Positivity
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Positivity

Paranormal measurable operators affiliated with a semifinite von Neumann algebra. II

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Abstract

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$ and τ be a faithful normal semifinite trace on $\mathcal{M}$. Let t_τ be the measure topology on the $*$-algebra $S(\mathcal{M}, \tau)$ of all τ-measurable operators. We define three t_τ-closed classes $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ of τ-measurable operators and investigate their properties. The class $\mathcal{P}_2$ contains $\mathcal{P}_1 \cup \mathcal{P}_3$. If a τ-measurable operator T is hyponormal, then T lies in $\mathcal{P}_1 \cap \mathcal{P}_3$; if an operator T lies in $\mathcal{P}_3$, then UTU^* belongs to $\mathcal{P}_3$ for all isometries U from $\mathcal{M}$. If a bounded operator T lies in $\mathcal{P}_1 \cup \mathcal{P}_3$ then T is normaloid. If an operator $T \in S(\mathcal{M}, \tau)$ is p-hyponormal with $0 < p \leq 1$ then $T \in \mathcal{P}_1$. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \mathrm{tr}$ is the canonical trace, then the class $\mathcal{P}_1$ (resp., $\mathcal{P}_3$) coincides with the set of all paranormal (resp., $*$-paranormal) operators on $\mathcal{H}$. Let $A, B \in S(\mathcal{M}, \tau)$ and A be p-hyponormal with $0 < p \leq 1$. If AB is τ-compact then A^*B is τ-compact.

Keywords Hilbert space · von Neumann algebra · Trace · Non-commutative integration · Measurable operator · Generalized singular value function · Paranormal operator · Hyponormal operator · Operator inequality

Mathematics Subject Classification 46L10 · 47C15 · 46L51

1 Introduction

It is well known that bounded hyponormal operators on a Hilbert space $\mathcal{H}$ have some interesting properties. For example, if A is a hyponormal operator then $\|A^n\|_\infty = \|A\|_\infty^n$ for every $n \in \mathbb{N}$ [20, Problem 162], here $\| \cdot \|_\infty$ denotes the uniform norm on $\mathcal{B}(\mathcal{H})$; every bounded hyponormal compact operator is normal [20, Problem 163].

This paper is dedicated to Professor P. G. Ovchinnikov.

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Fruitful generalizations of the notion of a hyponormal operator are the concepts of p-hyponormal [1], paranormal [17,23], and $*$-paranormal operators [3]. A number of modern authors study properties of such operators (see, for example, [29,30] and references in them).

In this article, we obtain analogs of certain properties of bounded p-hyponormal, paranormal, and $*$-paranormal operators on $\mathcal{H}$ for some unbounded ones. Let $\mathcal{M}$ be a von Neumann operator algebra on a Hilbert space $\mathcal{H}$, $\mathbf{1}$ be the unit of $\mathcal{M}$, τ be a faithful normal semifinite trace on $\mathcal{M}$, $S(\mathcal{M}, \tau)$ be the $*$-algebra of all τ-measurable operators, a number $0 < p < +\infty$ and $L_p(\mathcal{M}, \tau)$ be the space of integrable (with respect to τ) in pth degree operators. Let $\mathcal{M}_1 = \{X \in \mathcal{M} : \|X\|_\infty = 1\}$, $\mu(\cdot; X)$ be the generalized singular value function of operator $X \in S(\mathcal{M}, \tau)$ and let $|X| = \sqrt{X^*X}$. Assume that $\|X\|_\infty = +\infty$ for all $X \in S(\mathcal{M}, \tau)\backslash\mathcal{M}$.

In papers [6,8] we introduced two classes of τ-measurable operators

$$\mathcal{P}_1 = \{T \in S(\mathcal{M}, \tau) : \ \|T^2A\|_\infty \geq \|TA\|_\infty^2 \ \text{ for all } A \in \mathcal{M}_1 \ \text{ with } TA \in \mathcal{M}\},$$
$$\mathcal{P}_2 = \{T \in S(\mathcal{M}, \tau) : \ \mu(t; T^2) \geq \mu(t; T)^2 \ \text{ for all } t > 0\}$$

and investigated their properties. The classes $\mathcal{P}_1$ and $\mathcal{P}_2$ are closed in the topology of convergence in measure τ and $\mathcal{P}_1 \subset \mathcal{P}_2$ (Propositions 3.5 and 3.30 of [6]). In [6, Theorem 3.1] we gave an equivalent definition of the class $\mathcal{P}_1$ [i.e., $T \in \mathcal{P}_1$ if and only if $|T|^2 \leq (\lambda^{-1}|T^2|^2 + \lambda\mathbf{1})/2$ for all $\lambda > 0$], that allowed us to call $\mathcal{P}_1$ a class of all paranormal τ-measurable operators. A similar definition of paranormal elements for general normed algebras was introduced and investigated in [7].

If an operator $T \in S(\mathcal{M}, \tau)$ is hyponormal then $T \in \mathcal{P}_1$; if an operator $T \in \mathcal{P}_1$ has the inverse $T^{-1} \in \mathcal{M}$ then $T^{-1} \in \mathcal{P}_1$ [6, Theorem 3.6]. If an operator $T \in \mathcal{P}_k$ then $UTU^* \in \mathcal{P}_k$ for all isometries $U \in \mathcal{M}$ and $k = 1, 2$. If an operator $T \in \mathcal{P}_1 \cap \mathcal{M}$ then $T^n \in \mathcal{P}_1$ for all $n \in \mathbb{N}$ [6, Theorem 3.12]. Consider an operator $T \in \mathcal{P}_1 \cap \mathcal{M}$ and $n \in \mathbb{N}$. Then $\mu(t, T^n) \geq \mu(t; T)^n$ for all $t > 0$ [6, Theorem 3.16] and we have the equivalences: an operator T is τ-compact $\Leftrightarrow$ an operator T^n is τ-compact; $T \in L_{pn}(\mathcal{M}, \tau) \Leftrightarrow T^n \in L_p(\mathcal{M}, \tau)$, $0 < p < +\infty$ [6, Corollary 3.17]. Every operator $T \in \mathcal{P}_1 \cap \mathcal{M}$ is normaloid [6, Corollary 3.18]. Each τ-compact p-hyponormal operator is normal [12, Theorem 2.2]. If an operator $T \in S(\mathcal{M}, \tau)$ is hyponormal and T^n is τ-compact for some natural number n then T is both normal and τ-compact [6, Corollary 3.7]; it is a strengthening of item (i) of Corollary 3.2 [12]. If $T \in \mathcal{P}_1$ then $T^2 \in \mathcal{P}_1$ [6, Theorem 3.21].

Put $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and τ be the canonical trace tr. Then the class $\mathcal{P}_1$ coincides with the set of all paranormal operators on $\mathcal{H}$ [6, Corollary 3.3], is sequentially closed in the strong operator topology [6, Corollary 3.4] and contains a non-hyponormal operator [6, Corollary 3.13]. If $\mathcal{H}$ is separable and infinite-dimensional then $\mathcal{P}_1 \neq \mathcal{P}_2$ [6, Corollary 3.23].

