J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

When weak and local measure convergence implies norm convergence

A. Bikchentaev [a], F. Sukochev [b]

[a] Kazan Federal University, 18 Kremlyovskaya str., Kazan, 420008, Russia
[b] School of Mathematics and Statistics, University of New South Wales, Kensington, 2052, NSW, Australia

ARTICLE INFO

Article history:
Received 25 October 2018
Available online xxxx
Submitted by D. Blecher

Keywords:
Hilbert space
von Neumann algebra
Non-commutative integration
Measurable operator
Measure topology
Operator inequality

ABSTRACT

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$ and τ be a faithful normal semifinite trace on $\mathcal{M}$. Let t_τ be the measure topology on the $*$-algebra $S(\mathcal{M},\tau)$ of all τ-measurable operators. We prove that for $B \in S(\mathcal{M},\tau)^+$ the sets $I_B = \{A \in S(\mathcal{M},\tau)_h : -B \le A \le B\}$ and $K_B = \{A \in S(\mathcal{M},\tau) : A^*A \le B\}$ are convex and t_τ-closed in $S(\mathcal{M},\tau)$. In this case, we have $I_B = \{\sqrt{B}T\sqrt{B} : T \in \mathcal{M}_h \text{ and } \|T\| \le 1\}$ and, for invertible B, we describe the set of extreme points of the set I_B. Let $\mathcal{M}$ be an atomic von Neumann algebra. We prove that an operator $B \in S(\mathcal{M},\tau)^+$ is τ-compact if and only if the set I_B is t_τ-compact. The t_τ-compactness of I_B for all τ-compact operators B characterizes these algebras.

1. Introduction

A fundamental result in the theory of Lebesgue L_1-spaces on σ-finite measure spaces states that a combination of local measure convergence and weak convergence yields norm convergence, see [54, Theorem V.5, p. 122] and [26, Theorem IV.8.12, p. 295]. G. Ya. Lozanovskii (see Problems 654 and 1123 in [40]) suggested to thoroughly examine this property in the setting of Dedekind complete Banach lattices or KB-spaces. In this paper we shall discuss an analogue of this property in the setting of symmetric spaces of measurable operators (see e.g. [21], [36], [19], [25]). It should be stated from the outset that a direct noncommutative analogue of this property fails spectacularly already in the most familiar noncommutative L_1-space, that is, in the trace ideal $C_1(\mathcal{H})$ of compact operators on an infinite dimensional Hilbert space $\mathcal{H}$. Indeed, in this setting, local convergence in the measure (which seems to have been introduced firstly in [21]) reduces to convergence in the familiar weak operator topology (see [23, p. 482]). Assume, for simplicity, that $\mathcal{H}$ is separable and fix an orthonormal basis $\{e_k\}_{k=1}^\infty$ in $\mathcal{H}$. Consider those operators $\{x_{jk}\}_{j,k=1}^\infty$ whose matrix representation with respect to a basis $\{e_k\}_{k=1}^\infty$ contains a single non-zero (j,k)-th entry, namely 1. It is a

E-mail addresses: Airat.Bikchentaev@kpfu.ru (A. Bikchentaev), f.sukochev@unsw.edu.au (F. Sukochev).

https://doi.org/10.1016/j.jmaa.2019.01.028
0022-247X/© 2019 Elsevier Inc. All rights reserved.

