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Differences and Commutators of Projections on a Hilbert Space

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Abstract
We establish some new properties of pairs of idempotents and pairs of projections on a Hilbert space. The functional dependences of some operators associated with a pair of projections are found. Particular attention is paid to pairs of isoclinic projections. Such projections play an important role in non-commutative measure theory. Several relationships were obtained for the determinants. We also present an operator relation characterizing the non-trivial invariant subspace of such an operator.

Keywords Hilbert space · Idempotent · Projection · Commutator · C^*-algebra · Trace

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1 Introduction

Let P, Q be idempotents on a Hilbert space $\mathcal{H}$. Various properties (invertibility, Fredholm property, trace class property, positivity etc.) of the difference $X = P - Q$ have been actively studied in recent decades, see [1, 9, 10, 12, 25–28, 36, 39] and references therein. If X is a trace class operator, the traces of all odd degrees of X coincide:

$$\mathrm{tr}(P-Q) = \mathrm{tr}\left((P-Q)^{2n+1}\right) = \dim\ker(X-I) - \dim\ker(X+I) \in \mathbb{Z}, \tag{1}$$

where I is the identity operator on $\mathcal{H}$ [4, 25]. If X is a compact operator, the right-hand side of (1) gives a natural "regularization" for the trace, showing that it is always an integer [4, 21]. Pairs of idempotents play an important role in the Quantum Hall Effect [5]. For idempotents P, Q, R with trace class differences $P-Q$ and $Q-R$, the equality $\mathrm{tr}(P-Q) = \mathrm{tr}(P-R) + \mathrm{tr}(R-Q)$ together with (1) imply that

$$\mathrm{tr}\left((P-Q)^3\right) = \mathrm{tr}\left((P-R)^3\right) + \mathrm{tr}\left((R-Q)^3\right). \tag{2}$$

Physical sense of additivity in (2) comes from interpretation of $\mathrm{tr}((P-Q)^3)$ as the Hall conductance. Additivity of (cubic) equation in (2) can be seen as a variant of the Ohm's law

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on additivity of conductance [21]. In [11, Theorem 1], a C^*-analogue of the Quantum Hall Effect is obtained and it is proved there that the trace of the differences of a wide class of symmetries from a C^*-algebra is real [11, Corollaries 2 and 3]. Any tripotent ($A^3 = A$) in an algebra $\mathcal{A}$ is a difference $P - Q$ of some idempotents $P, Q \in \mathcal{A}$ with $PQ = QP = 0$ [7, Proposition 1]. Hence tripotents inherit some of the properties of idempotents [8, 10, 13].

In this article, we establish some new properties of pairs of idempotents and pairs of projections on a Hilbert space. The functional dependences of some operators associated with a pair of projections are found (Theorems 3.1, 3.15). Particular attention is paid to pairs of isoclinic projections. Such projections play an important role in non-commutative measure theory, see [17, 30, 35]. Several relationships are obtained for the determinants (Corollaries 3.17, 3.20). We also present an operator relation characterizing the non-trivial invariant subspace of such an operator (Theorem 3.10).

2 Preliminaries

Let $\mathcal{A}$ be an algebra, $\mathcal{A}^{\rm id} = \{A \in \mathcal{A} : A^2 = A\}$ be the set of all idempotents in $\mathcal{A}$. An element $X \in \mathcal{A}$ is a *commutator*, if $X = [A, B] = AB - BA$ for some $A, B \in \mathcal{A}$. If $A, B \in \mathcal{A}^{\rm id}$ then $(A-B)^2$ commutes with A and B. Since $[A, B]^2 = (A-B)^4 - (A-B)^2$, the element $[A, B]^2$ also commutes with A and B. If I is the unit of the algebra $\mathcal{A}$ and $P \in \mathcal{A}^{\rm id}$ then $P^{\perp} = I - P$, and $S_P = 2P - I$ is a symmetry.

A C^*-algebra is a complex Banach $*$-algebra $\mathcal{A}$ such that $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{A}$. For a C^*-algebra $\mathcal{A}$ by $\mathcal{A}^{\rm pr}$, and $\mathcal{A}^+$ we denote its subsets of projections ($A = A^* = A^2$), and positive elements, respectively. If $A \in \mathcal{A}$, then $|A| = \sqrt{A^*A} \in \mathcal{A}^+$. Projections $P, Q \in \mathcal{A}$ are called *isoclinic* with angle $\theta \in (0, \pi/2)$, if $PQP = \cos^2\theta\, P$ and $QPQ = \cos^2\theta\, Q$ [35, Definition 10.4].