In this paper we introduce the class

$$\mathcal{P}_3 = \{T \in S(\mathcal{M}, \tau) : \ \|T^2A\|_\infty \geq \|T^*A\|_\infty^2 \ \text{ for all } A \in \mathcal{M}_1 \ \text{ with } T^*A \in \mathcal{M}\}$$

of τ-measurable operators and investigate some properties of $\mathcal{P}_1$ and $\mathcal{P}_3$. In Theorem 3.1 we obtain an equivalent definition of the class $\mathcal{P}_3$ [i.e., $T \in \mathcal{P}_3$ if and only if

$|T^*|^2 \leq (\lambda^{-1}|T^2|^2 + \lambda\mathbf{1})/2$ for all $\lambda > 0$], that allows us to call $\mathcal{P}_3$ a class of all $*$-paranormal τ-measurable operators. The class $\mathcal{P}_3$ is closed in the measure topology t_τ (Corollary 3.2). If an operator $T \in S(\mathcal{M}, \tau)$ is hyponormal then $T \in \mathcal{P}_3$; if an operator $T \in \mathcal{P}_3$ then $UTU^* \in \mathcal{P}_3$ for all isometries $U \in \mathcal{M}$ (Theorem 3.6). If an operator $T \in S(\mathcal{M}, \tau)$ is p-hyponormal with $0 < p \leq 1$ then $T \in \mathcal{P}_1$ and $\mu(t; T^2) \geq \mu(t; T)^2$ for all $t > 0$ (Theorem 4.4 and Corollary 4.5). It is a strengthening of item (i) of Theorem 3.6 [6] and a generalization of Theorem 3 [28]. Methods of proof are new even for algebra $\mathcal{B}(\mathcal{H})$, endowed with the canonical trace tr. Let $A, B \in S(\mathcal{M}, \tau)$ and A be p-hyponormal with $0 < p \leq 1$. If AB is τ-compact then A^*B is τ-compact (Theorem 5.1). On τ-compactness of products of τ-measurable operators see [9].

2 Notation, definitions and preliminaries

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$. Let $\mathcal{P}(\mathcal{M})$ be the lattice of projections in $\mathcal{M}$, $\mathbf{1}$ be the unit of $\mathcal{M}$, and let $P^\perp = \mathbf{1} - P$ for $P \in \mathcal{P}(\mathcal{M})$. Also $\mathcal{M}^+$ denotes the cone of positive elements in $\mathcal{M}$, and $\|\cdot\|_\infty$ denotes the uniform norm on $\mathcal{M}$. A mapping $\varphi : \mathcal{M}^+ \to [0, +\infty]$ is called *a trace*, if $\varphi(X + Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$, $\lambda \geq 0$ [moreover, $0 \cdot (+\infty) \equiv 0$]; $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$. A trace φ is called *faithful*, if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$; *normal*, if $X_i \uparrow X$ $(X_i, X \in \mathcal{M}^+) \Rightarrow \varphi(X) = \sup \varphi(X_i)$; *semifinite*, if $\varphi(X) = \sup\{\varphi(Y) : \ Y \in \mathcal{M}^+, \ Y \leq X, \ \varphi(Y) < +\infty\}$ for every $X \in \mathcal{M}^+$.

A linear operator $X : \mathfrak{D}(X) \to \mathcal{H}$, where the domain $\mathfrak{D}(X)$ of X is a linear subspace of $\mathcal{H}$, is said to be *affiliated* with $\mathcal{M}$ if $YX \subseteq XY$ for all $Y \in \mathcal{M}'$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$. A linear operator $X : \mathfrak{D}(X) \to \mathcal{H}$ is termed *measurable* with respect to $\mathcal{M}$ if X is closed, densely defined, affiliated with $\mathcal{M}$ and there exists a sequence $\{P_n\}_{n=1}^\infty$ in the logic of all projections of $\mathcal{M}$, $\mathcal{P}(\mathcal{M})$, such that $P_n \uparrow \mathbf{1}$, $P_n(\mathcal{H}) \subseteq \mathfrak{D}(X)$ and $P_n^\perp$ is a finite projection (with respect to $\mathcal{M}$) for all n. It should be noted that the condition $P_n(\mathcal{H}) \subseteq \mathfrak{D}(X)$ implies that $XP_n \in \mathcal{M}$. The collection of all measurable operators with respect to $\mathcal{M}$ is denoted by $S(\mathcal{M})$, which is a unital $*$-algebra with respect to strong sums and products [denoted simply by $X + Y$ and XY for all $X, Y \in S(\mathcal{M})$] [27,31].

Let X be a self-adjoint operator affiliated with $\mathcal{M}$. We denote its spectral measure by $\{E^X\}$. It is well known that if X is a closed operator affiliated with $\mathcal{M}$ with the polar decomposition $X = U|X|$, then $U \in \mathcal{M}$ and $E \in \mathcal{M}$ for all projections $E \in \{E^{|X|}\}$. Moreover, $X \in S(\mathcal{M})$ if and only if X is closed, densely defined, affiliated with $\mathcal{M}$ and $E^{|X|}(\lambda, \infty)$ is a finite projection for some $\lambda > 0$. It follows immediately that in the case when $\mathcal{M}$ is a von Neumann algebra of type III or a type I factor, we have $S(\mathcal{M}) = \mathcal{M}$. For type II von Neumann algebras, this is no longer true. From now on, let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ.

For any closed and densely defined linear operator $X : \mathfrak{D}(X) \to \mathcal{H}$, the *null projection* $n(X) = n(|X|)$ is the projection onto its kernel $\mathrm{Ker}(X)$, the *range projection* $r(X)$ is the projection onto the closure of its range $\mathrm{Ran}(X)$ and the *support projection* $\mathrm{s}(X)$ of X is defined by $\mathrm{s}(X) = \mathbf{1} - n(X)$.

An operator $X \in S(\mathcal{M})$ is called τ-measurable if there exists a sequence $\{P_n\}_{n=1}^{\infty}$ in $P(\mathcal{M})$ such that $P_n \uparrow \mathbf{1}$, $P_n(\mathcal{H}) \subseteq \mathfrak{D}(X)$ and $\tau(P_n^{\perp}) < \infty$ for all n. The collection $S(\mathcal{M}, \tau)$ of all τ-measurable operators is a unital $*$-subalgebra of $S(\mathcal{M})$ denoted by $S(\mathcal{M}, \tau)$. It is well known that a linear operator X belongs to $S(\mathcal{M}, \tau)$ if and only if $X \in S(\mathcal{M})$ and there exists $\lambda > 0$ such that $\tau(E^{|X|}(\lambda, \infty)) < \infty$. Alternatively, an unbounded operator X affiliated with $\mathcal{M}$ is τ-measurable (see [16]) if and only if

$$\tau(E^{|X|}(n, +\infty)) \to 0 \quad \text{as } n \to \infty.$$

Let $\mathcal{L}^+$ and $\mathcal{L}_h$ denote the positive and Hermitian parts of a family $\mathcal{L} \subset S(\mathcal{M}, \tau)$, respectively. We denote by $\leq$ the partial order in $S(\mathcal{M}, \tau)_h$ generated by its proper cone $S(\mathcal{M}, \tau)^+$. If $X \in S(\mathcal{M}, \tau)$, then $|X| = \sqrt{X^* X} \in S(\mathcal{M}, \tau)^+$.

