International Journal of Theoretical Physics (2021) 60:585–596
https://doi.org/10.1007/s10773-019-04156-x

Studies on Noncommutative Measure Theory in Kazan University (1968–2018)

Airat M. Bikchentaev[1] · **Anatolij N. Sherstnev**[1]

Received: 29 October 2018 / Accepted: 14 May 2019 / Published online: 24 May 2019

Abstract
We describe the main results of the participants of the scientific seminar “Operator algebras and their applications” at the Kazan University for the years 1968–2018. We also provide a list of selected works of the seminar participants.

Keywords Hilbert space · von Neumann algebra · Projection · Idempotent · Quantum logic · Measure · State · Weight · Bilinear form · Trace · Measure topology · Measurable operator · Noncommutative integration theory

1 Introduction

Scientific seminar “Operator algebras and their applications” (supervisor Anatolij Sherstnev) arised within N.G. Chebotarev Research Institute of Mathematics and Mechanics in the late 60ths. Beginning 1974, the seminar relocated to the Chair of Mathematical Analysis, where it is now going on to be active. The research results were presented in reviews [127, 160], and in books [96, 131].

The successful activity of the seminar has promoted its growth as an authority among the mathematical community. A subject issue (1982, no. 8) of “Izv. Vuzov. Matematika” was dedicated to Noncommutative Integration and related topics in Probability Theory and Mathematical Physics. It was Kazan State University who, together with Moscow and St. Petersburg State Universities, organized and held Summer Mathematical Schools on Noncommutative Probability Theory in 1971 and 1978 [1, 122]. The initiator was Yakov Sinai.

At the seminar well-known mathematicians such as Grigorij Amosov, Vladimir Chilin, Anatolij Dvurečenskij, Alexander Khelemskij, Robin Hudson, Oleg Smolyanov, Alexander Stern, Fedor Sukochev, etc. gave talks.

✉ Airat M. Bikchentaev
Airat.Bikchentaev@kpfu.ru

Anatolij N. Sherstnev
anatolijsherstnev38@gmail.com

[1] Kazan Federal University, Kremlyovskaya str., 18 Kazan, 420008, Russia

2 The Noncommutative Measure and Integration Theory

Because of the progress in the theory of von Neumann algebras and the expansion of its applications during the late 60ths – early 70ths (that was stimulated by the fruitful Tomita-Takesaki theory and the theory of normal weights), extending the noncommutative integration theory by I. Segal (1953) has become urgent to normal weights being noncentral analogs to integrals by unbounded measures on the class of bounded functions. This attracted members of the seminar in the 70ths. For the first time, a solution to the above problem was obtained at our seminar in 1972–1978 [123, 125, 155, 156]. This integration scheme was essentially advanced during the subsequent 2 or 3 years. The principal idea, on which the construction of a noncommutative L_1 associated with a faithful normal semifinite weight on a von Neumann algebra M, acting in a Hilbert space H was based, is that of its representation as a space of integrable bilinear forms on the dense in H lineal of weight intrinsically related to the weight [123, 125]. The well-developed machinery of bilinear forms has also turned out to be extremely fruitful for solving other important problems of the general measure theory on projections in M [131]. Let us state the main results obtained when solving this problem.

1) A theory of L_1-spaces associated with a faithful normal semifinite weight has been built. Their representation as a space of integrable bilinear forms on the lineal of weight has been suggested. A dual description has been obtained for L_1 showing it to be isometrically isomorphic to the predual of the M. The isomorphism naturally preserves the natural order structures in L_1 and the predual [123, 125, 155, 156]. Characterization of integrable sesquilinear forms has been stated [132]. A theory of conditional expectation on the space L_1 of integrable bilinear forms has been developed. A relation of this concept to the usual one of a conditional expectation on von Neumann algebras has been stated [157], [155, II; 156].

2) A scale of spaces L_p, associated with a locally measurable weight on a semifinite M has been constructed. These spaces have been described in terms of locally measurable operators and bilinear forms [158]. Real spaces have been constructed and represented via integrals of real functions by generalized decompositions of identity [145].

Let τ be a faithful normal semifinite trace on a von Neumann algebra M, I be the unit of M, and $S(M,\tau)$ be the $*$-algebra of all τ-measurable operators. For every $\varepsilon,\delta > 0$, we define the set

$$V(\varepsilon, \delta) = \{X \in S(M, \tau) : \quad \exists P \in M^{pr} \text{such that } \| XP \| \leq \varepsilon, \quad \tau(I - P) \leq \delta\}.$$

The topology generated by the sets $V(\varepsilon,\delta)$, $\varepsilon,\delta > 0$, is called the measure topology t_τ on $S(M,\tau)$ [107]. It is well-known that the algebra $S(M,\tau)$ equipped with the measure topology is a complete metrizable topological algebra [107]. We say that $T_i \to T$ in the topology $t_{\tau l}$ (resp., $t_{w\tau l}$), if for any τ-finite projection $P \in M$, $T_iP \to TP$ (resp., $PT_iP \to PTP$) in measure.

Conditions for a real Borel function f, ensuring that the corresponding operator function $f(A)$ is continuous in the topology t_τ of convergence in measure and in the spaces $L_p(M,\tau)$ associated with a faithful normal semifinite trace τ on M has been found [146]. Some properties of the topology t_τ and its localizations $t_{\tau l}$ and $t_{w\tau l}$ were investigated in [6, 7, 11, 15, 52, 55, 60, 135].

Constructing and studying L_p-spaces associated with a state on M has also been implemented in [170] by applying the scale $L_{p,a}$, $0 \leq a \leq 1$, introduced in [159]. In this approach, the space $L_{p,a}$ for each fixed a in the complex interpolation scale constructed by the Banach couple ($L_{1,a}$,M) serves to be an analog to the space L_p. The spaces $L_{p,a}$ form a complex interpolation scale for every fixed p. A fairly general scheme for constructing L_p-type spaces has been suggested within an approach to the axiomatics of quantum mechanics that describes states of the system as elements in a convex set [147]. Measures on orthoideals in the set of projections in von Neumann algebra M, which appear within a framework of L_1-spaces associated with positive operators affiliated with M were studied in [109, 110]. Interpolation of positive operators in couples of L_1, L_∞ associated with weights and traces on von Neumann algebras has been stated [167].

