

## DIFFERENCES OF IDEMPOTENTS IN $C^*$ -ALGEBRAS AND THE QUANTUM HALL EFFECT

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Let  $\varphi$  be a trace on the unital  $C^*$ -algebra  $\mathcal{A}$  and  $\mathfrak{M}_\varphi$  be the ideal of the definition of the trace  $\varphi$ . We obtain a  $C^*$  analogue of the quantum Hall effect: if  $P, Q \in \mathcal{A}$  are idempotents and  $P - Q \in \mathfrak{M}_\varphi$ , then  $\varphi((P - Q)^{2n+1}) = \varphi(P - Q) \in \mathbb{R}$  for all  $n \in \mathbb{N}$ . Let the isometries  $U \in \mathcal{A}$  and  $A = A^* \in \mathcal{A}$  be such that  $I + A$  is invertible and  $U - A \in \mathfrak{M}_\varphi$  with  $\varphi(U - A) \in \mathbb{R}$ . Then  $I - A, I - U \in \mathfrak{M}_\varphi$  and  $\varphi(I - U) \in \mathbb{R}$ . Let  $n \in \mathbb{N}$ ,  $\dim \mathcal{H} = 2n + 1$ , the symmetry operators  $U, V \in \mathcal{B}(\mathcal{H})$ , and  $W = U - V$ . Then the operator  $W$  is not a symmetry, and if  $V = V^*$ , then the operator  $W$  is nonunitary.

**Keywords:** Hilbert space, linear operator, idempotent, symmetry, projection, unitary operator, trace-class operator,  $C^*$ -algebra, trace, quantum Hall effect

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### 1. Introduction

Let  $P$  and  $Q$  be idempotents in the Hilbert space  $\mathcal{H}$ . Various properties (invertibility, Fredholm properties, trace-class, positivity, etc.) of the difference  $X = P - Q$  were studied in [1]–[6]. Each tripotent ( $A = A^3$ ) is the difference  $P - Q$  of some idempotents  $P$  and  $Q$  with  $PQ = QP = 0$  (see Proposition 1 in [7]). Therefore, tripotents inherit some idempotent properties [8]. If  $X$  is a trace-class operator, then the traces of all odd powers of  $X$  coincide:

$$\mathrm{tr}(P - Q) = \mathrm{tr}((P - Q)^{2n+1}) = \dim \ker(X - I) - \dim \ker(X + I) \in \mathbb{Z}, \quad (1)$$

where  $I$  is the identity operator in  $\mathcal{H}$ . If  $X$  is a compact operator, then the right-hand side of (1) yields a natural “regularization” for the trace and shows that it is always an integer [5], [6].

Pairs of idempotents play an important role in the quantum Hall effect [9]. For idempotents  $P, Q$ , and  $R$  with the trace-class operators  $P - Q$  and  $Q - R$ , from the equality  $\mathrm{tr}(P - Q) = \mathrm{tr}(P - R) + \mathrm{tr}(R - Q)$  and (1), we obtain

$$\mathrm{tr}((P - Q)^3) = \mathrm{tr}((P - R)^3) + \mathrm{tr}((R - Q)^3). \quad (2)$$

The physical meaning of the additivity in Eq. (2) comes from the interpretation of  $\mathrm{tr}((P - Q)^3)$  as *Hall conductivity*. The additivity of (cubic) Eq. (2) can be considered a variant of Ohm’s law for the additivity of conductivity [10].

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Here, we obtain a  $C^*$  analogue of the quantum Hall effect (Theorem 1) and prove the realness for the trace of differences of a wide class of symmetries from the unital  $C^*$ -algebra (Corollaries 2 and 3). We show that in the complete matrix algebra  $\mathbb{M}_n(\mathbb{C})$  with odd  $n$ , the difference of the symmetries cannot be a symmetry (Theorem 3).