Definition 2.1 Let a semifinite von Neumann algebra $\mathcal{M}$ be equipped with a faithful normal semifinite trace τ and let $X \in S(\mathcal{M}, \tau)$. The generalized singular value function $\mu(X) : t \to \mu(t; X)$ of the operator X is defined by setting

$$\mu(s; X) = \inf\{\|XP\|_{\infty} : \ P \in \mathcal{P}(\mathcal{M}) \ \text{such that} \ \tau(P^{\perp}) \leq s\}. \tag{1}$$

An equivalent definition in terms of the distribution function of the operator X is the following. For every self-adjoint operator $X \in S(\mathcal{M}, \tau)$, setting

$$d_X(t) = \tau(E^X(t, \infty)), \quad t > 0,$$

we have (see e.g. [16] and [26])

$$\mu(t; X) = \inf\{s \geq 0 : \ d_{|X|}(s) \leq t\}.$$

Note that $d_X(\cdot)$ is a right-continuous function (see e.g. [16]).

For convenience of the reader, we also recall the definition of the *measure topology* t_τ on the algebra $S(\mathcal{M}, \tau)$. For every $\varepsilon, \delta > 0$, we define the set

$$V(\varepsilon, \delta) = \{X \in S(\mathcal{M}, \tau) : \ \exists P \in \mathcal{P}(\mathcal{M}) \ \text{such that} \ \|XP\|_{\infty} \leq \varepsilon, \ \tau(P^{\perp}) \leq \delta\}.$$

The topology generated by the sets $V(\varepsilon, \delta)$, $\varepsilon, \delta > 0$, is called the measure topology t_τ on $S(\mathcal{M}, \tau)$ [16,27]. It is well-known that the algebra $S(\mathcal{M}, \tau)$ equipped with the measure topology is a complete metrizable topological algebra [27]. We note that a sequence $\{X_n\}_{n=1}^{\infty} \subset S(\mathcal{M}, \tau)$ converges to zero with respect to measure topology t_τ (i.e. $X_n \xrightarrow{\tau} 0$) if and only if $\tau\left(E^{|X_n|}(\varepsilon, \infty)\right) \to 0$ as $n \to \infty$ for all $\varepsilon > 0$.

The space $S_0(\mathcal{M}, \tau)$ of τ-compact operators is the space associated to the algebra of functions from $S(0, \infty)$ vanishing at infinity, that is,

$$S_0(\mathcal{M}, \tau) = \left\{X \in S(\mathcal{M}, \tau) : \ \lim_{t \to +\infty} \mu(t; X) = 0\right\}.$$

The two-sided ideal $\mathcal{F}(\tau)$ in $\mathcal{M}$ consisting of all elements of τ-finite range is defined by

$$\mathcal{F}(\tau) = \{X \in \mathcal{M} \ : \ \tau(r(X)) < +\infty\} = \{X \in \mathcal{M} \ : \ \tau(s(X)) < +\infty\}.$$

Equivalently, $\mathcal{F}(\tau) = \{X \in \mathcal{M} : \mu(t; X) = 0 \text{ for some } t > 0\}$. Clearly, $S_0(\mathcal{M}, \tau)$ is the closure of $\mathcal{F}(\tau)$ with respect to the measure topology [14], which is a two-sided ideal in $S(\mathcal{M}, \tau)$.

Let m be Lebesgue measure on $\mathbb{R}$. The noncommutative L_p-Lebesgue space $(0 < p < +\infty)$ affiliated with $(\mathcal{M}, \tau)$ is defined as

$$L_p(\mathcal{M}, \tau) = \{X \in S(\mathcal{M}, \tau) : \ \mu(X) \in L_p(\mathbb{R}^+, m)\}$$

with the quasi-norm $\|X\|_p = \|\mu(X)\|_p$, $X \in L_p(\mathcal{M}, \tau)$. In particular, $\|\cdot\|_p$ is a norm when $1 \leq p < +\infty$. We have $\mathcal{F}(\tau) \subset L_p(\mathcal{M}, \tau) \subset S_0(\mathcal{M}, \tau)$ for all $0 < p < +\infty$.

If $\tau(\mathbf{1}) < +\infty$ then $S(\mathcal{M}, \tau) = S_0(\mathcal{M}, \tau)$ consists of all closed linear operators on $\mathcal{H}$ affiliated with $\mathcal{M}$ and $\mathcal{F}(\tau) = \mathcal{M}$. Furthermore, t_τ is independent of a concrete choice of a trace τ and is minimal among all metrizable topologies which agree with the ring structure of $S(\mathcal{M}, \tau)$ [13, Theorem 2].

Lemma 2.2 [16] *Let $X, Y, Z \in S(\mathcal{M}, \tau)$. Then*

(1) $\mu(t; X) = \mu(t; |X|) = \mu(t; X^*)$ *for all* $t > 0$*;*
(2) *if* $X, Y \in \mathcal{M}$ *then* $\mu(t; XZY) \leq \|X\|_\infty \|Y\|_\infty \mu(t; Z)$ *for all* $t > 0$*;*
(3) $\mu(t; |X|^p) = \mu(t, X)^p$ *for all* $p > 0$ *and* $t > 0$*;*
(4) *if* $|X| \leq |Y|$ *then* $\mu(t; X) \leq \mu(t; Y)$ *for all* $t > 0$*;*
(5) $\mu(s + t; X + Y) \leq \mu(s; X) + \mu(t; Y)$ *for all* $s, t > 0$*;*
(6) $\mu(t; \lambda X) = |\lambda| \mu(t; X)$ *for all* $\lambda \in \mathbb{C}$ *and* $t > 0$*;*
(7) $\lim_{t \to 0+} \mu(t; X) = \|X\|_\infty$ *if* $X \in \mathcal{M}$ *and* $\lim_{t \to 0+} \mu(t; X) = +\infty$ *if* $X \notin \mathcal{M}$*.*

An operator $A \in S(\mathcal{M}, \tau)$ is said to be *p-hyponormal* with $0 < p \leq 1$, if $(A^*A)^p \geq (AA^*)^p$; *hyponormal*, if it is 1-hyponormal; *cohyponormal*, if A^* is hyponormal; *quasinormal*, if A commutes with A^*A, i.e. $A \cdot A^*A = A^*A \cdot A$.