3) The main results in the integration theory with respect to a trace and the theory of trace inequalities in von Neumann algebras have been extended to the spaces in spectral duality of E. Alfsen and F. Schultz [148]. A non-commutative analog of A.V. Bukhvalov and G.Ya. Lozanovskiy theory on "near-compactness" properties of convex sets in Banach function spaces has been suggested [135, 150]. This has been applied to the study of weak sequential completeness of factor spaces [136].

Trace inequalities and new characterizations of traces among weights on von Neumann algebras have been obtained in [3, 10, 14, 18–21, 28, 35, 44, 57, 59, 128, 137, 149, 151]. A list of inequalities which characterize central elements in von Neumann algebras and C^*-algebras was presented in [108].

4)A general method for constructing noncommutative F-normed ideal spaces (in particular, Orlicz spaces) associated with a semiadditive measure on the projections of a von Neumann algebra has been suggested. A new majorization property of products of τ-measurable operators has been established [4]. Order, topological and geometrical properties of spaces $L_p(M,\tau)$ and Orlicz spaces $L_f(M,\tau)$ were investigated in [2, 3, 24, 25, 30, 34, 40–42, 49, 50, 54]. Hyponormal and paranormal measurable operators affiliated with a semifinite von Neumann algebra has been investigated [30, 45, 50, 51]. Conditions under which product of measurable operators is integrable or τ-compact have been established in [43, 46]. One additivity problem for mappings on measurable functions was considered in [32]. An analog of the M.G. Krein theorem for measurable operators has been established [53]. A noncommutative version of Nikishin's theorem was obtained [154].

5) Some equivalent definitions of operator monotone and operator convex functions can be found in [39, 58, 152]. The Haagerup problem on subadditive weights on W^*-algebras (1975) was solved for Abelian [22] and atomic [27] von Neumann algebras. An analog of the Hahn-Banach theorem for commutative semigroups was obtained [129].

6) Another direction of research at the seminar is the structure of measures on the orthoprojections in a von Neumann algebra. A fundamental result by A. Gleason (1957) on countably additive measures on the closed subspaces of a separable Hilbert space together with the possibility to apply that result in constructing nice axiomatic models for quantum mechanics) stimulated interest to the so-called linearity problem. That was the description of measures on projections and a problem of extension of a measure on projections on a von Neumann algebra to be a linear functional on the von Neumann algebra was formulated. At our seminar, it was M.S. Matveichuk who succeeded in solving this problem [92–95] at the same time with E. Christensen (1982) and F. Yeadon (1983–1984).

7) Developing an exhaustive measure theory on projections implies (as in the classical "commutative" case) the study of not necessarily bounded measures. In the classical theory, an unbounded measure is either a measure on a sigma-ring of sets taking nonnegative values or a measure on a sigma-algebra of sets with values in the extended real half-line. These approaches are substantially equivalent. A sigma-orthoadditive mapping from M^{pr} to the completed real half-line is a natural analog to the latter definition for measures on the

projections in M. In the noncommutative case, the first definition transforms to "a sigma-orthoadditive" mapping m on an "ideal" J in M^{pr} to the nonnegative half-line. But, these approaches are now not equivalent, as an unbounded measure on an ideal need not extend to a measure on all projections [121, 126]. The study of unbounded measures within the latter approach has turned out to be successful. The problem of describing the unbounded measures has completely been solved for the algebra of all bounded operators on a separable Hilbert space [83, 84]. This has been accomplished by applying methods of the bilinear forms theory. (For the first time, those methods were utilized in the study of measures on ideals [121]. Later on, these were essentially refined by applying a number of structural theorems by B. Simon (1978).) This result is a generalization of the aforementioned classical Gleason theorem to unbounded measures. The one has allowed for introducing an L_1-space for a wide class of unbounded measures on projections and representing the space as a predual of a suitable von Neumann algebra [85].

The extension problem of an unbounded measure on projections to a weight (or the linearity problem for unbounded measures) has also naturally arised. For all the sufficient conditions obtained for extending a measure to a weight, the problem has turned out to negative answer for arbitrary finitely additive measures. An example of an unbounded semifinite finitely additive measure on the projections in a von Neumann algebra with no direct summand of type I_2 nonextendable to a weight has been given [88, 89].

8) In the early 70ths, the study of an analog to the Gleason theorem for not necessarily positive signed measures on orthoprojections was begun at the seminar [124]. It was established that the assertion as the Gleason theorem was valid for signed measures on $B(H)$, the algebra of all bounded operators on a Hilbert space H ($\dim H > 2$) provided that the restriction of a signed measure to the one-dimensional orthoprojections was bounded (which was equivalent to the boundedness of the signed measure). More generally, the description of completely additive signed measures on a von Neumann algebra M, was immediately related to their boundedness. Let m be a completely additive signed measure. Is it bounded?

It was Gleason (1957) who observed that this question answers in the negative for the algebra $B(H)$ with finite-dimensional H. Only in 1990, we affirmatively answered the question for $B(H)$ with infinite-dimensional H [61] (see also [63]). Later on, S.V. Dorofeev proved [62] that every signed measure is bounded (and thus is a linear combination of measures) in case the von Neumann algebra M has no direct summand of type I_n (n is finite).

9) Unbounded analogs of vector orthoadditive measures on orthoprojections in a von Neumann algebra taking values in a Hilbert space have been introduced and studied [86, 87]. Also, order properties of orthogonal vector fields and their relationships with topological properties are studied. It turns out, in particular, that the order property of an orthogonal vector field being normal substantially differs from the corresponding scalar analog. A characterization of a broad class of orthogonal vector fields in the class of linear mappings is obtained by G. Lugovaya and A. Sherstnev [90, 91]. J. Hamhalter proved in [69] that if m is a measure on projections of a W^*-algebra M without a type I_2 direct summand, then there exists an H-valued orthogonal vector measure μ on M^{pr} such that $\|\mu(p)\|^2 = m(p)$ for every $p \in M^{pr}$. In [133] this assertion was obtained by A. Sherstnev for W^*-algebras of type I_2 and, therefore, for arbitrary W^*-algebras.