## 2. Definitions and notation

We call a complex Banach  $*$ -algebra  $\mathcal{A}$  such that  $\|A^*A\| = \|A\|^2$  for all  $A \in \mathcal{A}$  a  $C^*$ -algebra. For a unital  $C^*$ -algebra  $\mathcal{A}$ , we let  $\mathcal{A}^{\text{id}}$ ,  $\mathcal{A}^{\text{sa}}$ ,  $\mathcal{A}^{\text{sym}}$ ,  $\mathcal{A}^{\text{u}}$ ,  $\mathcal{A}^{\text{pr}}$ , and  $\mathcal{A}^+$  respectively denote its submanifolds of idempotents, Hermitian elements, symmetries, unitary elements, projectors, and positive elements. If  $I$  is the unit of the algebra  $\mathcal{A}$  and  $P \in \mathcal{A}^{\text{id}}$ , then  $P^\perp = I - P \in \mathcal{A}^{\text{id}}$ . We call the  $C^*$ -algebra  $\mathcal{A}$  with the predual Banach space  $\mathcal{A}_*$  the  $W^*$ -algebra:  $\mathcal{A} \simeq (\mathcal{A}_*)^*$ .

Let  $\mathcal{H}$  be a Hilbert space over the field  $\mathbb{C}$  and  $\mathcal{B}(\mathcal{H})$  be the  $*$ -algebra of all linear bounded operators on  $\mathcal{H}$ . Any  $C^*$ -algebra can be realized as a  $C^*$ -subalgebra in  $\mathcal{B}(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$  (Gelfand–Naimark theorem; see Theorem 3.4.1 in [11]).

We define the *trace* on the  $C^*$ -algebra  $\mathcal{A}$  as a map  $\varphi: \mathcal{A}^+ \rightarrow [0, +\infty]$  such that  $\varphi(X+Y) = \varphi(X) + \varphi(Y)$ ,  $\varphi(\lambda X) = \lambda\varphi(X)$  for all  $X, Y \in \mathcal{A}^+$ ,  $\lambda \geq 0$  (and  $0 \cdot (+\infty) \equiv 0$ );  $\varphi(Z^*Z) = \varphi(ZZ^*)$  for all  $Z \in \mathcal{A}$ . For the trace  $\varphi$ , we define

$$\mathfrak{M}_\varphi^+ = \{X \in \mathcal{A}^+ : \varphi(X) < +\infty\}, \quad \mathfrak{M}_\varphi^{\text{sa}} = \text{lin}_{\mathbb{R}} \mathfrak{M}_\varphi^+, \quad \mathfrak{M}_\varphi = \text{lin}_{\mathbb{C}} \mathfrak{M}_\varphi^+.$$

The restriction  $\varphi|_{\mathfrak{M}_\varphi^+}$  has a well-defined extension by linearity to a functional on  $\mathfrak{M}_\varphi$ , denoted by the same letter  $\varphi$ . Such an extension allows identifying finite traces (i.e.,  $\varphi(X) < +\infty$  for all  $X \in \mathcal{A}^+$ ) with positive functionals on  $\mathcal{A}$ .

## 3. Basic results

**Lemma 1** (Lemma 1 in [12]). *Let  $\varphi$  be a trace on a unital  $C^*$ -algebra  $\mathcal{A}$  and the elements  $A, B \in \mathcal{A}$  be such that  $A - B \in \mathfrak{M}_\varphi$ . Then  $AB - BA \in \mathfrak{M}_\varphi$  and  $\varphi(AB - BA) = 0$ .*

**Lemma 2** (Theorem 3 in [12]). *Let  $\varphi$  be a trace on a unital  $C^*$ -algebra  $\mathcal{A}$  and  $P, Q \in \mathcal{A}^{\text{id}}$ . If  $P - Q \in \mathfrak{M}_\varphi$ , then  $\varphi(P - Q) \in \mathbb{R}$ .*