Lemma 2.3 (See [15], p. 720) *If $X, Y \in S(\mathcal{M}, \tau)^+$ and $Z \in S(\mathcal{M}, \tau)$ then the inequality $X \leq Y$ implies that $ZXZ^* \leq ZYZ^*$.*

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, i.e. the $*$-algebra of all linear bounded operators on $\mathcal{H}$, and $\tau = \operatorname{tr}$ is the canonical trace then $S(\mathcal{M}, \tau)$ coincides with $\mathcal{B}(\mathcal{H})$. In this case the measure topology coincides with the $\|\cdot\|_\infty$-topology, $S_0(\mathcal{M}, \tau)$ is the ideal of all compact operators on $\mathcal{H}$, $\mathcal{F}(\tau)$ is the finite-dimensional operator ideal on $\mathcal{H}$ and

$$\mu(t; X) = \sum_{n=1}^{\infty} s_n(X) \chi_{[n-1,n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^{\infty}$ is the sequence of s-numbers of an operator X [19, Chap. 1]; here χ_A is the indicator function of a set $A \subset \mathbb{R}$. In this case, the space $L_p(\mathcal{M}, \tau)$ is a Schatten–von Neumann ideal $C_p(\mathcal{H})$, $0 < p < +\infty$.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *paranormal* (*$*$-paranormal*), if $\|T^2x\|_{\mathcal{H}} \geq \|Tx\|^2_{\mathcal{H}}$ (respectively, $\|T^2x\|_{\mathcal{H}} \geq \|T^*x\|^2_{\mathcal{H}}$) for all $x \in \mathcal{H}_1 = \{y \in \mathcal{H} : \ \|y\|_{\mathcal{H}} = 1\}$, see [17,24]; *normaloid*, if $\|T\|_\infty = \sup_{y\in\mathcal{H}_1} |\langle Tx, x\rangle|$. It is known that T is normaloid $\Leftrightarrow$ its spectral radius equals $\|T\|_\infty$, or, equivalently, $||T^n\|_\infty = \|T\|^n_\infty$ for all $n \in \mathbb{N}$ [20]. It is shown in [25, Problem 9.5] that an operator $T \in \mathcal{B}(\mathcal{H})$ is paranormal $\Leftrightarrow$ $|T|^2 \leq (\lambda^{-1}|T^2|^2 + \lambda\mathbf{1})/2$ for all $\lambda > 0$. It is shown in [4] that an operator $T \in \mathcal{B}(\mathcal{H})$ is $*$-paranormal $\Leftrightarrow$

$$|T^*|^2 \leq \frac{1}{2}(\lambda^{-1}|T^2|^2 + \lambda\mathbf{1}) \quad \text{for all } \lambda > 0. \tag{2}$$

Let (Ω, ν) be a measure space and $\mathcal{M}$ be the von Neumann algebra of multiplicator operators M_f by functions f from $L_\infty(\Omega, \nu)$ on a space $L_2(\Omega, \nu)$. The algebra $\mathcal{M}$ contains no compact operators $\Leftrightarrow$ the measure ν has no atoms [2, Theorem 8.4].

3 Three classes of τ-measurable operators

Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$. It is obvious that

$$T \in \mathcal{P}_k \Leftrightarrow \lambda T \in \mathcal{P}_k \quad \text{for all } \lambda \in \mathbb{C}\backslash\{0\},\ k = 1, 2, 3.$$

Theorem 3.1 *For an operator $T \in S(\mathcal{M}, \tau)$ the following conditions are equivalent:*

(i) *$T \in \mathcal{P}_3$;*
(ii) *T meets condition* (2).

Proof (i) $\Rightarrow$ (ii). Assume that for an operator $T \in \mathcal{P}_3$ condition (2) does not hold. Then there exists a number $\lambda > 0$ such that

$$\frac{1}{2}(\lambda^{-1}|T^2|^2 + \lambda\mathbf{1}) - |T^*|^2 = X - Y, \tag{3}$$

where $X, Y \in S(\mathcal{M}, \tau)^+$, $XY = 0$ and $Y \neq 0$. Let $Y = \int_0^{+\infty} t E^Y(\,\mathrm{d}t)$ be the spectral decomposition and $n \in \mathbb{N}$ be such that the projection

$$P = E^Y((n^{-1}, n)) \neq 0.$$

Then $PXP = 0$ and $PYP \geq n^{-1}P$.

Multiplying relation (3) by the projection P on both sides, leads us to

$$P|T^*|^2P = \frac{1}{2}(\lambda^{-1}P|T^2|^2P + \lambda P) + PYP \geq \frac{1}{2}(\lambda^{-1}P|T^2|^2P + (\lambda + 2n^{-1})P).$$

Since P is a unit in the reduced von Neumann algebra $\mathcal{M}_P$, we have

$$\|T^*P\|_\infty^2 = \|P|T^*|^2P\|_\infty \geq \frac{1}{2}\|\lambda^{-1}P|T^2|^2P + (\lambda + 2n^{-1})P\|_\infty$$
$$= \frac{1}{2}(\lambda^{-1}\|T^2P\|_\infty^2 + (\lambda + 2n^{-1})).$$

If $T^2P = 0$ then $\|T^*P\|_\infty^2 \geq \lambda 2^{-1} + n^{-1} > \|T^2P\|_\infty = 0$. If $T^2P \neq 0$ then by the inequality $a^2 + b^2 \geq 2|ab|$ for all $a, b \in \mathbb{R}$ we have

$$\|T^*P\|_\infty^2 \geq \frac{1}{2} \cdot 2\sqrt{\lambda^{-1}(\lambda + 2n^{-1})} \cdot \|T^2P\|_\infty > \|T^2P\|_\infty.$$

Thus, in both cases $T \notin \mathcal{P}_3$—a contradiction.

(ii)$\Rightarrow$(i). Consider an operator $A \in \mathcal{M}_1$ such that $T^*A \in \mathcal{M}$. Then $A^*A \leq \mathbf{1}$ and $|T^*|A \in \mathcal{M}$. If $T^2A \notin \mathcal{M}$ then the assertion is met. Let $T^2A \in \mathcal{M}$. Multiplying inequality (2) from the left-hand side by the operator A^* and from the right-hand side by the operator A, leads us to

$$A^*|T^*|^2A \leq \frac{1}{2}(\lambda^{-1}A^*|T^2|^2A + \lambda A^*A) \leq \frac{1}{2}(\lambda^{-1}A^*|T^2|^2A + \lambda\mathbf{1}) \quad \text{for all } \lambda > 0.$$

Therefore $\|A^*|T^*|^2A\|_\infty = \|T^*A\|_\infty^2 \leq \frac{1}{2}(\lambda^{-1}\|T^2A\|_\infty^2 + \lambda)$ for all $\lambda > 0$. Put here $\lambda = \|T^2A\|_\infty$ and obtain $\|T^*A\|_\infty^2 \leq \|T^2A\|_\infty$. □

Corollary 3.2 *The class $\mathcal{P}_3$ is closed in the measure topology t_τ.*

Proof Condition (2) is equivalent to the condition $T^{2*}T^2 - 2\lambda TT^* + \lambda^2\mathbf{1} \geq 0$ for all $\lambda > 0$. Hence t_τ-closedness of the class $\mathcal{P}_3$ follows from Theorem 3.1, t_τ-continuity of the involution, t_τ-continuity of the product operation on $S(\mathcal{M}, \tau)$ and t_τ-closedness of the cone $S(\mathcal{M}, \tau)^+$ in $S(\mathcal{M}, \tau)$. □