10) Classes of orthoclosed and splitting subspaces of a unitary space that are affiliated to a von Neumann algebra M have been introduced and studied in [128]. In this paper a topological definition of measures on the above classes of subspaces is given and results on the relationships of these measures to measures on M^{pr} are presented. See also [161, 162].

J. Hamhalter and E.A. Turilova studied some subspaces of inner product spaces invariant with respect to a given von Neumann algebra M and investigated the interplay between order properties of the poset of affiliated subspaces and the structure of M. Results on nonexistence of measures on incomplete structures were extended to invariant subspaces. Properties of inner product spaces as well as the structure of affiliated subspaces were reviewed in [70].

Structure of certain associative subalgebras of Jordan operator algebras were investigated in [71]. The authors found one characterization of finiteness of M. Moreover, it is shown that if the modular operator of a faithful normal state φ is bounded, then all important classes of affiliated subspaces in the GNS representation space of φ coincide. Orthogonally closed affiliated subspaces were characterized in terms of the supports of normal functionals. It was proved that complete affiliated subspaces correspond to left ideals generated by finite sums of orthogonal atomic projections [72]. It was shown that the structural properties of von Neumann algebras are connected with the metric and order-theoretic properties of various classes of affiliated subspaces. Properly infinite von Neumann algebras always admit an affiliated subspace for which (1) closed and orthogonally closed affiliated subspaces are different; (2) splitting and quasi-splitting affiliated subspaces do not coincide [73].

Any order isomorphism between the structures of unital associative JB subalgebras of JB algebras is given naturally by a partially linear Jordan isomorphism. The same holds for nonunital subalgebras and order isomorphisms preserving the unital subalgebra [74]. Any properly infinite von Neumann algebra M admits an affiliated subspace L such that all important subspace classes living on L are different. Moreover, it was shown that L can be chosen so that the set of σ-additive measures on subspace classes of L are empty [75]. For other results see [76–80].

A characterization of spectral order automorphisms of the lattice of positive (even unbounded) self-adjoint operators was obtained in [163]. It was shown that any spectral lattice orthoautomorphism of the structure of positive contractions on a von Neumann algebra M, endowed with the spectral order and the orthogonality relation, that preserves scalar operators is a composition of the function calculus with the natural transformation of spectral resolutions given by an orthoautomorphism of the projection lattice. In the case M is without type I_2 direct summand, any such map is a composition of the function calculus with a Jordan *-automorphism [164]. Spectral order on unbounded operators and their symmetries have been considered [81].

A state φ on a Jordan algebra A induces a complete inner product space if and only if φ is a convex combination of pure states. Inner product spaces generated by type I_n factor states and states on spin factors were described [165].

11) Here are some other results on operator algebras. Each element of a von Neumann algebra without a direct abelian summand is representable as a finite sum of products of at most three projections in the algebra. In a properly infinite algebra, the number of the product terms is at most two [5, 9, 12]. For a wide class of C*-algebras representation of elements as finite sums of products of projections has been established [8, 13]. It is proved that every skew-Hermitian element of any properly infinite von Neumann algebra can be represented in the form of a finite sum of commutators of projections of the algebra [16]. The properties of tripotents have been investigated in [23, 31]. The two-dimensional reductions of the cone of positive diagonal operators in ℓ^2 was considered in [130]. The convergence of functions of normal operators in a strong operator topology has been shown [153]. Invertibility conditions for bounded operators were establihed [19, 26, 29]. Every measure of non-compactness on a W^*-algebra is an ideal F-pseudonorm, and we have a criterion of the right Fredholm property of an element with respect to a W^*-algebra [36].

Differences of idempotents in C^*-algebras and the quantum Hall effect were considered in [47, 48]. Contiguity and entire separability of states on von Neumann algebras were investigated in [67]. Ultraproducts of von Neumann algebras and ergodicity have been investigated in [68].

3 Orthomodular Posets

The orthoprojections in a commutative von Neumann algebra naturally form a Boolean algebra. If the von Neumann algebra in question is noncommutative, we come to a more general structure that may naturally be viewed as an analog to the set of possible propositions about a suitable quantum-mechanical system. An abstract “Boolean” analog to this structure is called an orthomodular poset (abbr., OMP) or a quantum logic. The study of these was initiated by G. Birkhoff and J. von Neumann in the 30ths.

In Kazan, the study of OMPs was begun at our seminar in the late 60ths [120]. In that paper, it was proved that the quantum logic of all closed subspaces in a Hilbert space H ($\dim H > 2$) is not isomorphic to no concrete logic. Also in that paper the dimension theory by Loomis of orthomodular lattices was extended over the arbitrary OMPs. That approach was developed in [82]. The advanced research in OMPs has systematically been held by P.G. Ovchinnikov beginning the 80ths. He has suggested a new axiomatics for OMPs based on their combinatorial characterization [117], has described the automorphisms of the poset of skew projections on a Hilbert space [111], proved a well-known Ptak–Pulmannova conjecture on the Jauch-Piron property [112], has extended the Birkhoff theorem on doubly stochastic matrices to hypergraphs [113], has given the first known example of an atomistic nonorthoatomistic OMP [114], has found an exact topological analog to the concept of an orthoposet [115], and has developed a general theory of measures and signed measures on finite concrete logics [116, 118]. F.F. Sultanbekov has exhaustively studied a number of particular finite OMPs and concrete logics [138, 139] and has successfully applied a computer in the study of finite concrete logics [140]. In doing that, he has developed a series of specific routines [141].

Along with studying general OMPs, the search of analogs to the Gleason theorem for some interesting particular ones such as OMPs of skew projections on Banach spaces or vector spaces over “unusual” fields was held [100, 101, 103, 104]. The set-representable quantum logics closed under the formation of symmetric difference and states on them were investigated in [17, 33]. Quantum logics and measures (and states, charges) on them were studied in [37, 38, 65, 100, 103–106], [142–144, 166].