fundamental and easily verifiable fact that the sequence $\{x_{1k}\}_{k=1}^{\infty}$ is equivalent to a standard orthonormal basis in the space ℓ_2 and thus converges weakly to 0 in $C_1(\mathcal{H})$. The fact that it also converges to 0 in the weak operator topology is immediate whereas the trace norm of every element of this sequence is 1. Thus, there are two possible avenues to investigate the aforementioned Lozanovskii's problem in the setting of trace ideals of compact operators: either to replace the local convergence in measure with bona fide convergence in measure, or else to identify those subsets of the trace ideals (or more generally, noncommutative symmetric spaces) which still satisfy the original setting of the problem for local measure convergence. This paper belongs to the latter line of thought, however, prior to explaining our main results, we give a short update on the former direction, which has been developed into the study of symmetric function and symmetric operator spaces with the property that norm convergence of sequences is equivalent to weak convergence plus convergence for the measure topology. This study was initiated in [37,38], where the term (wm)-property was coined. In particular, [15, Corollary 1.4] asserts that every Lorentz space Λ_ϕ has the (wm)-property. Orlicz spaces on the interval $[0, 1]$ with property (wm) have been fully characterized in [3]. Finally, in [17, Proposition 6.10] it is shown that, in symmetric function spaces on measure spaces with finite measure possessing the property (wm), each relatively weakly compact subset is of uniformly absolutely continuous norm. The latter result does not hold when the measure space is equipped with an infinite measure. Furthermore, the just cited results hold also in a much greater generality when symmetric function spaces are replaced with their noncommutative counterparts [17].

We now briefly explain our main results in this article, which basically establish that, in the setting of quasi-Banach ideals of compact operators on $\mathcal{H}$, on every operator interval weak operator convergence and convergence with respect to the quasi-norm coincide.

The so-called intervals of linear bounded operators on Hilbert space $\mathcal{H}$ arise in the study of the range of Stieltjes transform over all operator-valued measures which generate a given Stieltjes Hermitian moment sequence [42]. Let $S(\mathcal{M}, \tau)$ be the $*$-algebra of all τ-measurable operators (see the following section for all unexplained notations). One of the main objects of the present paper is the operator interval [45] (such intervals were investigated also in [6,7,12,8,9] and [14])

$$I_B = \{A = A^* \in S(\mathcal{M}, \tau) : -B \le A \le B\}, \ 0 \le B \in S(\mathcal{M}, \tau),$$

which is an important component in noncommutative integration theory. The set of extreme points of operator intervals was studied in [43], [29]. In particular, the main result of [43] shows that for $B \in B(\mathcal{H})$, the interval $I_B := \{A \in B(\mathcal{H}) : -B \le A \le B\}$ is the closure of the convex hull of its extreme points in the weak operator topology. In the present paper, we characterize the extreme points of I_B, $B \in S(\mathcal{M}, \tau)^+$ (see Theorem 3.9). In addition, if $\mathcal{M}$ is atomic (with every atom having the same trace), then I_B ($0 \le B \in S(\mathcal{M}, \tau)$ is τ-compact) is the closure of the convex hull of its extreme points in the measure topology (Corollary 4.7), which generalizes the main result in [43].

In section 4, we characterize the compactness of operator intervals. In particular, if $\mathcal{M}$ is atomic, then an operator $B \in S(\mathcal{M}, \tau)^+$ is τ-compact if and only if I_B is compact in the measure topology (Corollary 4.6). As an application, we show that, if $\mathcal{M}$ is atomic and $E(\mathcal{M}, \tau)$ is a quasi-(or even Δ-)normed operator space, then the compactness of the operator interval I_B, $B \in E^{oc}(\mathcal{M}, \tau)$, in the (local) measure topology coincides with that in the quasi-(or Δ-)norm topology (Corollaries 4.9 and 4.11), where $E^{oc}(\mathcal{M}, \tau)$ stands for the set of all elements of order continuous quasi-(or Δ-)norm in $E(\mathcal{M}, \tau)$.

Our final result, Theorem 4.12, presents a wide class of subsets in every symmetrically normed operator space with order continuous norm (in particular, in the trace ideal $C_1(\mathcal{H})$) in which local convergence in measure implies norm convergence, thus providing a noncommutative analogue for [54, Theorem V.5, p. 122] and [26, Theorem IV.8.12, p. 295]. Some of these results without proofs were announced in the brief note [11].

The authors sincerely thank Jinghao Huang and Dmitriy Zanin for detailed discussions of the results and proofs presented in this paper.