Corollary 3.3 *Consider operators $T \in \mathcal{P}_3$, $A \in S(\mathcal{M}, \tau)$ and numbers $k \in \mathbb{N}$, $0 < p, q, r < +\infty$ with $1/p + 1/q = 1/r$. Then*

(i) *if $T^2T^{*k}A, T^{*k}A \in \mathcal{M}$ then $(T^*)^{k+1}A \in \mathcal{M}$;*
(ii) *if $T^2T^{*k}A \in \mathcal{M}$, $T^{*k}A \in \mathcal{F}(\tau)$ or $T^2T^{*k}A \in \mathcal{F}(\tau)$, $T^{*k}A \in \mathcal{M}$ then $(T^*)^{k+1}A \in \mathcal{F}(\tau)$;*
(iii) *if $T^2T^{*k}A \in L_p(\mathcal{M}, \tau)$, $T^{*k}A \in L_q(\mathcal{M}, \tau)$ then $(T^*)^{k+1}A \in L_{2r}(\mathcal{M}, \tau)$.*

Proof A slight modification of the proof of [6, Corollary 3.1] leads to the goal. □

Corollary 3.4 *Every operator $T \in \mathcal{M} \cap \mathcal{P}_3$ is $*$-paranormal, hence it is normaloid. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ then the class $\mathcal{P}_3$ coincides with the class of all $*$-paranormal operators on $\mathcal{H}$ and is closed in $\|\cdot\|_\infty$-topology.*

Proof Every $*$-paranormal operator is normaloid [3, Theorem 1.1]. □

Remark 3.5 If an operator $T \in S(\mathcal{M}, \tau)$ is hyponormal or cohyponormal then $\mu(t; T^2) = \mu(t; T)^2$ for all $t > 0$ [12, Theorem 3.1] and $T \in \mathcal{P}_2$. If $T \in S(\mathcal{M}, \tau)$ is nilpotent of second order ($T \neq 0 = T^2$) then $T \notin \mathcal{P}_2$.

Theorem 3.6 (i) *If an operator $T \in S(\mathcal{M}, \tau)$ is hyponormal then $T \in \mathcal{P}_3$.*
(ii) *If an operator $T \in \mathcal{P}_3$ then $UTU^* \in \mathcal{P}_3$ for all isometries $U \in \mathcal{M}$.*

Proof (i) For a hyponormal operator $T \in S(\mathcal{M}, \tau)$ and every number $\lambda > 0$ by Lemma 2.3 we have

$$T^* \cdot T^*T \cdot T - 2\lambda TT^* + \lambda^2\mathbf{1} \geq T^* \cdot TT^* \cdot T - 2\lambda T^*T + \lambda^2\mathbf{1} = (T^*T - \lambda\mathbf{1})^2 \geq 0.$$

Applying Theorem 3.1 we conclude the proof.

(ii) Consider operators $T \in \mathcal{P}_3$ and $A \in \mathcal{M}_1$ such that $(UTU^*)^* \cdot A \in \mathcal{M}$ for an isometry $U \in \mathcal{M}$. If $(UTU^*)^2 \cdot A \notin \mathcal{M}$ or $U^*A = 0$ then the assertion is obvious. Let $(UTU^*)^2 \cdot A \in \mathcal{M}$ and $U^*A \neq 0$. Then $0 < \|U^*A\|_\infty \leq 1$ and

$$\begin{aligned} \|(UTU^*)^2 \cdot A\|_\infty &= \|UT^2U^* \cdot A\|_\infty \geq \|U^* \cdot UT^2U^* \cdot A\|_\infty = \|T^2U^*A\|_\infty \\ &= \left\| T^2 \frac{U^*A}{\|U^*A\|_\infty} \right\|_\infty \cdot \|U^*A\|_\infty \\ &\geq \left\| T \frac{U^*A}{\|U^*A\|_\infty} \right\|_\infty^2 \cdot \|U^*A\|_\infty = \frac{\|T \cdot U^*A\|_\infty^2}{\|U^*A\|_\infty} \\ &\geq \|T^* \cdot U^*A\|_\infty^2 \geq \|UT^*U^* \cdot A\|_\infty^2 = \|(UTU^*)^* \cdot A\|_\infty^2. \end{aligned}$$

□

Corollary 3.7 *If an operator $T \in S(\mathcal{M}, \tau)$ is quasinormal then $T \in \mathcal{P}_3$.*

Proof Every quasinormal operator $T \in S(\mathcal{M}, \tau)$ is hyponormal [11, Theorem 2.9]. □

Remark 3.8 If an operator $T \in S(\mathcal{M}, \tau)$ is quasinormal then T^n is also quasinormal [10, Proposition 2.10] and $\mu(t; T^n) = \mu(t; T)^n$ for all $t > 0$ and $n \in \mathbb{N}$ [10, Theorem 2.6].

Proposition 3.9 *Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$. Then $\mathcal{P}_3 \subset \mathcal{P}_2$. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ for separable and infinite dimensional $\mathcal{H}$ then $\mathcal{P}_3 \neq \mathcal{P}_2$.*

Proof Let $t > 0$ be fixed. From relation (1) for $X = T^2$ we have

$$\forall \varepsilon > 0 \;\; \exists P_\varepsilon \in \mathcal{P}(\mathcal{M}) \;\; (\tau(P_\varepsilon^\perp) \leq t, \;\; \varepsilon + \mu(t; T^2) > \|T^2 P_\varepsilon\|_\infty \geq \mu(t; T^2)),$$

thereby $\|T^* P_\varepsilon\|_\infty^2 \leq \varepsilon + \mu(t; T^2)$. Note that a projection P_ε is included in the right-hand side of (1) for $X = T^*$. Therefore $\mu(t; T) = \mu(t; T^*) \leq \|T^* P_\varepsilon\|_\infty$ and because of the arbitrariness of the number $\varepsilon > 0$ we get $\mu(t; T^2) \geq \mu(t; T)^2$. Thus $\mathcal{P}_3 \subset \mathcal{P}_2$.

For $T \in \widetilde{\mathcal{M}}$ we have $T \in \mathcal{P}_2 \Leftrightarrow T^* \in \mathcal{P}_2$ [6, Proposition 3.22]. Let $\{e_n\}_{n=0}^\infty$ be an orthonormal basis in $\mathcal{H}$. The unilateral shift $Te_n = e_{n+1} \quad (n = 0, 1, 2, \ldots)$ is a hyponormal operator (an isometry) and $T \in \mathcal{P}_3$ by item (i) of Theorem 3.5. The null-space $\mathrm{Ker} T^*$ is generated by vector e_0, and the null-space $\mathrm{Ker}(T^*)^2$ is generated by vectors e_0 and e_1. We have for the one-dimensional projection $A = \langle \cdot, e_1 \rangle e_1$ the relations

$$0 = \|(T^*)^2 A\|_\infty < \|(T^*)^* A\|_\infty^2 = \|T^* A\|_\infty^2 = 1$$

and $T^* \notin \mathcal{P}_3$. The assertion is proved. □

Now by Proposition 3.24 of [6] we have

Corollary 3.10 *For $T \in \mathcal{P}_3$ we have the equivalences:*

(i) $T \in \mathcal{M} \Leftrightarrow T^2 \in \mathcal{M}$;
(ii) $T \in \mathcal{F}(\tau) \Leftrightarrow T^2 \in \mathcal{F}(\tau)$;
(iii) $T \in S_0(\mathcal{M}, \tau) \Leftrightarrow T^2 \in S_0(\mathcal{M}, \tau)$;
(iv) $T \in L_{2p}(\mathcal{M}, \tau) \Leftrightarrow T^2 \in L_p(\mathcal{M}, \tau)$, $0 < p < +\infty$.