4 Probability Measures on Banach Spaces

The problem of (sigma-)additivity of a cylindrical probability is one of the most important and difficult in the theory of probability distributions on infinite-dimensional Banach spaces. This problem is closely related to trajectories of the most important stochastic processes. The classical theorems by N. Wiener and A.N. Kolmogorov were the first to be obtained in this field. The most efficient criteria of the (sigma)-additivity are topological ones that are the infinite-dimensional analogs to the Bochner theorem. The description problem of the Banach spaces admitting an analog of the Bochner theorem has been motivated by the basic research by V.V. Sazonov (1958), R.A. Minlos (1959), and Yu.V. Prokhorov (1960).

In solving this and related problems, D.Kh. Mushtari has developed a topological technique for studying weak compactness and (sigma-)additivity of cylindrical probabilities in

Banach spaces. This technique allowed him to give the complete description of Banach spaces admitting a topological solution to the problem. He has extended a number of major results from Banach spaces to the topological or metric linear spaces. Some probability characterizations of the Frechet nuclear spaces have also been obtained. Stable probabilities on Banach spaces have been studied [97–99, 102].

References

1. Begovatov, E.A., Minlos, R.A., Sherstnev, A.N.: A report on summer mathematical school "non-commutative theory of probability". (Russian) Uspehi mat. nauk **34**(5), 242–246 (1978)
2. Bikchentaev, A.M.: On noncommutative function spaces. Selected papers in K-theory. Amer. Math. Soc. Transl. (2) **154**, 179–187 (1992)
3. Bikchentaev, A.M.: On a property of L_p-spaces on semifinite von Neumann algebras. Math. Notes **64**(1–2), 159–163 (1998)
4. Bikchentaev, A.: Majorization for products of measurable operators. Internat. J. Theoret. Phys. **37**(1), 571–576 (1998)
5. Bikchentaev, A.M.: Representation of linear operators in a Hilbert space in the form of finite sums of products of projectors. Doklady Math. **68**(3), 376–379 (2003)
6. Bikchentaev, A.M.: Minimality of the convergence in measure topology on finite von Neumann algebras. Math. Notes **75**(3–4), 315–321 (2004)
7. Bikchentaev, A.M.: The continuity of multiplication for two topologies associated with a semifinite trace on von Neumann algebra. Lobachevskii J. Math. **14**, 17–24 (2004)
8. Bikchentaev, A.M., Sherstnev, A.N.: Projective convex combinations in C^*-algebras with the unitary factorization property. Math. Notes **76**(3–4), 578–581 (2004)
9. Bikchentaev, A.M.: On the representation of elements of a von Neumann algebra in the form of finite sums of products of projections. Sib. Math. J. **46**(1), 24–34 (2005)
10. Bikchentaev, A.M., Tikhonov, O.E.: Characterization of the trace by Young's inequality. JIPAM. J. Inequal. Pure Appl. Math. **6**(2), Article 49, 3 pp (2005)
11. Bikchentaev, A.M.: Local convergence in measure on semifinite von Neumamn algebras. Proc. Steklov Inst. Math. **255**, 35–48 (2006)
12. Bikchentaev, A.M.: Representation of elements of von Neumann algebras in the form of finite sums of products of projections. II. Operator Theory **20**, 15–23 (2006). Theta Ser. Adv Math., vol. 6. Theta, Bucharest
13. Bikchentaev, A.M.: Projective convex combinations in C^*-algebras and the invariant subspace problem. I. Math. Notes **79**(1–2), 285–290 (2006)
14. Bikchentaev, A.M., Tikhonov, O.E.: Characterization of the trace by monotonicity inequalities. Linear Algebra Appl. **422**(1), 274–278 (2007)
15. Bikchentaev, A.M.: Local convergence in measure on semifinite von Neumann algebras. II. Math. Notes **82**(5–6), 703–707 (2007)
16. Bikchentaev, A.M.: On the representation of elements of a von Neumann algebra in the form of finite sums of products of projections. III. Commutators in C^*-algebras. Sb. Math. **199**(3–4), 477–493 (2008)
17. Bikchentaev, A.M.: States on symmetric logics: Conditional probability and independence. Lobachevskii J. Math. **30**(2), 101–106 (2009)
18. Bikchentaev, A.M.: Commutativity of projections and trace characterization on von Neumann algebras. Sib. Math. J. **51**(6), 971–977 (2010)
19. Bikchentaev, A.M.: On a lemma of F. A. Berezin. Math. Notes **87**(5–6), 768–773 (2010)
20. Bikchentaev, A.M.: Commutativity of projections and trace characterization on von Neumann algebras. II. Math. Notes **89**(3–4), 461–471 (2011)
21. Bikchentaev, A.M.: The Peierls-Bogoliubov inequality in C^*-algebras and characterization of tracial functionals. Lobachevskii J. Math. **32**(3), 175–179 (2011)
22. Bikchentaev, A.M.: The Haagerup problem on subadditive weights on W^*-algebras. Russian Math. (Iz. VUZ) **55**(10), 82–85 (2011)
23. Bikchentaev, A.M., Yakushev, R.S.: Representation of tripotents and representations via tripotents. Linear Algebra Appl. **435**(9), 2156–2165 (2011)
24. Bikchentaev, A.M., Sabirova, A.A.: Dominated convergence in measure on semifinite von Neumann algebras and arithmetic averages of measurable operators. Sib. Math. J. **53**(2), 207–216 (2012)