**Theorem 1.** *Let  $\varphi$  be a trace on a unital  $C^*$ -algebra  $\mathcal{A}$  and  $P, Q \in \mathcal{A}^{\text{id}}$ . If  $P - Q \in \mathfrak{M}_\varphi$ , then  $\varphi((P - Q)^{2n+1}) = \varphi(P - Q) \in \mathbb{R}$  for all  $n \in \mathbb{N}$ .*

**Proof.** Step 1. We show that

$$(P - Q)^{2n+1} = P - Q + \lambda_1(PQP - QPQ) + \cdots + \lambda_n(\underbrace{PQP \cdots QP}_{2n+1} - \underbrace{QPQ \cdots PQ}_{2n+1}) \quad (3)$$

for some  $\lambda_k \in \mathbb{Z}$ ,  $k = 1, 2, \dots, n$ . We use induction. For  $n = 1$ , we have

$$(P - Q)^3 = P - Q - (PQP - QPQ), \quad (4)$$

and equality (3) is satisfied with  $\lambda_1 = -1$ . We suppose that (3) holds for  $k = n$  and show that it holds for  $k = n + 1$ . We note that  $(P - Q)^{2n+3} = (P - Q)(P - Q)^{2n+1}(P - Q)$  and

$$\begin{aligned} (P - Q)(\underbrace{PQP \cdots QP}_{2m+1} - \underbrace{QPQ \cdots PQ}_{2m+1})(P - Q) &= \\ &= (\underbrace{PQP \cdots QP}_{2m+1} - \underbrace{QPQ \cdots PQ}_{2m+1}) - (\underbrace{PQP \cdots QP}_{2m+3} - \underbrace{QPQ \cdots PQ}_{2m+3}) \end{aligned}$$

for all  $m = 1, 2, \dots, n-1$ . We have thus established equality (3).

Step 2. We show that

$$X_m \equiv \underbrace{PQP \cdots QP}_{2m+1} - \underbrace{QPQ \cdots PQ}_{2m+1} \in \mathfrak{M}_\varphi \quad \text{and} \quad \varphi(X_m) = 0 \quad \text{for all } m \in \mathbb{N}.$$

We set  $A = PQ$  and  $B = \underbrace{QPQ \cdots QP}_{2m}$ . Then

$$X_m = PQ \cdot \underbrace{QPQ \cdots QP}_{2m} - \underbrace{QPQ \cdots QP}_{2m} \cdot PQ = AB - BA.$$

Further,

$$\begin{aligned} A - B &= PQ - \underbrace{QPQ \cdots QP}_{2m} = \\ &= (P^2Q - QPQ) + (QP \cdot Q^2 - QPQP) + \cdots + (\underbrace{QPQ \cdots PQ^2}_{2m-1} - \underbrace{QPQ \cdots QP}_{2m}) = \\ &= (P - Q)PQ + QPQ(Q - P) + \cdots + \underbrace{QPQ \cdots PQ}_{2m-1}(Q - P) \end{aligned}$$

belongs to  $\mathfrak{M}_\varphi$ , and by Lemma 1, the element  $AB - BA$  also belongs to  $\mathfrak{M}_\varphi$  and  $\varphi(AB - BA) = 0$ . From (3), we obtain

$$\varphi((P - Q)^{2n+1}) = \varphi(P - Q) + \lambda_1 \cdot 0 + \lambda_2 \cdot 0 + \cdots + \lambda_n \cdot 0 = \varphi(P - Q)$$

for all  $n \in \mathbb{N}$ . We now note that  $\varphi(P - Q) \in \mathbb{R}$  by Lemma 2. The theorem is proved.