A mapping $\varphi : \mathcal{A}^+ \to [0, +\infty]$ is called *a trace* on a C^*-algebra $\mathcal{A}$, if $\varphi(X + Y) = \varphi(X) + \varphi(Y)$, $\quad \varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{A}^+$, $\lambda \geq 0$ $(0 \cdot (+\infty) \equiv 0)$; $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{A}$. A trace φ is called *faithful*, if $\varphi(X) > 0$ for any nonzero $X \in \mathcal{A}^+$.

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$, $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all linear bounded operators on $\mathcal{H}$. If $P, Q \in \mathcal{B}(\mathcal{H})^{\rm pr}$, then the projection $P \wedge Q$ is defined by equality $(P \wedge Q)\mathcal{H} = P\mathcal{H} \cap Q\mathcal{H}$, and $P \vee Q = (P^{\perp} \wedge Q^{\perp})^{\perp}$ projected onto $\overline{\mathrm{lin}(P\mathcal{H} \cup Q\mathcal{H})}$. If an operator $A \in \mathcal{B}(\mathcal{H})$ is selfadjoint then $\|A\| = \sup_{\|\xi\|=1} |\langle A\xi, \xi\rangle|$ [34, Chapter VI, Exer. 9]. By Gelfand–Naimark Theorem every C^*-algebra is isometrically $*$-isomorphic to a concrete C^*-algebra of operators on a Hilbert space $\mathcal{H}$ [19, II.6.4.10].

An *antiunitary operator* is a bijective antilinear map $U : \mathcal{H} \to \mathcal{H}$ such that $\langle U\xi, U\eta\rangle = \overline{\langle \xi, \eta\rangle}$ for all $\xi, \eta \in \mathcal{H}$. For antiunitary U the definition of the *adjoint* operator U^* is changed to compensate the complex conjugation, becoming $\langle U\xi, \eta\rangle = \overline{\langle \xi, U^*\eta\rangle}$ for all $\xi, \eta \in \mathcal{H}$. The adjoint of an antiunitary is also antiunitary and $UU^* = U^*U = I$.

Let $\mathcal{H}$ be a separable Hilbert space, $\dim \mathcal{H} = \infty$. An operator $A \in \mathcal{B}(\mathcal{H})$ is non-commutator if and only if

$$A = aI + K \tag{3}$$

for some $a \in \mathbb{C} \setminus \{0\}$ and a compact operator $K \in \mathcal{B}(\mathcal{H})$ [20, Theorem 3], [24, Chapter 19, Problem 182]. For $\dim \mathcal{H} = n < \infty$ the algebra $\mathcal{B}(\mathcal{H})$ can be identified with the full matrix algebra $\mathbb{M}_n(\mathbb{C})$.

3 Projection Differences and Commutators

An operator $T \in \mathcal{B}(\mathcal{H})$ for a separable space $\mathcal{H}$ is a commutator of projections if and only if $T^* = -T$, $\|T\| \le \frac{1}{2}$ and T is unitary equivalent to T^*, see [29]. Every skew-Hermitian element of any properly infinite von Neumann algebra $\mathcal{A}$ can be represented in the form of a finite sum of commutators of projections of the algebra $\mathcal{A}$ [6, 18].

Theorem 3.1 *Let $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$, and $f(t) = t\sqrt{1-t^2}$ $(0 \le t \le 1)$. Then $|[P, Q]| = f(|P - Q|)$.*

Proof We have $\|P - Q\| = \frac{1}{2}\|S_P - S_Q\| \le 1$ by the triangle inequality for the norm $\|\cdot\|$. Hence $|P-Q| \le I$ and $(P-Q)^2 \le I$. Via the identity (see [10, the proof of Proposition 1])

$$(P - Q)^2 = (P - Q)^4 + |[P, Q]|^2$$

we obtain

$$\begin{aligned} |[P, Q]| &= \sqrt{(P-Q)^2 - (P-Q)^4} = \sqrt{(P-Q)^2(I - (P-Q)^2)} \\ &= |P - Q| \cdot \sqrt{I - |P - Q|^2} = f(|P - Q|). \end{aligned}$$

Theorem is proved. □

Corollary 3.2 *Let $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ and $0 \le t \le \frac{1}{\sqrt{2}}$ be so that $\|P - Q\| = t$. Then $\|[P, Q]\| = f(t)$.*