25. Bikchentaev, A.M.: A block projection operator in normed ideal spaces of measurable operators. Russian Math. (Iz. VUZ) **56**(2), 75–79 (2012)
26. Bikchentaev, A.M.: On invertibility of some operator sums. Lobachevskii J. Math. **33**(3), 216–222 (2012)
27. Bikchentaev, A.M.: The Haagerup problem on subadditive weights on W^*-algebras. II. Russian Math. (Iz. VUZ) **57**(12), 66–69 (2013)
28. Bikchentaev, A.M.: Commutativity of operators and the characterization of traces on C^*-algebras. Dokl. Math. **87**(1), 79–82 (2013)
29. Bikchentaev, A.M.: On Hermitian operators X and Y meeting the condition $-Y\leq X\leq Y$. Lobachevskii J. Math. **34**(3), 227–233 (2013)
30. Bikchentaev, A.M.: On normal τ-measurable operators affiliated with semifinite von Neumann algebras. Math. Notes **96**(3–4), 332–341 (2014)
31. Bikchentaev, A.M.: Tripotents in algebras: Invertibility and hyponormality. Lobachevskii J. Math. **35**(3), 281–285 (2014)
32. Bikchentaev, A.M.: On the additivity of mappings on measurable functions. Sib. Math. J. **55**(1), 7–11 (2014)
33. Bikchentaev, A., Yakushev, R.: States on symmetric logics: Conditional probability and independence. II. Internat. J. Theoret. Phys. **53**(2), 397–408 (2014)
34. Bikchentaev, A.M.: Concerning the theory of τ-measurable operators affiliated to a semifinite von Neumann algebra. Math. Notes **98**(3–4), 382–391 (2015)
35. Bikchentaev, A.M.: Commutation of projections and characterization of traces on von Neumann algebras. III. Internat. J. Theoret. Phys. **54**(12), 4482–4493 (2015)
36. Bikchentaev, A.M.: Ideal F-norms on C^*-algebras. Russian Math. (Iz. VUZ) **59**(5), 58–63 (2015)
37. Bikchentaev, A., Navara, M., Yakushev, R.: Quantum logics of idempotents of unital rings. Internat. J. Theoret. Phys. **54**(6), 1987–2000 (2015)
38. Bikchentaev, A., Navara, M.: States on symmetric logics: Extensions. Math. Slovaca **66**(2), 359–366 (2016)
39. Bikchentaev, A.M.: On operator monotone and operator convex functions. Russian Math. (Iz. VUZ) **60**(5), 61–65 (2016)
40. Bikchentaev, A.M.: A trace and integrable operators affiliated with a semifinite von Neumann algebra. Dokl. Math. **93**(1), 16–19 (2016)
41. Bikchentaev, A.M.: On idempotent τ-measurable operators affiliated to a von Neumann algebra. Math. Notes **100**(3–4), 515–525 (2016)
42. Bikchentaev, A.M.: On the convergence of integrable operators affiliated to a finite von Neumann algebra. Proc. Steklov Inst. Math. **293**(1), 67–76 (2016)
43. Bikchentaev, A.M.: Integrable products of measurable operators. Lobachevskii J. Math. **37**(4), 397–403 (2016)
44. Bikchentaev, A.M.: Inequality for a trace on a unital C*-algebra. Math. Notes **99**(4), 487–491 (2016)
45. Bikchentaev, A.M.: Two classes of τ-measurable operators affiliated with a von Neumann algebra. Russian. Math. (Iz. VUZ) **61**(1), 76–80 (2017)
46. Bikchentaev, A.M.: On τ-compactness of products of τ-measurable operators. Internat. J. Theoret. Phys. **56**(12), 3819–3830 (2017)
47. Bikchentaev, A.M.: Differences of idempotents in C^*-algebras. Sib. Math. J. **58**(2), 183–189 (2017)
48. Bikchentaev, A.M.: Differences of idempotents in C^*-algebras and the quantum Hall effect. Theoret. Math. Phys. **195**(1), 557–562 (2018)
49. Bikchentaev, A.M.: Ideal spaces of measurable operators affiliated to a semifinite von Neumann algebra. Sib. Math. J. **59**(2), 243–251 (2018)
50. Bikchentaev, A.M.: Paranormal measurable operators affiliated with a semifinite von Neumann algebra. Lobachevskii J. Math. **39**(6), 731–741 (2018)
51. Bikchentaev, A.M., Abed, S.A.: Paranormal elements in normed algebras. Russian Math. (Iz. VUZ) **62**(5), 10–15 (2018)
52. Bikchentaev, A.M.: On sets of measurable operators convex and closed in topology of convergence in measure. Doklady Math. **98**(3), 545–548 (2018)
53. Bikchentaev, A.M.: On an analog of the M.G. Krein theorem for measurable operators. Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki **160**(2), 243–249 (2018)
54. Bikchentaev, A.M.: Ideal F-norms on C^*-algebras. II. Russian Math. (Iz. VUZ) **63**(3), 78–82 (2019)
55. Bikchentaev, A.M., Sukochev, F.A.: When weak and local measure convergence implies norm convergence. J. Math. Anal. Appl. **473**(2), 1414–1431 (2019)
56. Christensen, E.: Measures on projections and physical states. Comm. Math. Phys. **86**(4), 529–538 (1982)