2. Notation, definitions and preliminaries

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$, $\mathcal{U}(\mathcal{M})$ be the unitary part of $\mathcal{M}$. Let $\mathcal{P}(\mathcal{M})$ be the lattice of projections in $\mathcal{M}$, $\mathbf{1}$ be the unit of $\mathcal{M}$, and let $P^\perp = \mathbf{1} - P$ for $P \in \mathcal{P}(\mathcal{M})$. Also $\mathcal{M}^+$ denotes the cone of positive elements in $\mathcal{M}$, and $\|\cdot\|_\infty$ denotes the uniform norm on $\mathcal{M}$. A mapping $\varphi : \mathcal{M}^+ \to [0, +\infty]$ is called *a trace*, if $\varphi(X + Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$, $\lambda \geq 0$ (moreover, $0 \cdot (+\infty) \equiv 0$); $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$. A trace φ is called *faithful*, if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$; *normal*, if $X_i \uparrow X$ $(X_i, X \in \mathcal{M}^+) \Rightarrow \varphi(X) = \sup \varphi(X_i)$; *semifinite*, if $\varphi(X) = \sup\{\varphi(Y) : Y \in \mathcal{M}^+, Y \leq X, \varphi(Y) < +\infty\}$ for every $X \in \mathcal{M}^+$.

A linear operator $X : \mathfrak{D}(X) \to \mathcal{H}$, where the domain $\mathfrak{D}(X)$ of X is a linear subspace of $\mathcal{H}$, is said to be *affiliated* with $\mathcal{M}$ if $YX \subseteq XY$ for all $Y \in \mathcal{M}'$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$. A linear operator $X : \mathfrak{D}(X) \to \mathcal{H}$ is termed *measurable* with respect to $\mathcal{M}$ if X is closed, densely defined, affiliated with $\mathcal{M}$ and there exists a sequence $\{P_n\}_{n=1}^\infty$ in the logic of all projections of $\mathcal{M}$, $\mathcal{P}(\mathcal{M})$, such that $P_n \uparrow \mathbf{1}$, $P_n(\mathcal{H}) \subseteq \mathfrak{D}(X)$ and $P_n^\perp$ is a finite projection (with respect to $\mathcal{M}$) for all n. It should be noted that the condition $P_n(\mathcal{H}) \subseteq \mathfrak{D}(X)$ implies that $XP_n \in \mathcal{M}$. The collection of all measurable operators with respect to $\mathcal{M}$ is denoted by $S(\mathcal{M})$, which is a unital $*$-algebra with respect to strong sums and products (denoted simply by $X + Y$ and XY for all $X, Y \in S(\mathcal{M})$) [47,44].

Let X be a self-adjoint operator affiliated with $\mathcal{M}$. We denote its spectral measure by $\{E^X\}$. It is well known that if X is a closed operator affiliated with $\mathcal{M}$ with the polar decomposition $X = U|X|$, then $U \in \mathcal{M}$ and $E \in \mathcal{M}$ for all projections $E \in \{E^{|X|}\}$. Moreover, $X \in S(\mathcal{M})$ if and only if X is closed, densely defined, affiliated with $\mathcal{M}$ and $E^{|X|}(\lambda, \infty)$ is a finite projection for some $\lambda > 0$. It follows immediately that in the case when $\mathcal{M}$ is a von Neumann algebra of type III or a type I factor, we have $S(\mathcal{M}) = \mathcal{M}$. For type II von Neumann algebras, this is no longer true. From now on, let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ.

For any closed and densely defined linear operator $X : \mathfrak{D}(X) \to \mathcal{H}$, the *null projection* $n(X) = n(|X|)$ is the projection onto its kernel $\mathrm{Ker}(X)$, the *range projection* $r(X)$ is the projection onto the closure of its range $\mathrm{Ran}(X)$ and the *support projection* $\mathrm{supp}(X)$ of X is defined by $\mathrm{supp}(X) = \mathbf{1} - n(X)$.