Proof Consider the commutative unital C^*-subalgebra $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$, generated by the operators $|P - Q|$ and I. Then $|[P, Q]| = f(|P - Q|) \in \mathcal{A}$. By the classical Gelfand Theorem on representation of an Abelian unital C^*-algebra (see, for example, [38, Chapter 3, Theorem 1.18]) it follows that the C^*-algebra $\mathcal{A}$ is isometrically $*$-isomorphic to the C^*-algebra $C(\Omega)$ of all complex valued continuous functions on the Stonean space Ω of all characters of the algebra $\mathcal{A}$. Let $\pi : \mathcal{A} \to C(\Omega)$ be this $*$-isomorphism. We have

$$\pi(|[P, Q]|) = \pi(f(|P - Q|)) = f(\pi(|P - Q|)),$$

cf. [15, Lemma 2.1(i)]. Finally, note that the function $f(t) = t\sqrt{1-t^2}$ possesses the positive derivative for $t \in \left[0, \frac{1}{\sqrt{2}}\right)$ and strongly increase in this interval. Thus $\|[P, Q]\| = f(\|\pi(|P - Q|)\|) = f(\|P - Q\|)$. □

Corollary 3.3 *Let $\mathcal{A}$, $\mathcal{B}$ be C^*-algebras and a mapping $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be so that $\mathcal{F}(\mathcal{A}^{\mathrm{pr}}) \subset \mathcal{B}^{\mathrm{pr}}$. If $\|\mathcal{F}(P) - \mathcal{F}(Q)\| = \|P - Q\|$ for all $P, Q \in \mathcal{A}^{\mathrm{pr}}$ with $\|P - Q\| \le \frac{1}{\sqrt{2}}$ then $\|[\mathcal{F}(P), \mathcal{F}(Q)]\| = \|[P, Q]\|$ for all $P, Q \in \mathcal{A}^{\mathrm{pr}}$ with $\|P - Q\| \le \frac{1}{\sqrt{2}}$.*

Consider a unital C^*-algebra $\mathcal{A}$ and put $\mathcal{B} = \mathcal{A}$. Then 1) $\mathcal{F}(A) = I - A$ for all $A \in \mathcal{A}$, and 2) $\mathcal{F}(A) = UAU^*$ for all $A \in \mathcal{A}$ and a fixed isometry $U \in \mathcal{A}$ are examples of such mappings. 3) For $\mathcal{A} = \mathcal{B} = \mathcal{B}(\mathcal{H})$ consider the mapping $\mathcal{F}(A) = UAU^*$ for any $A \in \mathcal{A}$ and a fixed antiunitary operator $U : \mathcal{H} \to \mathcal{H}$. Then $\mathcal{F}$ is linear and $\mathcal{F}(\mathcal{A}^{\mathrm{pr}}) \subset \mathcal{A}^{\mathrm{pr}}$: for $P \in \mathcal{A}^{\mathrm{pr}}$ we have $UPU^* \cdot UPU^* = UPU^*$ and

$$\langle UPU^*\xi, \eta\rangle = \overline{\langle PU^*\xi, U^*\eta\rangle} = \overline{\langle U^*\xi, PU^*\eta\rangle} = \langle PU^*\eta, U^*\xi\rangle = \overline{\langle UPU^*\eta, \xi\rangle} = \langle \xi, UPU^*\eta\rangle$$

for all $\xi, \eta \in \mathcal{H}$. Also

$$\|\mathcal{F}(P)-\mathcal{F}(Q)\| = \|\mathcal{F}(P-Q)\| = \|U(P-Q)U^*\| = \sup_{\|\xi\|=1} |\langle U(P-Q)U^*\xi, \xi\rangle|$$
$$= \sup_{\|\xi\|=1} |\overline{\langle (P-Q)U^*\xi, U^*\xi\rangle}| = \sup_{\|\eta\|=1} |\langle (P-Q)\eta, \eta\rangle| = \|P-Q\|$$

for all $P, Q \in \mathcal{A}^{\mathrm{pr}}$.

Example 3.4 If $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ are isoclinic with angle $\theta \in (0, \pi/2)$, then $|P-Q| = \sin\theta\, P \vee Q$ [35, Theorem 10.5(iii)] and $|[P, Q]| = \cos\theta\, |P-Q|$. Thus $\|P-Q\| = \sin\theta$ and $\|[P, Q]\| = \sin\theta\, \cos\theta$. We have $\cos\theta\, |P-Q| = f(|P-Q|)$, for the real function $f(t) = t\sqrt{1-t^2}$, $0 \le t \le 1$.