57. Hoa, D.T., Tikhonov, O.E.: Weighted trace inequalities of monotonicity. Lobachevskii J. Math. **26**, 63–67 (2007)
58. Hoa, D.T., Tikhonov, O.E.: On the theory of operator-monotone and operator-convex functions. Russian Math. (Iz. VUZ) **54**(3), 7–11 (2010)
59. Hoa, D.T., Tikhonov, O.E.: Weighted monotonicity inequalities for traces on operator algebras. Math. Notes **88**(1–2), 177–182 (2010)
60. Dodds, P.G., Dodds, T.K., Sukochev, F.A., Tikhonov, O.Ye.: A non-commutative Yosida-Hewitt theorem and convex sets of measurable operators closed locally in measure. Positivity **9**(3), 457–484 (2005)
61. Dorofeev, S.V., Sherstnev, A.N.: Functions of frame type and their applications. Soviet Math. (Iz. VUZ) **34**(4), 25–31 (1990)
62. Dorofeev, S.: On the problem of boundedness of a signed measure on projections of a von Neumann algebra. J. Funct. Anal. **103**(1), 209–216 (1992)
63. Dorofeev, S.V., Sherstnev, A.N.: Frame-type functions and their applications. In: Measures on Projections and Orthomodular Posets (Dedicated to A.N. Sherstnev on the occasion of his 60th Anniversary). 118 pp. Kazan Math. Soc., Kazan (1998), pp. 2–13 (1998)
64. Gleason, A.M.: Measures on the closed subspaces of a Hilbert space. J. Math. Mech. **6**(6), 885–893 (1957)
65. Grigoryan, S.A., Mushtari, D.H., Ovchinnikov, P.G.: Disjunctivity and alternativity in projection logics. Internat. J. Theoret. Phys. **39**(3), 705–709 (2000)
66. Haagerup, U.: Normal weights on W^*-algebras. J. Functional Anal. **19**(3), 302–317 (1975)
67. Haliullin, S.: Contiguity and entire separability of states on von Neumann algebras. Internat. J. Theoret. Phys. **56**(12), 3889–3894 (2017)
68. Haliullin, S.G.: Ultraproducts of von Neumann algebras and ergodicity. Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki **160**(2), 287–292 (2018)
69. Hamhalter, J.: States on W^*-algebras and orthogonal vector measures. Proc. Am. Math. Soc. **110**(3), 803–806 (1990)
70. Hamhalter, J., Turilova, E.: Subspace structures in inner product spaces and von Neumann algebras. Internat. J. Theoret. Phys. **50**(12), 3812–3820 (2011)
71. Hamhalter, J., Turilova, E.: Structure of associative subalgebras of Jordan operator algebras. Q. J. Math. **64**(2), 397–408 (2013)
72. Hamhalter, J., Turilova, E.: Affiliated subspaces and the structure of von Neumann algebras. J. Operator Theory **69**(1), 101–115 (2013)
73. Hamhalter, J., Turilova, E.: Affiliated subspaces and infiniteness of von Neumann algebras. Math. Nachr. **286**(10), 976–985 (2013)
74. Hamhalter, J., Turilova, E.: Automorphisms of order structures of Abelian parts of operator algebras and their role in quantum theory. Internat. J. Theoret. Phys. **53** (10), 3333–3345 (2014)
75. Hamhalter, J., Turilova, E.: Classes of invariant subspaces for some operator algebras. Internat. J. Theoret. Phys. **53**(10), 3397–3408 (2014)
76. Hamhalter, J., Turilova, E.: Orthogonal measures on state spaces and context structure of quantum theory. Internat. J. Theoret. Phys. **55**(7), 3353–3365 (2016)
77. Hamhalter, J., Turilova, E.: Spectral order on AW^*-algebras and its preservers. Lobachevskii J. Math. **37**(4), 439–448 (2016)
78. Hamhalter, J., Turilova, E.: Completeness of Gelfand-Neumark-Segal inner product space on Jordan algebras. Math. Slovaca **66**(2), 459–468 (2016)
79. Hamhalter, J., Turilova, E.: Quantum spectral symmetries. Internat. J. Theoret. Phys. **56**(12), 3807–3818 (2017)
80. Hamhalter, J., Turilova, E.: Choquet order and Jordan maps. Lobachevskii J. Math. **39**(3), 340–347 (2018)
81. Hamhalter, J., Turilova, E.A.: Spectral order on unbounded operators and their symmetries. Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki **160**(2), 293–299 (2018)
82. Kalinin, V.V.: Orthomodular partially ordered sets with dimension. Algebra i Logika **15**(5), 535–557 (1976)
83. Lugovaya, G.D., Sherstnev, A.N.: The Gleason theorem for unbounded measures. (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. **12**, 30–32 (1980)
84. Lugovaya, G.D.: Bilinear forms determining measures on projections. (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. **2**, 88 (1983)
85. Lugovaya, G.D., Sherstnev, A.N.: Realization of the space L_1 with respect to an unbounded measure on projectors. Izv. Vyssh. Uchebn. Zaved. Mat. **12**, 35–42 (1984)
86. Lugovaya, G.D.: On unbounded orthogonal measures on projections on a Hilbert space. (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. **12**, 43–46 (1993)

87. Lugovaya, G., Sherstnev, A.: On extending an unbounded orthogonal vector measure in a semifinite von Neumann algebra to a weight. Demonstratio Math. **30**(3), 591–598 (1997)
88. Lugovaya, G.D., Sherstnev, A.N.: On the problem of the extension of an unbounded measure on projections to a weight. Dokl. Akad. Nauk **365**(2), 165–166 (1999)
89. Lugovaya, G.D., Sherstnev, A.N.: On the extension problem for unbounded measures on projections. Math. Slovaca **50**(4), 473–481 (2000)
90. Lugovaya, G.D., Sherstnev, A.N.: Order properties of orthogonal vector fields and a representation theorem. Russ. J. Math. Phys. **8**(2), 172–179 (2001)
91. Lugovaya, G.D., Sherstnev, A.N.: Description of orthogonal vector fields over W^*-algebras of type I_2. Russian Math. (Iz. VUZ) **59**(4), 28–37 (2015)
92. Matveichuk, M.S.: A theorem on the states of quantum logics. I. (Russian) Teoret. Mat. Fiz. **45**(2), 244–250 (1981)
93. Matveichuk, M.S.: A theorem on the states of quantum logics. II. (Russian) Teoret Mat. Fiz. **48**(2), 261–265 (1981)
94. Matveichuk, M.S.: A description of finite measures in semifinite algebras. (Russian) Funktsional. Anal. i Prilozhen. **15**(3), 41–53 (1981)
95. Matveichuk, M.S., Nessonov, N.I.: A description of finite measures in W^*-factors of type III. (Russian) Izv. Vyssh. Ucebn. Zaved. Mat. **2**, 68–71 (1984)
96. Measures on Projections and Orthomodular Posets (Dedicated to A.N. Sherstnev on the occasion of his 60th Anniversary). 118 pp. Kazan Math. Soc. Kazan (1998)
97. Mushtari, D.: Certain general questions of the theory of probability measures in linear spaces. (Russian) Teoriya Veroyatn. i Primen. **18**(1), 66–77 (1973)
98. Mouchtari, D.: Sur l'existence d'une topologie du type de Sazonov sur un espace de Banach. (French) Sémin. Maurey-Schwartz 1975–1976: Espaces L^p, applications radonifiantes et géométrie des espaces de Banach, Exp. No. 17, 12 pp. Centre Math., Ècole Polytech., Palaiseau (1976)
99. Mushtari, D.Kh., Chuprunov, A.N.: Sufficient topologies and norms. (Russian) Teor. Veroyatnost. i Primenen. **28**(4), 700–714 (1983)
100. Mushtari, D.Kh., Matveichuk, M.S.: Charges on the logic of skew projectors. Dokl. Akad. Nauk SSSR **283**(1), 43–46 (1985)
101. Mushtari, D.: Logics of projectors in Banach spaces. Soviet. Math. (Iz. VUZ) **33**(8), 59–70 (1989)
102. Mushtari, D.H.: Probabilities and topologies on linear spaces. Kazan Math. Foundation, Kazan, 248 pp. ISBN: 5–900975-01–0 (1996)
103. Mushtari, D.: Gleason-type theorem for linear spaces over the field of four elements. Internat. J. Theoret. Phys. **37**(1), 127–130 (1998)
104. Mushtari, D.H.: Orthoadditive functions on idempotents. In: Measures on Projections and Orthomodular Posets (Dedicated to A.N. Sherstnev on the occasion of his 60th Anniversary). 118 pp. Kazan Math. Soc., Kazan (1998), pp. 38–83 (1998)
105. Mushtari, D.: The Gleason theorem for the field of rational numbers and residue fields. Math. Notes **64**(3–4), 506–512 (1999)
106. Mushtari, D.Kh.: Some remarks on measures on orthogonal rational projections and the rational sphere. Lobachevskii J. Math. **9**, 47–53 (2001)
107. Nelson, E.: Notes on non-commutative integration. J. Funct. Anal. **15**(2), 103–116 (1974)
108. Novikov, An.An., Tikhonov, O.E.: Characterization of central elements of operator algebras by inequalities. Lobachevskii. J. Math. **36**(2), 208–210 (2015)
109. Novikov, An.An., Tikhonov, O.E.: Measures on orthoideals and L_1-spaces associated with positive operators. Lobachevskii J. Math. **37**(4), 497–499 (2016)
110. Novikov, A.: L_1-space for a positive operator affiliated with von Neumann algebra. Positivity **21**(1), 359–375 (2017)
111. Ovchinnikov, P.G.: Automorphisms of the poset of skew projections. J. Funct. Anal. **115**(1), 184–189 (1993)
112. Ovchinnikov, P.G.: Countable Jauch-Piron logics. Internat. J. Theoret. Phys. **32**(6), 885–890 (1993)
113. Ovchinnikov, P.G.: A theorem on pure states on a hypergraph. Math. Nachr. **167**, 275–278 (1994)
114. Ovchinnikov, P.G.: On alternative orthomodular posets. Demonstratio Math. **27**(1), 89–93 (1994)
115. Ovchinnikov, P.G.: Exact topological analogs to orthoposets. Proc. Amer. Math. Soc. **125**(10), 2839–2841 (1997)
116. Ovchinnikov, P.G., Sultanbekov, F.F.: Finite concrete logics: Their structure and measures on them. Internat. J. Theoret. Phys. **37**(1), 147–153 (1998)
117. Ovchinnikov, P.G.: Another combinatorial characterization of orthomodular posets. In: Measures on Projections and Orthomodular Posets (Dedicated to A.N. Sherstnev on the occasion of his 60th Anniversary). 118 pp. Kazan Math. Soc., Kazan (1998), pp. 87–94 (1998)