An operator $X \in S(\mathcal{M})$ is called τ-measurable if there exists a sequence $\{P_n\}_{n=1}^\infty$ in $\mathcal{P}(\mathcal{M})$ such that $P_n \uparrow \mathbf{1}$, $P_n(\mathcal{H}) \subseteq \mathfrak{D}(X)$ and $\tau(P_n^\perp) < \infty$ for all n. The collection $S(\mathcal{M}, \tau)$ of all τ-measurable operators is a unital $*$-subalgebra of $S(\mathcal{M})$ denoted by $S(\mathcal{M}, \tau)$. It is well known that a linear operator X belongs to $S(\mathcal{M}, \tau)$ if and only if $X \in S(\mathcal{M})$ and there exists $\lambda > 0$ such that $\tau(E^{|X|}(\lambda, \infty)) < \infty$. Alternatively, an unbounded operator X affiliated with $\mathcal{M}$ is τ-measurable (see [27]) if and only if

$$\tau\left(E^{|X|}(n, \infty)\right) \to 0, \quad n \to \infty.$$

For any $X = X^* \in S(\mathcal{M}, \tau)$, we set $X_+ = XE^X[0, \infty)$ and $X_- = XE^X(-\infty, 0]$; see [25], remarks following Theorem II.2.16.

Let $\mathcal{L}^+$ and $\mathcal{L}_h$ denote the positive and Hermitian parts of a family $\mathcal{L} \subset S(\mathcal{M}, \tau)$, respectively. We denote by $\leq$ the partial order in $S(\mathcal{M}, \tau)_h$ generated by its proper cone $S(\mathcal{M}, \tau)^+$. If $X \in S(\mathcal{M}, \tau)$, then $|X| = \sqrt{X^*X} \in S(\mathcal{M}, \tau)^+$.

Definition 2.1. Let a semifinite von Neumann algebra $\mathcal{M}$ be equipped with a faithful normal semi-finite trace τ and let $X \in S(\mathcal{M}, \tau)$. The generalized singular value function $\mu(X) : t \to \mu(t; X)$ of the operator X is defined by setting

$$\mu(s; X) = \inf\{\|XP\|_\infty : P = P^* \in \mathcal{M} \text{ is a projection}, \tau(P^\perp) \leq s\}.$$

An equivalent definition in terms of the distribution function of the operator X is the following. For every self-adjoint operator $X \in S(\mathcal{M}, \tau)$, setting

$$d_X(t) = \tau(E^X(t, \infty)), \quad t > 0,$$

we have (see e.g. [27] and [39])

$$\mu(t; X) = \inf\{s \geq 0 : \ d_{|X|}(s) \leq t\}.$$

Note that $d_X(\cdot)$ is a right-continuous function (see e.g. [27]).

For convenience of the reader, we also recall the definition of the *measure topology* t_τ on the algebra $S(\mathcal{M}, \tau)$. For every $\varepsilon, \delta > 0$, we define the set

$$V(\varepsilon, \delta) = \{X \in S(\mathcal{M}, \tau) : \ \exists P \in \mathcal{P}(\mathcal{M}) \text{ such that } \|XP\|_\infty \leq \varepsilon, \ \tau(P^\perp) \leq \delta\}.$$

The topology generated by the sets $V(\varepsilon, \delta)$, $\varepsilon, \delta > 0$, is called the measure topology t_τ on $S(\mathcal{M}, \tau)$ [27,44]. It is well-known that the algebra $S(\mathcal{M}, \tau)$ equipped with the measure topology is a complete metrizable topological algebra [44]. We note that a sequence $\{X_n\}_{n=1}^\infty \subset S(\mathcal{M}, \tau)$ converges to zero with respect to measure topology t_τ (i.e. $X_n \xrightarrow{\tau} 0$) if and only if $\tau\big(E^{|X_n|}(\varepsilon, \infty)\big) \to 0$ as $n \to \infty$ for all $\varepsilon > 0$.