Lemma 3.5 *Let J be an ideal in an algebra $\mathcal{A}$. If $A, B \in \mathcal{A}$ and $A-B \in J$, then $[A, B] = (A-B)B - B(A-B) \in J$. Moreover, for $A \in \mathcal{A}^{\mathrm{id}}$ we have:*

(i) *If $A-B \in J$ then $B - ABA = A(A-B)A - (A-B) \in J$;*
(ii) *If $B - ABA \in J$ then $[A, B] = A(B-ABA) - (B-ABA)A \in J$.*

Proposition 3.6 *Assume that $A, B \in \mathcal{A}^{\mathrm{id}}$ and $ABA = \lambda A$, $BAB = \lambda B$ for some $\lambda \in \mathbb{C}\setminus\{0\}$. Then*

(i) *$A - B \in J \Leftrightarrow [A, B] \in J$;*
(ii) *$AB + BA \in J \Leftrightarrow A, B \in J$;*
(iii) *$(AB+BA)^2 = \lambda(A+B)^2 = \lambda(AB+BA) + \lambda(A+B)$;*
(iv) *If an algebra $\mathcal{A}$ is unital then $S_A S_B S_A = S_B S_A S_B \Leftrightarrow \lambda = \frac{1}{\sqrt{2}}$.*

Proof (i). If $[A, B] \in J$ then $\lambda(A-B) = ABA - BAB = [A, B]A - B[A, B] \in J$. (ii). If $AB + BA \in J$ then $2\lambda B = B(AB+BA) + (AB+BA)B - (AB+BA) \in J$. □

Example 3.7 Let $\mathcal{H}$ be a separable Hilbert space and $\dim \mathcal{H} = \infty$, let J be the p-Schatten ideal $\mathfrak{S}_p(\mathcal{H})$ $(0 < p < +\infty)$ in $\mathcal{B}(\mathcal{H})$, see [22, 37]. The inverse implications in (i) and (ii) of Lemma 3.5 do not hold. For (i) put $A = I$ and consider $B \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ with $\dim B\mathcal{H} = \dim B^{\perp}\mathcal{H} = \infty$. Then $B - ABA = [A, B] = 0 \in J$, but $A - B = B^{\perp} \notin J$. For (ii) consider B as above and put $A = B^{\perp}$. Then $[A, B] = 0 \in J$, but $B - ABA \notin J$. On p-Schatten commutators of projections see [2].

Corollary 3.8 *Let J be an ideal in a unital C^*-algebra $\mathcal{A}$ such that $X \in J \Leftrightarrow |X| \in J$, $X \in \mathcal{A}$. Then for $P, Q \in \mathcal{A}^{\mathrm{pr}}$ with $\|P-Q\| < 1$ the following conditions are equivalent:*

(i) *$P - Q \in J$;*
(ii) *$[P, Q] \in J$.*

Proof We show that the element $I - |P-Q|^2$ is invertible if and only if $\|P-Q\| < 1$. We have $\|A-B\| \le 1$ for all $A, B \in \mathcal{A}^{\mathrm{pr}}$. Hence, $|P-Q| \le I$ and $|P-Q|^2 \le I$. We have a chain of equivalences:

“ the element $I - |P-Q|^2$ is invertible in $\mathcal{A}$” $\Leftrightarrow$ “$\exists\varepsilon > 0 : I - |P-Q|^2 \ge \varepsilon I$ ”$\Leftrightarrow$

$\Leftrightarrow$ “$\exists\varepsilon > 0 :\ (1-\varepsilon)I \ge |P-Q|^2$ ” $\Leftrightarrow$ “$\exists\varepsilon > 0 :\ \sqrt{1-\varepsilon}I \ge |P-Q|$ ” $\Leftrightarrow$
$\Leftrightarrow$ “$\exists\varepsilon > 0 :\ \|P-Q\| \le \sqrt{1-\varepsilon}$ ”.

Then we apply Theorem 3.1. □

A Rickart C^*-algebra is a C^*-algebra within which the right annihilator of any element equals the principal right ideal generated by some projection. If $\mathcal{A}$ is a Rickart C^*-algebra then $\mathcal{A}$ is unital and for every ideal J in $\mathcal{A}$ we have $X \in J \Leftrightarrow |X| \in J$, $X \in \mathcal{A}$. Note that $\mathcal{A}$ satisfies polar decomposition ([3, Corollary 3.5], [23, Corollary 7.4]). Curiously, if a unital C^*-algebra $\mathcal{A}$ admits polar decomposition and possesses "good" faithful tracial states then $\mathcal{A}$ is a Rickart C^*-algebra [15, Theorem 3.11].