118. Ovchinnikov, P.G.: Measures on finite concrete logics. Proc. Amer. Math. Soc. **127**(7), 1957–1966 (1999)
119. Segal, I.E.: A non-commutative extension of abstract integration. Ann. Math. **57**(3), 401–457 (1953)
120. Sherstnev, A.N.: Boolean logics. (Russian) Kazan. Gos. Univ. Uchen. Zap. **128** (2), 48–62 (1968)
121. Sherstnev, A.N.: The representation of measures that are defined on the orthoprojectors of Hilbert space by bilinear forms. Izv. Vyssh. Ucebn. Zaved. Mat. **9**, 90–97 (1970)
122. Sherstnev, A.N.: Summer mathematical school on noncoomutative probability theory. (Russian) Uspehi mat. nauk **27**(1), 287–288 (1972)
123. Sherstnev, A.N.: On the general theory of states on von Neumann algebras. Funct. Anal. i Prilozhen. **8**(3), 89–90 (1974)
124. Sherstnev, A.N.: The concept of charge in a noncommutative framework of measurement theory. (Russian) Probabilistic methods and cybernetics, No. X-XI (Russian), pp. 68–72. Izdat. Kazan Univ., Kazan (1974)
125. Sherstnev, A.N.: Every smooth weight is an l-weight. (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. **8**, 88–91 (1977)
126. Sherstnev, A.N.: Measures on the ideals of the orthoprojectors of a von Neumann algebra. (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. **10**, 108–109 (1978)
127. Sherstnev, A.N.: On certain problems in the theory of unbounded measures on projections. Atti Sem. Mat. Fis. Univ. Modena **42**(2), 357–366 (1994)
128. Sherstnev, A.N., Turilova, E.A.: Classes of subspaces affiliated with a von Neumann algebra. Russ. J. Math. Phys. **6**(4), 426–434 (1999)
129. Sherstnev, A.N.: An analog of the Hahn-Banach theorem for commutative semigroups. Russ. J. Math. Phys. **9**(2), 198–201 (2002)
130. Sherstnev, A.N.: Two-dimensional reductions of the cone of positive diagonal operators in l^2. Linear Algebra Appl. **370**, 375–378 (2003)
131. Sherstnev, A.N.: Methods of bilinear forms in non-commutative measure and integral theory. (Russian) Moskva: Fizmatlit (ISBN 978–5-9221–0911-6/hbk) 259 p (2008)
132. Sherstnev, A.N., Tikhonov, O.E.: On characterization of integrable sesquilinear forms. Math. Slovaca **62**(6), 1167–1172 (2012)
133. Sherstnev, A.N.: Measures on projections in a W^*-algebra of type I_2. Funct. Anal. Appl. **47**(4), 302–314 (2013)
134. Simon, B.: A canonical decomposition for quadratic forms with applications to monotone convergence theorems. J. Funct. Anal. **28**(3), 377–385 (1978)
135. Skvortsova, G.Sh., Tikhonov, O.E.: Convex sets in noncommutative L_1-spaces that are closed in the topology of local convergence in measure. Russian. Math. (Iz. VUZ) **42**(8), 46–52 (1998)
136. Skvortsova, G.: On the weak sequential completeness of quotient spaces of the space of integrable operators. Russian. Math. (Iz. VUZ) **46**(9), 68–71 (2002)
137. Stolyarov, A.I., Tikhonov, O.E., Sherstnev, A.N.: Characterization of normal traces on von Neumann algebras by inequalities for the modulus. Math. Notes **72**(3–4), 411–416 (2002)
138. Sultanbekov, F.: Set logics and their representations. Internat. J. Theoret. Phys. **32**(11), 2177–2186 (1993)
139. Sultanbekov, F.: Operation associated to partitions and best extension of signed measures on set logics. Internat. J. Theor. Phys. **34**(8), 1735–1739 (1995)
140. Sultanbekov, F.: Solving problems on finite concrete logics with the help of a PC. Internat. J. Theoret. Phys. **37**(1), 199–202 (1998)
141. Sultanbekov, F.: Routines for investigation of concrete logics. In: Measures on Projections and Orthomodular Posets (Dedicated to A.N. Sherstnev on the occasion of his 60th Anniversary). 118 pp. Kazan Math. Soc., Kazan (1998), pp. 95–113 (1998)
142. Sultanbekov, F.F.: On (3,3)-homogeneous Greechie orthomodular posets. Internat. J. Theoret. Phys. **44**(7), 957–963 (2005)
143. Sultanbekov, F.: Automorphism groups of small (3,3)-homogeneous logics. Internat. J. Theoret. Phys. **49**(12), 3271–3278 (2010)
144. Sultanbekov, F.F.: (3,3)-homogeneous quantum logics with 18 atoms. I. Russian Math. (Iz. VUZ) **56**(11), 62–66 (2012)
145. Tikhonov, O.E.: Spaces of L_p type with respect to a weight on a von Neumann algebra. Izv. Vyssh. Uchebn. Zaved. Mat. **8**, 76–78 (1982)
146. Tikhonov, O.E.: Continuity of operator functions in topologies connected with a trace on a von Neumann algebra. Soviet Math. (Iz. VUZ) **31**(1), 110–114 (1987)
147. Tikhonov, O.E.: Banach spaces associated with a state space, and the information function. Constructive theory of functions and functional analysis, No. 7 (Russian), 67–90, Kazan. Gos. Univ. Kazan (1990)
148. Tikhonov, O.E.: Trace inequalities for spaces in spectral duality. Studia Math. **104**(1), 99–110 (1993)