If $\varepsilon, \delta > 0$ and if P is a projection in $\mathcal{M}$ with $\tau(P) < \infty$, then the family of all sets $N_{\varepsilon,\delta,P}$ consisting of all $X \in S(\mathcal{M}, \tau)$ such that $\mu_\delta(PXP) < \varepsilon$ form a neighborhood base at 0 for a Hausdorff linear topology on $S(\mathcal{M}, \tau)$. This topology (cf. [21, p. 746]) will be called the topology of local convergence in measure. Convergence with the respect to the topology of local convergence in measure coincides with convergence for the measure topology relative to $(P\mathcal{M}P, \tau(P \cdot P))$, for each projection $P \in \mathcal{M}$ with $\tau(P) < \infty$ [22, p. 492].

Remark 2.2. We warn the reader that in [53, Definition 3.1], Yeadon introduces the *topology of convergence locally in measure* in the algebra $LS(\mathcal{M})$ of all locally measurable operators affiliated with a general von Neumann algebra $\mathcal{M}$. This is an unfortunate clash of terminology.

The space $S_0(\mathcal{M}, \tau)$ of τ-compact operators is the space associated to the algebra of functions from $S(0, \infty)$ vanishing at infinity, that is,

$$S_0(\mathcal{M}, \tau) = \{x \in S(\mathcal{M}, \tau) : \ \mu(\infty; X) = 0\}.$$

The two-sided ideal $\mathcal{F}(\tau)$ in $\mathcal{M}$ consisting of all elements of τ-finite range is defined by

$$\mathcal{F}(\tau) = \{X \in \mathcal{M} \ : \ \tau(r(X)) < \infty\} = \{X \in \mathcal{M} \ : \ \tau(s(X)) < \infty\}.$$

Equivalently, $\mathcal{F}(\tau) = \{X \in \mathcal{M} : \mu(t; X) = 0 \text{ for some } t > 0\}$. Clearly, $S_0(\mathcal{M}, \tau)$ is the closure of $\mathcal{F}(\tau)$ with respect to the measure topology [19], which is a two-sided ideal in $S(\mathcal{M}, \tau)$.

Let m be Lebesgue measure on $\mathbb{R}$. The noncommutative L_p-Lebesgue space $(0 < p < \infty)$ affiliated with $(\mathcal{M}, \tau)$ is defined as

$$L_p(\mathcal{M}, \tau) = \{X \in S(\mathcal{M}, \tau) : \ \mu(X) \in L_p(\mathbb{R}^+, m)\}$$

with the quasi-norm $\|X\|_p = \|\mu(X)\|_p$, $X \in L_p(\mathcal{M}, \tau)$. In particular, $\|\cdot\|_p$ is a norm when $1 \leq p < \infty$. We have $\mathcal{F}(\tau) \subset L_p(\mathcal{M}, \tau) \subset S_0(\mathcal{M}, \tau)$ for all $0 < p < +\infty$.

Lemma 2.3. *[27] Let X, $Y \in S(\mathcal{M}, \tau)$ and $U \in \mathcal{U}(\mathcal{M})$. Then,*

(i) $\mu(t; X) = \mu(t; |X|) = \mu(t; X^*) = \mu(t; UXU^*)$ *for all* $t > 0$*;*
(ii) *if* $|X| \le |Y|$*, then* $\mu(t; X) \le \mu(t; Y)$ *for all* $t > 0$*;*
(iii) $\mu(t; AXB) \le \|A\|_\infty \|B\|_\infty \mu(t; X)$ *for all* $A, B \in \mathcal{M}$ *and* $t > 0$*;*
(iv) $\mu(s + t; X + Y) \le \mu(s; X) + \mu(t; Y)$ *for all* $s, t > 0$*;*
(v) $\mu(s + t; XY) \le \mu(s; X)\mu(t; Y)$ *for all* $s, t > 0$*;*
(vi) $\mu(t; f(|X|)) = f(\mu(t; X))$ *for all continuous increasing functions* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $f(0) = 0$ *and* $t > 0$.

Lemma 2.4 *([53, p. 261], [19, Proposition 1]). If* $A, B \in S(\mathcal{M}, \tau)^+$ *and* $A \le B$*, then there exists an operator* $Z \in \mathcal{M}$ *with* $\|Z\|_\infty \le 1$ *such that* $\sqrt{A} = Z\sqrt{B}$ *and* $A = ZBZ^*$.