Lemma 3.9 *Let $\mathcal{A}$ be an algebra, $A, B \in \mathcal{A}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. If $ABA = \lambda A$ then $\lambda^{-1}AB, \lambda^{-1}BA \in \mathcal{A}^{\mathrm{id}}$. If, moreover, $B \in \mathcal{A}^{\mathrm{id}}$ then $\lambda^{-1}BAB \in \mathcal{A}^{\mathrm{id}}$.*

Proof Multiply both sides of the equality $ABA = \lambda A$ from the left (resp., the right) by the element $\lambda^{-2}B$ and obtain $\lambda^{-1}BA \in \mathcal{A}^{\mathrm{id}}$ (resp., $\lambda^{-1}AB \in \mathcal{A}^{\mathrm{id}}$).

Assume now that $B \in \mathcal{A}^{\mathrm{id}}$. Multiply both sides of the equality $ABA = \lambda A$ from the left and the right by the element $\lambda^{-1}B$ and obtain $\lambda^{-1}BAB \in \mathcal{A}^{\mathrm{id}}$. □

For an arbitrary $A \in \mathbb{M}_n(\mathbb{C})$ there exists a pseudo-inverse $B \in \mathbb{M}_n(\mathbb{C})$ such that $ABA = A$ (see [32, Theorem 1.4.15]).

Theorem 3.10 *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda, \mu, \nu \in \mathbb{C}$, $\lambda \neq 0$. Put $A_{\nu,\mu} = \nu I + \mu A$. If $A_{\nu,\mu}BA_{\nu,\mu} = \lambda A_{\nu,\mu}$ for some ν, μ, λ and $A_{\nu,\mu}B \neq \lambda I$ then the operator A possesses a non-trivial invariant subspace.*

Proof It is well known that the subspace $P\mathcal{H}$, $P \in \mathcal{B}(\mathcal{H})^{\mathrm{id}}$, is invariant under the operator $X \in \mathcal{B}(\mathcal{H})$ if and only if $XP = PXP$ [33, Chap. 0, Theorem 0.1].

Step 1. If $P \in \mathcal{B}(\mathcal{H})^{\mathrm{id}} \setminus \{0, I\}$ with $PA_{\nu,\mu} = A_{\nu,\mu}$ then $A_{\nu,\mu}P = PA_{\nu,\mu}P$ and $P\mathcal{H}$ is a non-trivial invariant subspace of the operator $A_{\nu,\mu}$. Therefore $AP = PAP$ and the operator A possesses a non-trivial invariant subspace $P\mathcal{H}$.

Step 2. By Lemma 3.9 we have $P = \lambda^{-1}A_{\nu,\mu}B \in \mathcal{B}(\mathcal{H})^{\mathrm{id}}$ and $PA_{\nu,\mu} = A_{\nu,\mu}$. Then we apply Step 1. □

In particular, every partial isometry $A \in \mathcal{B}(\mathcal{H})$ has a non-trivial invariant subspace $AA^*\mathcal{H}$ (hint: $A = AA^*A$, see [24, Corollary 3 of Problem 98]).

Corollary 3.11 *Let an invertible operator $X \in \mathcal{B}(\mathcal{H})$ satisfy the following condition:*

$$(x - \lambda + 2)I + X^{-1} + (2x - \lambda + 1)X + xX^2 = 0 \tag{4}$$

for some $x, \lambda \in \mathbb{C}$, $\lambda \neq 0$ and $X^{-1} + xX \neq (x - \lambda + 1)I$. Then the operator X possesses a non-trivial invariant subspace.

Proof For $A = I + X$, $B = xI + X^{-1}$ we have $ABA = \lambda A$ with $AB \neq \lambda I$ and apply Theorem 3.10 with $\nu = 0$, $\mu = 1$. □

For $x = 0$ we have a quadratic (4) (after multiplication by X) for X. Such operators were studied in [31]. In [14, Theorem 1], we presented an operator inequality characterizing the invariant subspace of an operator $A \in \mathcal{B}(\mathcal{H})$.

Theorem 3.12 *Let $\mathcal{H}$ be a separable Hilbert space, $\dim \mathcal{H} = \infty$, $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda, \mu, \nu \in \mathbb{C}$, $\lambda \neq 0$. Put $A_{\nu,\mu} = \nu I + \mu A$. If $A_{\nu,\mu}BA_{\nu,\mu} = \lambda A_{\nu,\mu}$ for some ν, μ, λ and A is non-commutator with $\nu + a\mu \neq 0$ (the number a from (3)) then B is non-commutator.*