149. Tikhonov, O.E.: On matrix-subadditive functions and a relevant trace inequality. Linear and Multilinear Algebra **44**(1), 25–28 (1998)
150. Tikhonov, O.E.: On the Lozanovskii class of condensing operators and its applications to noncommutative integration. Function spaces, interpolation spaces, and related topics (Haifa, 1995), 204–208, Israel Math. Conf Proc., vol. 13. Bar-Ilan Univ., Ramat Gan (1999)
151. Tikhonov, O.E.: Subadditivity inequalities in von Neumann algebras and characterization of tracial functionals. Positivity **9**(2), 259–264 (2005)
152. Tikhonov, O.E.: A note on definition of matrix convex functions. Linear Algebra Appl. **416**(2–3), 773–775 (2006)
153. Tikhonov, O.E.: On the convergence of functions of normal operators in a strong operator topology. Funct. Anal. Appl. **41**(3), 245–246 (2007)
154. Tikhonov, O.E., Veselova, L.V.: A non-commutative version of Nikishin's theorem. Indag. Math. (N.S.) **26**(1), 142–146 (2015)
155. Trunov, N.V., Sherstnev, A.N.: On the general theory of integration with respect to a weight in algebras of operators. I, II. Soviet Math. (Iz. VUZ) **22**(7), 65–72 (1978)
156. Trunov, N.V., Sherstnev, A.N.: . Izv. Vyssh. Uchebn. Zaved. Mat. **12**, 88–98 (1978)
157. Trunov, N.V., Sherstnev, A.N.: Conditional expectation in a scheme of noncommutative probability theory. In: Trans.VIII Prague Conf. on Inform. Theory, Stat. Decis. Funct., Random Processes (Prague, 1978), vol. B, pp. 287–299. Reidel, Dordrecht (1978)
158. Trunov, N.V.: L_p Spaces Associated with a Weight on a Semifinite von Neumann Algebra. Constructive Theory of Functions and Functional Analysis, vol. 3, pp. 88–93. Kazan. Gos. Univ., Kazan (1981)
159. Trunov, N.V.: On the Theory of Noncommutative L_1 and L_2 Spaces. (Russian) Constructive Theory of Functions and Functional Analysis, vol. IV, pp. 96–105. Kazan. Gos. Univ., Kazan (1983)
160. Trunov, N.V., Sherstnev, A.N.: Introduction to the Theory of Noncommutative Integration. Current Problems in Mathematics. Newest Results, vol. 27, pp. 167–190, 232. Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow (1985)
161. Turilova, E.A.: On some classes of subspaces affiliated with a von Neumann algebra in the representation space of the algebra $B(H)$ associated with a weight. Russian Math. (Iz. VUZ) **50**(9), 55–62 (2006)
162. Turilova, E.: Measures on classes of subspaces affiliated with a von Neumann algebra. Internat. J. Theoret. Phys. **48**(11), 3083–3091 (2009)
163. Turilova, E.: Automorphisms of spectral lattices of unbounded positive operators. Lobachevskii. J. Math. **35**(3), 259–263 (2014)
164. Turilova, E.: Automorphisms of spectral lattices of positive contractions on von Neumann algebras. Lobachevskii. J. Math. **35**(4), 355–359 (2014)
165. Turilova, E.: Completeness of inner product spaces induced by states on Jordan and C^*-algebras. Internat. J. Theoret. Phys. **54**(12), 4229–4236 (2015)
166. Turilova, E.: Probability structures in subspace lattice approach to foundations of quantum theory. Lith. Math. J. **55**(2), 263–269 (2015)
167. Veselova, L.V., Sukochev, F.A., Tikhonov, O.E.: Interpolation of positive operators. Math. Notes **81**(1–2), 37–50 (2007)
168. Yeadon, F.J.: Measures on projections in W^*-algebras of type II_1. Bull. London Math. Soc. **15**(2), 139–145 (1983)
169. Yeadon, F.J.: Finitely additive measures on projections in finite W^*-algebras. Bull. London Math. Soc. **16**(2), 145–150 (1984)
170. Zolotarev, A.A.: L^p spaces with respect to a state on a von Neumann algebra, and interpolation. (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. **8**, 36–43 (1982)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.