A linear subspace $\mathcal{E}$ in $S(\mathcal{M}, \tau)$ is called an ideal (or, solid) space on $(\mathcal{M}, \tau)$ if (1) $X \in \mathcal{E}$ implies that $X^* \in \mathcal{E}$; (2) $X \in \mathcal{E}$, $Y \in S(\mathcal{M}, \tau)$ and $|Y| \le |X|$ imply that $Y \in \mathcal{E}$ [10]. The algebra $\mathcal{M}$, the set $\mathcal{F}(\tau)$, $S_0(\mathcal{M}, \tau)$, $(L_1 + L_\infty)(\mathcal{M}, \tau)$, and $L_p(\mathcal{M}, \tau)$ for $0 < p < +\infty$ are examples of such solid spaces.

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, i.e. the $*$-algebra of all linear bounded operators on $\mathcal{H}$, and $\tau = \mathrm{tr}$ is the canonical trace then $S(\mathcal{M}, \tau)$ coincides with $\mathcal{B}(\mathcal{H})$. In this case the measure topology coincides with the $\|\cdot\|_\infty$-topology, the topology of local convergence in measure coincides with the weak operator topology [23, p. 460], $S_0(\mathcal{M}, \tau)$ is the compact operator ideal on $\mathcal{H}$, $\mathcal{F}(\tau)$ is the finite-dimensional operator ideal on $\mathcal{H}$ and

$$\mu(t; X) = \sum_{n=1}^{\infty} s_n(X)\chi_{[n-1,n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^\infty$ is the sequence of s-numbers of an operator X [28, Chap. 1]; here χ_A is the indicator function of a set $A \subset \mathbb{R}$. In this case, the space $L_p(\mathcal{M}, \tau)$ is a Schatten–von Neumann ideal $C_p(\mathcal{H})$, $0 < p < +\infty$.

The following result is well known (see e.g. [46] and [25, Corollary I.2.28]).

Lemma 2.5. *The function* $f(t) = \sqrt{t}$ $(t \ge 0)$ *is operator monotone, that is* $f(A) \ge f(B)$ *whenever* τ*-measurable operators* A *and* B *such that* $A \ge B \ge 0$.

3. Convex sets of τ-measurable operators

Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$. For every $B \in S(\mathcal{M}, \tau)^+$, we consider the following operator intervals

$$K_B := \{A \in S(\mathcal{M}, \tau) : \ A^*A \le B\},$$
$$M_B := \{A \in S(\mathcal{M}, \tau) : \ |A| \le B\},$$

and

$$I_B := \{A \in S_h(\mathcal{M}, \tau) : \ -B \le A \le B\}.$$

Theorem 3.1. *If an operator* B *belongs to* $S(\mathcal{M}, \tau)^+$*, then*

(i) $K_B \subseteq M_{\sqrt{B}}$ *with the equality for abelian* $\mathcal{M}$*;*
(ii) $I_B \supset M_B \cap S(\mathcal{M}, \tau)_h$*;*
(iii) *if* $B \in \mathcal{P}(\mathcal{M})$*, then* $I_B \subset M_B = K_B$*;*
(iv) *if* $A \in I_B$*, then* $A + B \in S(\mathcal{M}, \tau)^+ \cap M_{2B}$*;*
(v) *if* $A \in I_B$*, then there exists* $S \in \mathcal{M}_h \cap \mathcal{U}(\mathcal{M})$ *such that* $A \in M_{(B+SBS)/2}$.

Proof. (i). The assertion follows immediately from Lemma 2.5.

(ii). Let $A \in I_B$. Note that $A \leq |A| \leq B$ and $-A \leq |A| \leq B$. Hence, $-B \leq A \leq B$ for $A \in M_B \cap S(\mathcal{M},\tau)_h$.