Proposition 3.11 *If a τ-measurable operator T belongs to $\mathcal{P}_k$ and $P \in \mathcal{P}(\mathcal{M})$ is such that $TP = PTP$ then the restriction $T|_{P\mathcal{H}}$ belongs to $\mathcal{P}_k$, $k = 1, 3$.*

Proof For $k = 1$, $P \in \mathcal{P}(\mathcal{M})$ and $A \in \mathcal{M}_1$ with $PA \neq 0$ we have $0 < \|PA\|_\infty \leq 1$ and

$$\begin{aligned} &\|(T|_{P\mathcal{H}})^2 A\|_\infty \\ &\quad = \|(PTP)^2 A\|_\infty = \|T^2 PA\|_\infty = \left\| T^2 \frac{PA}{\|PA\|_\infty} \right\|_\infty \cdot \|PA\|_\infty \geq \\ &\quad \geq \left\| T \frac{PA}{\|PA\|_\infty} \right\|_\infty^2 \cdot \|PA\|_\infty = \|T|_{P\mathcal{H}} A\|_\infty^2 \cdot \frac{1}{\|PA\|_\infty} \geq \|T|_{P\mathcal{H}} A\|_\infty^2. \end{aligned}$$

For $k = 3$ we apply the equality $(T|_{P\mathcal{H}})^* = PT^*P$. □

Proposition 3.12 *Let $T \in S(\mathcal{M}, \tau)$ and a unitary operator $S \in \mathcal{M}_h$ be so that $ST = TS$. Then $T \in \mathcal{P}_3 \Leftrightarrow ST \in \mathcal{P}_3$.*

Proof We have $S^2 = \mathbf{1}$ and $(ST)^2 = T^2$, $ST^* = T^*S$.

($\Rightarrow$) Let $A \in \mathcal{M}_1$ be so that $(ST)^* A \in \mathcal{M}$. Then

$$\begin{aligned} \|(ST)^2 A\|_\infty = \|T^2 A\|_\infty \geq \|T^* A\|_\infty^2 = \|A^* T T^* A\|_\infty = \|A^* T S^2 T^* A\|_\infty \\ = \|ST^* A\|_\infty^2 = \|T^* SA\|_\infty^2 = \|(ST)^* A\|_\infty^2. \end{aligned}$$

($\Leftarrow$) If $ST \in \mathcal{P}_3$ then by the above proved result $T = S \cdot ST \in \mathcal{P}_3$. □

4 Every p-hyponormal τ-measurable operator lies in $\mathcal{P}_1$

Lemma 4.1 *For all operators $Y \in S(\mathcal{M}, \tau)^+$, $X \in \mathcal{M}_1$ and $1 \leq r \leq 2$ we have $(X^* Y X)^r \leq X^* Y^r X$.*

Proof Let $Y = \int_0^{+\infty} t E^Y(\,\mathrm{d}t)$ be the spectral decomposition. Put $Y_n = \int_0^n t E^Y(\,\mathrm{d}t)$ for all $n \in \mathbb{N}$. Since the function $f(t) = t^r$ $(t \geq 0)$ is operator convex, we apply [22, Theorem 2.1] and obtain $(X^*Y_nX)^r \leq X^*Y_n^rX$ for all $n \in \mathbb{N}$. By t_τ-continuity of operator functions [33] and the product operation we have $(X^*Y_nX)^r \xrightarrow{\tau} (X^*YX)^r$ and $X^*Y_n^rX \xrightarrow{\tau} X^*Y^rX$ as $n \to \infty$. Finally, we apply the t_τ-closedness of the cone $S(\mathcal{M}, \tau)^+$ in $S(\mathcal{M}, \tau)$. □

Lemma 4.2 *For all operators $Y \in S(\mathcal{M}, \tau)^+$, $X \in \mathcal{M}_1$ and $t > 0$, $q \geq 1$ we have $\mu(t; X^*Y^qX) \geq \mu(t; X^*YX)^q$. In particular, we have $\|X^*Y^qX\|_\infty \geq \|X^*YX\|_\infty^q$.*

Proof Let $1 < q = p_1p_2\ldots p_k$ with some $1 < p_n \leq 2$, $n = 1, 2, \ldots, k$. By Lemma 4.1 and by items (3), (4) of Lemma 2.2 for all $t > 0$ we have

$$\begin{aligned}\mu(t; X^*Y^qX) &= \mu(t; X^*(Y^{q/p_1})^{p_1}X) \geq \mu(t; (X^*Y^{q/p_1}X)^{p_1}) \\ &= \mu(t; X^*Y^{q/p_1}X)^{p_1} = \mu(t; X^*(Y^{q/p_1p_2})^{p_2}X)^{p_1} \geq \cdots \\ &\geq \mu(t; X^*Y^{q/p_1p_2p_3\ldots p_k}X)^{p_1p_2p_3\ldots p_k} = \mu(t; X^*YX)^q.\end{aligned}$$

We apply item (7) of Lemma 2.2 and obtain $\|X^*Y^qX\|_\infty \geq \|X^*YX\|_\infty^q$. □

Lemma 4.3 *Let an operator $T \in S(\mathcal{M}, \tau)$ be p-hyponormal with $0 < p \leq 1$ and $T = U|T|$ be the polar decomposition of T. Then*

(i) *$U^*|T|^{1/2^n}U \geq |T|^{1/2^n} \geq U|T|^{1/2^n}U^*$ for some $n \in \mathbb{N}$;*
(ii) *the operator $T_p = U|T|^p$ is hyponormal.*

Proof (i) We have $|T^*| = U|T|U^*$ and $|T|^{2p} = (T^*T)^p \geq (TT^*)^p = |T^*|^{2p} = U|T|^{2p}U^*$. Let $n \in \mathbb{N}$ be such that $q = \dfrac{1}{p2^{n-1}} \in (0, 1)$. Then by Hansen's Theorem ([21]; [5, Lemma 3.1.1]), we have the relations

$$|T|^{1/2^n} = (|T|^{2p})^q \geq (U|T|^{2p}U^*)^q \geq U|T|^{2pq}U^* = U|T|^{1/2^n}U^*,$$

i.e., $|T|^{1/2^n} \geq U|T|^{1/2^n}U^*$. Multiplication of this relation from the left-hand side by the operator U^* and from the right-hand side by the operator U and Lemma 2.3 lead us to

$$U^*|T|^{1/2^n}U \geq U^*U|T|^{1/2^n}U^*U = |T|^{1/2^n}.$$

(ii) We have $U|T|^{2p}U^* = (U|T|^2U^*)^p \leq |T|^{2p} \leq U^*|T|^{2p}U$. □

The following statement strengthens item (i) of Theorem 3.6 [6] and is a generalization of Theorem 3 [28].