**Corollary 1.** *Let  $\varphi$  be a trace on a unital  $C^*$ -algebra  $\mathcal{A}$ . If  $P, Q, R \in \mathcal{A}^{\text{id}}$  and  $P - Q, Q - R \in \mathfrak{M}_\varphi$ , then*

$$\varphi((P - R)^{2n+1}) = \varphi((P - Q)^{2n+1}) + \varphi((Q - R)^{2n+1}) \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

**Corollary 2.** *Let  $\varphi$  be a trace on a unital  $C^*$ -algebra  $\mathcal{A}$ . If  $U, V, W \in \mathcal{A}^{\text{sym}}$  and  $U - V, V - W \in \mathfrak{M}_\varphi$ , then  $\varphi(U - V) \in \mathbb{R}$  and*

$$\varphi((U - W)^{2n+1}) = \varphi((U - V)^{2n+1}) + \varphi((V - W)^{2n+1}) \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

**Proof.** The formula  $U = 2P - I$  ( $P \in \mathcal{A}^{\text{id}}$ ) establishes a bijection between the manifolds  $\mathcal{A}^{\text{id}}$  and  $\mathcal{A}^{\text{sym}}$ . Let  $V = 2Q - I$  and  $W = 2R - I$  with  $Q, R \in \mathcal{A}^{\text{id}}$ . Then

$$\varphi(U - V) = \varphi(2P - 2Q) = 2\varphi(P - Q) \in \mathbb{R}$$

by Lemma 2. By Corollary 1, we have

$$\begin{aligned} \varphi((U - W)^{2n+1}) &= 2^{2n+1} \varphi((P - R)^{2n+1}) = \\ &= 2^{2n+1} (\varphi((P - Q)^{2n+1}) + \varphi((Q - R)^{2n+1})) = \\ &= \varphi((U - V)^{2n+1}) + \varphi((V - W)^{2n+1}) \end{aligned}$$

for each  $n \in \mathbb{N} \cup \{0\}$ . The statement is proved.

**Remark 1.** The condition

$$\varphi((P - Q)^3) = \varphi(P - Q) \quad \text{for all } P, Q \in \mathcal{A}^{\text{pr}}$$

characterizes the traces in the class of all positive functionals on the  $W^*$ -algebra  $\mathcal{A}$ . From (4), we have  $\varphi(PQP) = \varphi(QPQ)$  for all  $P, Q \in \mathcal{A}^{\text{pr}}$ , and we can apply Theorem 1 in [13] (see [14]–[16] and the references therein for other characterizations of the trace using the commutation relations for projectors).

**Theorem 2.** *Let  $\mathcal{J}$  be an ideal in a unital  $*$ -algebra  $\mathcal{A}$ , let an isometry  $U \in \mathcal{A}$  (i.e.,  $U^*U = I$ ), and let the operator  $A \in \mathcal{A}^{\text{sa}}$  be such that  $I + A$  is invertible in  $\mathcal{A}$ . Then the following conditions are equivalent:*

1.  $U - A \in \mathcal{J}$  and
2.  $I - A, I - U \in \mathcal{J}$ .

**Proof.**  $1 \Rightarrow 2$ . We have  $U^* - A = (U - A)^* \in \mathcal{J}$  and

$$-U^*A + AU = U^*(U - A) - (U^* - A)U \in \mathcal{J}.$$

Consequently,

$$I - A^2 = (U^* - A)(U + A) - U^*A + AU \in \mathcal{J}, \quad I - A = (I - A^2)(I + A)^{-1} \in \mathcal{J}.$$

Therefore,  $I - U = I - A - (U - A) \in \mathcal{J}$ .

$2 \Rightarrow 1$ . We have  $U - A = (I - A) - (I - U) \in \mathcal{J}$ . The theorem is proved.

**Corollary 3.** *Let  $\varphi$  be a trace on a unital  $C^*$ -algebra  $\mathcal{A}$ , let an isometry  $U \in \mathcal{A}$ , and let  $A \in \mathcal{A}^{\text{sa}}$  be such that the operator  $I + A$  is invertible in  $\mathcal{A}$  and  $U - A \in \mathfrak{M}_\varphi$  with  $\varphi(U - A) \in \mathbb{R}$ . Then  $I - A, I - U \in \mathfrak{M}_\varphi$ , and  $\varphi(I - U) \in \mathbb{R}$ .*