(iii). Assume that B is a projection. For $I_B \subset M_B$, see [8, Theorem 2.4]. The inclusion $K_B \subset M_B$ follows from (i). Let $A \in M_B$, i.e., $|A| \leq B$. Then, $|A|B = B|A| = |A|$ (see e.g. [48, Chap. 2, item 2.17]). Hence,

$$A^*A = |A|^2 = |A|^{\frac{1}{2}}|A||A|^{\frac{1}{2}} \leq |A|^{\frac{1}{2}}B|A|^{\frac{1}{2}} = B|A|^{\frac{1}{2}}|A|^{\frac{1}{2}}B \leq B$$

and therefore, $M_B \subset K_B$.

(iv). Note that $0 \leq A + B \leq 2B$, i.e., $A + B \in M_{2B} \cap S(\mathcal{M},\tau)^+ \subset I_{2B}$.

(v). See [7, Theorem 1]. □

Example 3.2. Let $X, Y \in S(\mathcal{M},\tau)$.

(i) Since $(X \pm Y)^*(X \pm Y) \geq 0$, we have $X^*Y + Y^*X \in I_{X^*X+Y^*Y}$;
(ii) since $(XY \pm I)(XY \pm I)^* \geq 0$, we have $XY + Y^*X^* \in I_{X|Y^*|^2X^*+I}$;
(iii) since $(\sqrt{X} \pm Y\sqrt{X})(\sqrt{X} \pm Y\sqrt{X})^* \geq 0$ for $X \in S(\mathcal{M},\tau)^+$, we have $XY^* + YX \in I_{X+YXY^*}$;
(iv) since $(X \pm YX^{-1})(X \pm YX^{-1})^* \geq 0$ for invertible $X \in S_h(\mathcal{M},\tau)$ with $X^{-1} \in S(\mathcal{M},\tau)$, we have $Y + Y^* \in I_{X^2+YX^{-2}Y^*}$.

Lemma 3.3. *[12, Theorem 5.1] Let $E, G, E_n, G_n \in L_1(\mathcal{M},\tau)_h$ and $F, F_n \in S_h(\mathcal{M},\tau)$ with $E_n \leq F_n \leq G_n$ for any $n \in \mathbb{N}$. Assume that*

$$E_n \xrightarrow{\tau} E,\ F_n \xrightarrow{\tau} F,\ G_n \xrightarrow{\tau} G \ \text{ and } \ \tau(E_n) \to \tau(E), \tau(G_n) \to \tau(G) \ \text{ as } \ n \to \infty.$$

Then, $F, F_n \in L_1(\mathcal{M},\tau)$ and $\tau(F_n) \to \tau(F)$ as $n \to \infty$. If, in addition, $E_n \leq 0 \leq G_n$ and $E_n \leq (F_n)^p \leq G_n$, where $0 < p < +\infty$ is such that the function $\mathbb{R} \ni \lambda \mapsto \lambda^p \in \mathbb{R}$ is defined, then $F_n, F \in L_p(\mathcal{M},\tau)$ and $\|F_n - F\|_p \to 0$ as $n \to \infty$.

Proposition 3.4. *Let $X, Y, X_n, Y_n \in S(\mathcal{M},\tau)$ be such that $X^*X + Y^*Y, X_n^*X_n + Y_n^*Y_n \in L_1(\mathcal{M},\tau)$ for any $n \in \mathbb{N}$. Assume that $\|X_n^*X_n + Y_n^*Y_n - X^*X - Y^*Y\|_1 \to 0$ and $X_n^*Y_n + Y_n^*X_n \xrightarrow{\tau} X^*Y + Y^*X$ as $n \to \infty$. Then, $X_n^*Y_n + Y_n^*X_n, X^*Y + Y^*X \in L_1(\mathcal{M},\tau)$ and $\|X_n^*Y_n + Y_n^*X_n - X^*Y - Y^*X\|_1 \to 0$ as $n \to \infty$.*

Proof. Let $E_n := -(X_n^*X_n + Y_n^*Y_n)$, $G_n := X_n^*X_n + Y_n^*Y_n$, $F_n := X_n^*Y_n + Y_n^*X_n$ and $E := -(X^*X + Y^*Y)$, $G := X^*X + Y^*Y$, $F := X^*Y + Y^*X$. We have $E_n \xrightarrow{\tau} E$ and $G_n \xrightarrow{\tau} G$ (see e.g. [19, Proposition 20]). By item (i) of Example 3.2, we obtain that $E_n \leq F_n \leq G_n$ for every n.