Theorem 4.4 *If an operator $T \in S(\mathcal{M}, \tau)$ is p-hyponormal with $0 < p \leq 1$ then $T \in \mathcal{P}_1$.*

Proof Let $T = U|T|$ be the polar decomposition of the p-hyponormal operator $T \in S(\mathcal{M}, \tau)$ with $0 < p \leq 1$ and $n \in \mathbb{N}$ be as in item (i) of Lemma 4.3. For $A \in \mathcal{M}_1$ with $TA \in \mathcal{M}\setminus\{0\}$ we have

$$\|T^2A\|_\infty^2 = \|A^*T^{*2}T^2A\|_\infty = \|A^*T^* \cdot |T|^2 \cdot TA\|_\infty$$
$$= \left\| \frac{A^*T^*}{\|A^*T^*\|_\infty} \cdot (|T|^{1/2^n})^{2^{n+1}} \cdot \frac{TA}{\|TA\|_\infty} \right\|_\infty \cdot \|TA\|_\infty^2.$$

Then by Lemma 4.2 we obtain

$$\|T^2A\|_\infty^2 \geq \left\| \frac{A^*T^*}{\|A^*T^*\|_\infty} \cdot |T|^{1/2^n} \cdot \frac{TA}{\|TA\|_\infty} \right\|_\infty^{2^{n+1}} \cdot \|TA\|_\infty^2$$
$$= \frac{\|A^*T^* \cdot |T|^{1/2^n} \cdot TA\|_\infty^{2^{n+1}} \cdot \|TA\|_\infty^2}{\|TA\|_\infty^{2^{2n+2}}}$$
$$= \frac{\|A^*|T|U^* \cdot |T|^{1/2^n} \cdot U|T|A\|_\infty^{2^{n+1}} \cdot \|TA\|_\infty^2}{\|TA\|_\infty^{2^{2n+2}}}.$$

Therefore by item (i) of Lemmas 4.3 and 2.3 we have

$$\|T^2A\|_\infty^2 \geq \frac{\|A^*|T| \cdot |T|^{1/2^n} \cdot |T|A\|_\infty^{2^{n+1}} \cdot \|TA\|_\infty^2}{\|TA\|_\infty^{2^{2n+2}}}$$
$$= \frac{\|A^* \cdot |T|^{1/2^n+2} \cdot A\|_\infty^{2^{n+1}} \cdot \|TA\|_\infty^2}{\|TA\|_\infty^{2^{2n+2}}}$$
$$= \frac{\|A^* \cdot (|T|^2)^{1/2^{n+1}+1} \cdot A\|_\infty^{2^{n+1}} \cdot \|TA\|_\infty^2}{\|TA\|_\infty^{2^{2n+2}}}$$

Thus by Lemma 4.2 we obtain

$$\|T^2A\|_\infty^2 \geq \frac{\|A^* \cdot |T|^2 \cdot A\|_\infty^{2^{n+1}+1} \cdot \|TA\|_\infty^2}{\|TA\|_\infty^{2^{2n+2}}}$$
$$= \frac{\||T| \cdot A\|^{2^{2n+2}+2} \cdot \|TA\|_\infty^2}{\|TA\|_\infty^{2^{2n+2}}}$$
$$\geq \frac{\|U|T| \cdot A\|_\infty^{2^{2n+2}+2} \cdot \|TA\|_\infty^2}{\|TA\|_\infty^{2^{2n+2}}} = \|TA\|_\infty^4$$

and Theorem 4.4 is proved. □

Corollary 4.5 *If an operator $T \in S(\mathcal{M}, \tau)$ is p-hyponormal with $0 < p \leq 1$ then $\mu(t; T^2) \geq \mu(t; T)^2$ for all $t > 0$.*

Proof We have $\mathcal{P}_1 \subset \mathcal{P}_2$ by Proposition 3.5 of [6]. □

Corollary 4.6 *If an operator $T \in \mathcal{M}$ is p-hyponormal with $0 < p \leq 1$ then $\mu(t; T^n) \geq \mu(t; T)^n$ for all $t > 0$ and $n \in \mathbb{N}$.*

Proof We apply Theorem 3.16 of [6]. □

5 On *p*-hyponormal *τ*-measurable operators

Let a semifinite von Neumann algebra $\mathcal{M}$ be equipped with a faithful normal semifinite trace τ.

Theorem 5.1 *Let $A, B \in S(\mathcal{M}, \tau)$ and A be p-hyponormal with $0 < p \leq 1$.*

(i) *If $AB \in S_0(\mathcal{M}, \tau)$ then $A^*B \in S_0(\mathcal{M}, \tau)$.*
(ii) *If $A, B \in \mathcal{M}$ and $AB \in \mathcal{F}(\tau)$ then $A^*B \in \mathcal{F}(\tau)$.*
(iii) *If $A, B \in \mathcal{M}$ and $AB \in L_q(\mathcal{M}, \tau)$ then $A^*B \in L_{q/p}(\mathcal{M}, \tau)$.*

Proof (i) Let $A^* = U|A^*|$ be the polar decomposition of an operator A^*. Every operator $B \in S(\mathcal{M}, \tau)$ can be represented as a sum $B = S + T$ with $S \in \mathcal{M}$ and $T \in S_0(\mathcal{M}, \tau)$, see [32]. Hence we may assume that $B \in \mathcal{M}$. By items (1), (2), (3), (4) and (6) of Lemma 2.2 and by the Hansen's inequality ([21]; [5, Lemma 3.1.1]) for $B_1 = B/\|B\|_\infty$ for all $t > 0$ we have

$$\begin{aligned} \mu(t; AB)^2 &= \mu(t; B^*A^*)^2 = \mu(t; B^*A^*AB) = \|B\|_\infty^2 \mu(t; B_1^*A^*AB_1) \\ &= \|B\|_\infty^2 \mu(t; (B_1^*A^*AB_1)^p)^{1/p} \geq \|B\|_\infty^2 \mu(t; B_1^*(A^*A)^p B_1)^{1/p} \\ &\geq \|B\|_\infty^2 \mu(t; B_1^*(AA^*)^p B_1)^{1/p} = \|B\|_\infty^2 \mu(t; |A^*|^p B_1)^{2/p}. \end{aligned} \tag{3}$$

Therefore $|A^*|^p B \in S_0(\mathcal{M}, \tau)$ and $A^*B = U|A^*|^{1-p} \cdot |A^*|^p B \in S_0(\mathcal{M}, \tau)$.

(ii) We apply (3) and conclude that $|A^*|^p B \in \mathcal{F}(\tau)$. Thus $A^*B = U|A^*|^{1-p} \cdot |A^*|^p B \in \mathcal{F}(\tau)$.

