.$$

Therefore, the inequality $\varphi(PQ + QP) \leq \varphi(PQP + QPQ)$ holds only if $s = 0$. This completes the proof of the theorem. □

For other characterizations of trace, see [23] and [24] and the references therein.

Proposition 4. *Let $P, Q \in \mathbb{M}_2(\mathbb{C})^{\mathrm{id}}$ be such that $PQ = QP = P$. If $P \notin \{0, Q\}$, then $Q = I$.*

Proof. Obviously, $PQ = QP = P \Leftrightarrow P \preceq Q$ and $P^* \preceq Q^*$. Let $P = \widetilde{P} + Z$, and let $Q = \widetilde{Q} + T$ be the decompositions described above, where $\widetilde{P}, \widetilde{Q} \in \mathbb{M}_2(\mathbb{C})^{\mathrm{pr}}$, $Z, T \in \mathbb{M}_2(\mathbb{C})$ are nilpotent, $Z^2 = T^2 = 0$, and relations (1) hold. Let $Q \neq I$; then it follows from $QP = P$ that $\widetilde{P} \leq \widetilde{Q}$ by Lemma 1 (i). Since the projections $\widetilde{P}$ and $\widetilde{Q}$ are one-dimensional, we have $\widetilde{P} = \widetilde{Q}$. It follows from $PQ = P$ that the equality

$$\widetilde{P}\widetilde{Q} + \widetilde{P}T + Z\widetilde{Q} + ZT = \widetilde{P} + Z$$

holds; hence, taking into account $\widetilde{P} = \widetilde{Q}$, (1), and (2), we have

$$T + ZT = Z. \tag{9}$$

Multiplying both sides of (9) on the right by T and taking into account the equation $T^2 = 0$, we obtain $ZT = 0$, and it follows from (9) that $T = Z$. Hence $P = Q$, and we have arrived at a contradiction. Thus, $Q = I$, which completes the proof of the proposition. □

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