Proof Assume that $A = aI + K$ for some $a \in \mathbb{C} \setminus \{0\}$ and a compact operator $K \in \mathcal{B}(\mathcal{H})$, see (3). Then the relation $A_{\nu,\mu} B A_{\nu,\mu} = \lambda A_{\nu,\mu}$ turns into

$$(\nu + a\mu)^2 B + \mu(\nu + a\mu)(BK + KB) + \mu^2 KBK = \lambda(\nu + a\mu)I + \lambda\mu K,$$

and $B = \frac{\lambda}{\nu + a\mu} I + Z$ for the compact operator

$$Z = \frac{\lambda\mu}{(\nu + a\mu)^2} K - \frac{\mu}{\nu + a\mu}(BK + KB) - \frac{\mu^2}{(\nu + a\mu)^2} KBK.$$

Theorem is proved. □

Corollary 3.13 *Assume that a noninvertible operator $A \in \mathcal{B}(\mathcal{H})$ has a left inverse $B \in \mathcal{B}(\mathcal{H})$. Then the operator A possesses a non-trivial invariant subspace $AB\,\mathcal{H}$. If, moreover, $\mathcal{H}$ is separable and A is non-commutator then B is non-commutator.*

Proof We have $BA = I$, $AB \in \mathcal{B}(\mathcal{H})^{\mathrm{id}} \setminus \{0, I\}$ and apply Theorem 3.10 with $\nu = 0$, $\mu = \lambda = 1$. Thus the operator A has a non-trivial invariant subspace $AB\,\mathcal{H}$.

For a separable space $\mathcal{H}$ we apply Theorem 3.12 with $\nu = 0$, $\mu = \lambda = 1$. □

Example 3.14 Consider the following complex 2×2 matrices

$$P = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ z & 0 \end{pmatrix}, \quad A = \begin{pmatrix} \lambda & \mu \\ 0 & \nu \end{pmatrix}.$$

Then $P, Q \in \mathbb{M}_2(\mathbb{C})^{\mathrm{id}}$ and $PAP = \lambda P$, $PQP = (1 + z^2)P$, $QPQ = (1 + z^2)Q$. We have $|P - Q| = |z|I$ for all $z \in \mathbb{C}$ and $|PQ - QP| = \sqrt{2}I$ for $z = 1$.

If $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ are isoclinic with angle $\theta \in (0, \pi/2)$ then the C^*-subalgebra $\mathcal{A}$, generated by P and Q, is $*$-isomorphic to $\mathbb{M}_2(\mathbb{C})$ [35, Theorem 10.7].

Theorem 3.15 *Let $\theta \in (0, \pi/2)$ and consider the real function $g(t) = \cos^{-2}\theta\, t^2 - t$, $-1 \le t \le 2$. Then for $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ the following conditions are equivalent:*

(i) *P, Q are isoclinic with angle θ;*
(ii) *$P + Q = g(PQ + QP)$;*
(iii) *$\cos\theta(P + Q) = |PQ + QP|$;*
(iv) *$\cos\theta|P - Q| = |PQ - QP|$.*

Proof (i)⇒(ii). Since $(P - Q)^2 \ge 0$, we have $PQ + QP \le P + Q$. Since $(P - Q^{\perp})^2 \ge 0$, we have $P + Q - I \le PQ + QP$. Thus $-I \le PQ + QP \le 2I$ and we apply item (iii) of Proposition 3.6 with $\lambda = \cos^2\theta$. Hence $P + Q = g(PQ + QP)$. In particular, $[P + Q, PQ + QP] = 0$.

(ii)⇒(i). Multiplying both sides of (ii) from the left by the projection $Q^{\perp}$, we obtain

$$P - PQ = \cos^{-2}\theta(QPQP - QPQPQ + PQP - PQPQ) - QP + QPQ. \quad (5)$$

Taking adjoints in (5) yields

$$P - QP = \cos^{-2}\theta(PQPQ - QPQPQ + PQP - QPQP) - PQ + QPQ. \quad (6)$$

Adding term by term (5) and (6), we obtain

$$P = \cos^{-2}\theta(PQP - QPQPQ) + QPQ. \quad (7)$$

Multiplying both sides of (7) from the left by the projection $Q^{\perp}$, we obtain

$$P - PQ = \cos^{-2}\theta(PQP - PQPQ). \tag{8}$$

Multiply both sides of (8) from the left by the projection P and denote the operator PQP by A. Then

$$\cos^{-2}\theta A^2 - (1 + \cos^{-2}\theta)A + P = 0. \tag{9}$$

Consider the commutative unital C^*-subalgebra $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$, generated by the operators A and P, with the unit P. By the Gelfand Theorem on representation of an Abelian unital C^*-algebra (see, for example, [38, Chapter 3, Theorem 1.18]) the C^*-algebra $\mathcal{A}$ is isometrically $*$-isomorphic to the C^*-algebra $C(\Omega)$ of all complex valued continuous functions on the Stonean space Ω of all characters of the algebra $\mathcal{A}$. Let $\pi : \mathcal{A} \to C(\Omega)$ be this $*$-isomorphism. Then $\pi(P)(\omega) = 1$ and $0 \le \pi(A)(\omega) \le 1$ for all $\omega \in \Omega$. By (8) we have the equation