(iii) For $q > 0$ by (3) we have $|A^*|^p B \in L_{q/p}(\mathcal{M}, \tau)$. Thus $A^*B = U|A^*|^{1-p} \cdot |A^*|^p B \in L_{q/p}(\mathcal{M}, \tau)$. Moreover, for all $t > 0$ and for $C = \|B\|_\infty^q \cdot \|U|A^*|^{1-p}\|_\infty^{-q/p}$ by (3) and items (2), (3) and (4) of Lemma 2.2 we have

$$\begin{aligned} \mu(t; AB)^q &\geq \|B\|^q \mu(t; |A^*|^p B_1)^{q/p} = C\|U|A^*|^{1-p}\|_\infty^{q/p} \mu(t; |A^*|^p B_1)^{q/p} \\ &\geq C\mu(t; U|A^*|^{1-p} \cdot |A^*|^p B_1)^{q/p} = C\mu(t; A^*B_1)^{q/p} \\ &= C\|B\|_\infty^{-q/p} \mu(t; A^*B)^{q/p}. \end{aligned} \tag{4}$$

Theorem is proved. □

Corollary 5.2 *Let $A, B \in \mathcal{M}$ and A, B^* be p-hyponormal with $1/2 < p \leq 1$.*

(i) *If $AB \in S_0(\mathcal{M}, \tau)$ then $BA \in S_0(\mathcal{M}, \tau)$.*
(ii) *If $AB \in \mathcal{F}(\tau)$ then $BA \in \mathcal{F}(\tau)$.*
(iii) *If $AB \in L_q(\mathcal{M}, \tau)$ then $BA \in L_{2q/p}(\mathcal{M}, \tau)$.*

Proof (i), (ii). Dividing suitably if need be, we may assume that $A, B \in \{X \in \mathcal{M} : \|X\|_\infty \le 1\}$. Also, by Löwner's inequality, both A and B^* are $\frac{1}{2}$-hyponormal. Hence, by Hansen's inequality [21] we conclude that

$$A^*|B|^2A \le A^*|B|A \le A^*|B^*|A = A^*(|B^*|^2)^{1/2}A \le (A^*|B^*|^2A)^{1/2}. \tag{5}$$

Then we apply items (1), (3) and (4) of Lemma 2.2 and Theorem 5.1.

(iii) We have $A^*B \in L_{q/p}(\mathcal{M}, \tau)$ by Theorem 5.1. Hence $B^*A = (A^*B)^* \in L_{q/p}(\mathcal{M}, \tau)$. Inequality (5) yields $|BA|^2 \le |B^*A|$ and we apply items (1), (3) and (4) of Lemma 2.2. Moreover, for all $t > 0$ and for $C = \|B\|_\infty^q \cdot \|U|A^*|^{1-p}\|_\infty^{-q/p}$ by (4), (5) and items (1), (2), (3) and (4) of Lemma 2.2 we have

$$\begin{aligned}\mu(t; AB)^q &\ge C\|B\|_\infty^{-q/p}\mu(t; A^*B)^{q/p} = C\|B\|_\infty^{-q/p}\mu(t; B^*A)^{q/p}\\ &\ge C\|B\|_\infty^{-q/p}\mu(t; BA)^{2q/p}.\end{aligned}$$

The assertion is proved. □

Theorem 5.3 (cf [1, Theorem 1]) *Let $T \in S(\mathcal{M}, \tau)$ be p-hyponormal with $\frac{1}{2} \le p < 1$ and $T = U|T|$ be the polar decomposition of T. Then the operator $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is hyponormal.*

Proof We have $|T|^{2p} \ge (U|T|^2U^*)^p$. By operator monotonicity of the function $t \mapsto t^{\frac{1}{2p}}$ $(t \ge 0)$ and by Hansen's inequality ([21]; [5, Lemma 3.1.1]) we obtain

$$|T| \ge (U|T|^2U^*)^{p\cdot\frac{1}{2p}} = (U|T|^2U^*)^{\frac{1}{2}} \ge U|T|U^*.$$

Thus by Lemma 2.3 we have

$$\begin{aligned}\widetilde{T}^*\widetilde{T} &= |T|^{\frac{1}{2}}U^*\cdot|T|\cdot U|T|^{\frac{1}{2}} \ge |T|^{\frac{1}{2}}U^*\cdot U|T|U^*\cdot U|T|^{\frac{1}{2}} = |T|^2 = |T|^{\frac{1}{2}}\cdot|T|\cdot|T|^{\frac{1}{2}}\\ &\ge |T|^{\frac{1}{2}}\cdot U|T|U^*\cdot|T|^{\frac{1}{2}} = \widetilde{T}\widetilde{T}^*.\end{aligned}$$

Theorem is proved. □

Lemma 5.4 *Let $A, B \in \widetilde{\mathcal{M}}^+$ with $A \ge B$. Then for each $r > 0$,*

$$(B^rA^pB^r)^{\frac{1}{q}} \ge B^{(p+2r)/q} \tag{6}$$

and

$$A^{(p+2r)/q} \ge (A^rB^pA^r)^{\frac{1}{q}} \tag{7}$$

hold for each p and q such that $p \ge 0$, $q \ge 1$, and $(1 + 2r)q \ge p + 2r$.

Proof Let $A = \int_0^{+\infty} \lambda E^A(d\lambda)$ and $B = \int_0^{+\infty} \lambda E^B(d\lambda)$ be the spectral decompositions of A and B. Put

$$A_n = \int_0^n \lambda E^A(d\lambda) \quad \text{and} \quad B_n = \int_0^n \lambda E^B(d\lambda)$$

for all $n \in \mathbb{N}$. Then A_n, B_n belong to $\mathcal{M}^+$ and meet inequalities (6) and (7) for all $n \in \mathbb{N}$ by [18]. We have $A_n \xrightarrow{\tau} A$, $B_n \xrightarrow{\tau} B$ as $n \to \infty$ and apply t_τ-continuity of real continuous functions [33] and t_τ-continuity of the product operation. Inequalities (6), (7) follow by t_τ-closedness of the cone $\widetilde{\mathcal{M}}^+$ in $\widetilde{\mathcal{M}}$. ◻

Theorem 5.5 (cf [1, Theorem 2]) *Let $T \in S(\mathcal{M}, \tau)$ be p-hyponormal with $0 < p < \frac{1}{2}$ and $T = U|T|$ be the polar decomposition of T. Then the operator $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ is $(p + \frac{1}{2})$-hyponormal.*

Proof We apply Lemma 5.4 and repeat the proof of [1, Theorem 2]. Theorem is proved. ◻

Corollary 5.6 *If an operator $T \in S(\mathcal{M}, \tau)$ is p-hyponormal with $0 < p < \frac{1}{2}$ and has the polar decomposition $T = U|T|$, then the τ-measurable operator $|\widetilde{T}|^{\frac{1}{2}} \widetilde{U} |\widetilde{T}|^{\frac{1}{2}}$ is hyponormal, where $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ and $\widetilde{T} = \widetilde{U}|\widetilde{T}|$ is the polar decomposition of $\widetilde{T}$.*

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