**Lemma 3.** *Let  $n \in \mathbb{N}$  and  $\dim \mathcal{H} = 2n + 1$ . If  $P, Q, R \in \mathcal{B}(\mathcal{H})^{\text{id}}$ , then  $P + Q + R \neq 3I/2$ .*

**Proof.** We assume that  $P + Q + R = 3I/2$ . For the canonical trace, we then have  $\text{tr}(P) \in \mathbb{N} \cup \{0\}$  and

$$\frac{3}{2}(2n + 1) = \text{tr}(P + Q + R) = \text{tr}(P) + \text{tr}(Q) + \text{tr}(R) \in \mathbb{N},$$

i.e., we obtain a contradiction. The lemma is proved.

**Theorem 3.** *Let  $n \in \mathbb{N}$ ,  $\dim \mathcal{H} = 2n + 1$ , the operators  $U, V \in \mathcal{B}(\mathcal{H})^{\text{sym}}$ , and  $W \equiv U - V$ . Then*

1. *the operator  $W \notin \mathcal{B}(\mathcal{H})^{\text{sym}}$ , and*
2. *if  $V = V^*$ , then  $W \notin \mathcal{B}(\mathcal{H})^{\text{u}}$ .*

**Proof.** 1. We assume that  $W \in \mathcal{B}(\mathcal{H})^{\text{sym}}$ . Then the operators

$$P = \frac{U + I}{2}, \quad Q = \frac{V + I}{2}, \quad R = \frac{W + I}{2}$$

are in  $\mathcal{B}(\mathcal{H})^{\text{id}}$ , and the equality  $U - V = W$  is equivalent to the equality  $2P + 2Q^\perp + 2R^\perp = 3I$ , which contradicts Lemma 3.

2. We now assume that  $W \in \mathcal{B}(\mathcal{H})^u$  and let

$$U = 2P - I \quad \text{with } P \in \mathcal{B}(\mathcal{H})^{\text{id}} \quad \text{and} \quad V = 2Q - I \quad \text{with } Q \in \mathcal{B}(\mathcal{H})^{\text{pr}}.$$

Because

$$\frac{W + 2Q}{2} = P = P^2 = \frac{W^2 + 4Q + 2WQ + 2QW}{4},$$

we have  $2W = W^2 + 2WQ + 2QW$ . Multiplying both sides of this equality by the operator  $W^*$  from the left, we obtain

$$2I = W + 2Q + 2W^*QW. \tag{5}$$

Therefore,  $2I - W \in \mathcal{B}(\mathcal{H})^{\text{sa}}$  and  $W = W^*$ . Consequently,  $W = 2R - I$  with  $R = (W + I)/2 \in \mathcal{B}(\mathcal{H})^{\text{pr}}$ . From (5), we obtain

$$3I = 2R + 2Q + 2W^*QW,$$

which contradicts Lemma 3. The theorem is proved.

**Corollary 4.** *Let  $n \in \mathbb{N}$ ,  $\dim \mathcal{H} = 2n + 1$ , and the operators  $U, V \in \mathcal{B}(\mathcal{H})^u \cap \mathcal{B}(\mathcal{H})^{\text{sa}}$ . Then the operator  $U - V \notin \mathcal{B}(\mathcal{H})^u$ .*

**Example 1.** The condition  $\dim \mathcal{H} = 2n + 1$  is important in Theorem 2 and Corollary 4. We set  $x = 1/2 + \sqrt{3}/16$  and consider the operators  $U = 2P - I$  and  $V = 2Q - I$  in  $\mathbb{M}_2(\mathbb{C})$ , where

$$P = \begin{pmatrix} x & 1/4 \\ 1/4 & 1 - x \end{pmatrix}, \quad Q = \begin{pmatrix} x & -1/4 \\ -1/4 & 1 - x \end{pmatrix}.$$

Then  $P, Q \in \mathbb{M}_2(\mathbb{C})^{\text{pr}}$ , and the operator  $U - V$  is unitary.

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