$$\cos^{-2}\theta\pi(A)^2 - (1 + \cos^{-2}\theta)\pi(A) + 1 = 0.$$

By solving it, we obtain the functions $\pi(A)_1 = 1$ and $\pi(A)_2 = \cos^2\theta\, 1$. For $\pi(A)_1$ we have $A = PQP = P$, hence $PQ \cdot PQ = PQ$ and $PQ \in \mathcal{A}^{\mathrm{id}}$. Since $\|PQ\| \le 1$, we conclude that $PQ \in \mathcal{A}^{\mathrm{pr}}$. Thus $PQ = (PQ)^* = QP = P$ and $P \le Q$. Now (ii) turns into

$$P + Q = g(2P) = (4\cos^{-2}\theta - 2)P.$$

Therefore, $P = Q$ and $\cos^2\theta = 1$, we have a contradictions.

For $\pi(A)_2$ we have $PQP = \cos^2\theta P$. By symmetry of conditions (i), (ii) on P and Q, we obtain $QPQ = \cos^2\theta Q$.

(i)⇒(iii). Since $(PQ+QP)^2 = \cos^2\theta(P+Q)^2$, we have $|PQ+QP| = \cos\theta(P+Q)$. Hence $\|PQ + QP\| = \cos\theta\|P + Q\|$.

(iii)⇒(i). Multiplying both sides of $(PQ + QP)^2 = \cos^2\theta(P + Q)^2$ from the left by the projection $Q^{\perp}$, we obtain

$$QPQP - QPQPQ + PQP - PQPQ = \cos^2\theta(P - PQ + QP - QPQ). \tag{10}$$

Taking adjoints in (10) yields

$$PQPQ - QPQPQ + PQP - QPQP = \cos^2\theta(P - QP + PQ - QPQ). \tag{11}$$

Adding term by term (10) and (11), we obtain (7). Then we repeat the rest part of the proof of implication (ii)⇒(i).

(i)⇒(vi). We have $|PQ - QP|^2 = \cos^2\theta(P - Q)^2$.

(vi)⇒(i). Multiplying both sides of $|PQ - QP|^2 = \cos^2\theta(P - Q)^2$ from the left by the projection P, denote PQP by A, we obtain (9). Then we repeat the rest part of the proof of implication (ii)⇒(i). □

Corollary 3.16 *If $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ are isoclinic with angle $\theta \in (0, \pi/2)$ then $|PQ+QP| = \cos\theta\, g(PQ + QP)$, for the function $g(t) = \cos^{-2}\theta\, t^2 - t$, $-1 \le t \le 2$.*

For the function $h(t) = \lambda|t|$, where $\lambda \in \mathbb{R}$ is fixed and $-1 \le t \le 2$, we have $h(PQ+QP) = \lambda|PQ+QP|$ for all $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$. Consider in $\mathbb{M}_2(\mathbb{C})$ the non-isoclinic projections $P = \mathrm{diag}(1, 0)$ and $Q = \mathrm{diag}(1, 1) = I$. Then $PQ+QP = 2P = |PQ+QP|$. If $\theta \in (0, \pi/2)$ and

$$(4\cos^{-2}\theta - 2)P = g(PQ + QP) = \cos\theta\, |PQ + QP| = 2\cos\theta\, P$$

then we have $2 = \cos^2\theta + \cos^3\theta$, a contradiction.

Corollary 3.17 *Let* $\dim \mathcal{H} = n < \infty$ *and* $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ *be isoclinic with angle* $\theta \in (0, \pi/2)$, $i \in \mathbb{C}$ *with* $i^2 = -1$. *Then*

(i) $\det(PQ + QP) \in \{-\cos^n \theta \det(P + Q), \cos^n \theta \det(P + Q)\}$;
(ii) $\det(i(PQ - QP)) \in \{-\cos^n \theta \det(P - Q), \cos^n \theta \det(P - Q)\}$ *and* $\det(P - Q) = 0$ *for all odd* $n \in \mathbb{N}$.

Proof (i). Since the determinant of a Hermitian matrix is a real number, the assertion follows from Theorem 3.15(iii) and via the theorem on determinant of matrix product.

(ii). Follows from Theorem 3.15(iv) and via the theorem on determinant of matrix product. For odd $n \in \mathbb{N}$ we apply Corollary 6 from [16]. □

Theorem 3.18 *Let* $\mathcal{A}$ *be a* C^**-algebra,* $P, Q \in \mathcal{A}^{\mathrm{pr}}$ *and* $PQP = \lambda P$ *with some* $0 < \lambda < 1$. *If* $\varphi(Q) \le \varphi(P) < +\infty$ *for a faithful trace* φ *on* $\mathcal{A}$, *then* P *and* Q *are isoclinic with angle* $\theta \in (0, \pi/2)$, *such that* $\cos^2 \theta = \lambda$.

Proof An element $R = \lambda^{-1} QPQ \in \mathcal{A}^{\mathrm{pr}}$ by Lemma 3.9. Since $R = QR = RQ$, we have $R \le Q$. Assume that $R \ne Q$. The trace φ is positive homogeneous and faithful, so

$$\lambda\varphi(Q) \le \lambda\varphi(P) = \varphi(PQP) = \varphi((QP)^* QP) = \varphi(QP(QP)^*) = \varphi(QPQ) = \lambda\varphi(R) < \lambda\varphi(Q).$$

We have a contradiction. Thus $R = Q$. □

Theorem 3.19 *Let* $\mathcal{A}$ *be a unital algebra, let* $P, Q \in \mathcal{A}^{\mathrm{id}}$ *be such that* $Q \ne I$ *and* $PQP = \lambda P$, $P^\perp Q P^\perp = \lambda P^\perp$ *for some* $\lambda \in \mathbb{C}$. *Then* $\lambda = \frac{1}{2}$ *and* $Q^\perp = S_P Q S_P$. *Moreover,* $QPQ = QP^\perp Q = \frac{1}{2} Q$ *and* $Q^\perp P Q^\perp = Q^\perp P^\perp Q^\perp = \frac{1}{2} Q^\perp$, $P^\perp = S_Q P S_Q$.

Proof Adding term by term the equalities $PQP = \lambda P$, $P^\perp Q P^\perp = \lambda P^\perp$, we obtain

$$(P - Q)^2 = \lambda I + (1 - 2\lambda) P. \tag{12}$$

Recall that $(P - Q)^2$ commutes with Q. If $\lambda \ne \frac{1}{2}$ then $PQ = QP$ and $PQP \in \mathcal{A}^{\mathrm{id}}$. Therefore, $\lambda = 1$ by the equality $PQP = \lambda P$ and $I = PQP + P^\perp Q P^\perp = PQ + P^\perp Q = Q$, we have a contradiction. Thus $\lambda = \frac{1}{2}$. Since $2PQP + 2P^\perp Q P^\perp = Q + S_P Q S_P = I$, we obtain $Q^\perp = S_P Q S_P$, i.e., Q and $Q^\perp$ are similar.

Multiplying both sides of (12) with $\lambda = \frac{1}{2}$ from the left and the right by the idempotent P (resp., $P^\perp$; Q; $Q^\perp$), we obtain $PQP = \frac{1}{2} P$ (resp., $P^\perp Q P^\perp = \frac{1}{2} P^\perp$; $QPQ = \frac{1}{2} Q$; $Q^\perp P Q^\perp = \frac{1}{2} Q^\perp$). Therefore, $PQ^\perp P = \frac{1}{2} P$, $QP^\perp Q = \frac{1}{2} Q$ and

$$P^\perp Q^\perp P^\perp = P^\perp - P^\perp Q P^\perp = P^\perp - \frac{1}{2} P^\perp = \frac{1}{2} P^\perp.$$

Analogously, we have $Q^\perp P^\perp Q^\perp = Q^\perp - Q^\perp P Q^\perp = \frac{1}{2} Q^\perp$. Multiplying both sides of (12) with $\lambda = \frac{1}{2}$ from the left and the right by the symmetry S_Q, we obtain $P^\perp = S_Q P S_Q$. □

Corollary 3.20 *Let* $\dim \mathcal{H} = n < \infty$ *and* $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{id}}$ *be such that* $Q^\perp = S_P Q S_P$. *Then* n *is even and* $\det(P - Q)^2 = 2^{-n}$.

Proof We have $\dim Q^\perp \mathcal{H} = \mathrm{tr}(Q^\perp) = \mathrm{tr}(S_P Q S_P) = \mathrm{tr}(Q) = \dim Q\mathcal{H} = \frac{1}{2} \dim \mathcal{H}$. Muliply both sides of the equality $Q^\perp = S_P Q S_P$ from the left by the symmetry S_P and obtain (12) with $\lambda = \frac{1}{2}$. Thus $\det(P - Q)^2 = 2^{-n}$ via the theorem on determinant of matrix product. □

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