By Lemma 3.3, we obtain that $F_n, F \in L_1(\mathcal{M},\tau)$ with $\|F_n - F\|_1 \to 0$ as $n \to \infty$. □

Proposition 3.5. *For all operators $A, B \in S(\mathcal{M},\tau)_h$ and numbers $t \in [0,1]$, we have $(\sqrt{t}A + \sqrt{1-t}B)^2 \leq A^2 + B^2$. It turns into equality if and only if $\sqrt{1-t}A = \sqrt{t}B$. Hence, $\sqrt{t}A + \sqrt{1-t}B \in M_{\sqrt{A^2+B^2}}$.*

Proof. It is clear that

$$(\sqrt{t}A + \sqrt{1-t}B)^2 + (\sqrt{1-t}A - \sqrt{t}B)^2 = A^2 + B^2.$$

Hence, $(\sqrt{t}A + \sqrt{1-t}B)^2 \leq A^2 + B^2$. Now, by Lemma 2.5, we have $|\sqrt{t}A + \sqrt{1-t}B| \leq \sqrt{A^2 + B^2}$. It turns into equality if and only if $\sqrt{1-t}A = \sqrt{t}B$. □

Example 3.6. Let $\mathcal{E}$ be $S_0(\mathcal{M},\tau)$ or $\mathcal{F}(\tau)$ and $B \in \mathcal{E}^+$. It is clear that K_B, M_B and I_B are subsets of $\mathcal{E}$.

Please cite this article in press as: A. Bikchentaev, F. Sukochev, When weak and local measure convergence implies norm convergence, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.01.028

Indeed, $M_B \subset \mathcal{E}$ follows from item (ii) of Lemma 2.3.

Let $A \in K_B$. It follows from item (ii) of Lemma 2.3 that $\mu(A^*A) \le \mu(B)$. Then, by items (i) and (vi) of Lemma 2.3, we obtain that $K_B \subset \mathcal{E}$.

For every $A \in I_B$, by (v) of Theorem 3.1, $A \in M_{\frac{B+SBS}{2}}$ for a selfadjoint unitary operator S. By (i) of Lemma 2.3, we have $SBS \in \mathcal{E}$. We obtain that $A \in M_{\frac{B+SBS}{2}} \subset \mathcal{E}$. Hence, $I_B \subset \mathcal{E}$ (the case when $\mathcal{E} = \mathcal{F}(\tau)$ can be obtained by [33, Lemma 4.2] immediately).

Remark 3.7. We note that if $\mathcal{E}$ is a solid space on $(\mathcal{M}, \tau)$ and $B \in \mathcal{E}^+$, then M_B and I_B are subsets of $\mathcal{E}$.

Proposition 3.8. *Consider $A \in S(\mathcal{M}, \tau)$ and $U \in \mathcal{M}$ such that $\|U\|_\infty \le 1$.*

(i) *If $A \in K_B$, then $UA \in K_B$.*

(ii) *If $AU \in M_B$, then $U^*|A|U \in M_B$.*

Proof. (i). If $A \in K_B$, then $A^*A \le B$ and therefore, $A^*U^*UA \le A^*\mathbf{1}A = A^*A$. That is, $UA \in K_B$.

(ii). By Lemma 2.5 and Hansen's Theorem ([30]; [4, Lemma 3.1.1]), we have

$$B \ge |AU| = \sqrt{U^*A^*AU} \ge U^*\sqrt{A^*A}U = U^*|A|U. \quad \square$$

Theorem 3.9. *If $B \in S(\mathcal{M}, \tau)^+$ then*

(i) *$K_B = \{T\sqrt{B} : \ T \in \mathcal{M}$ and $\|T\|_\infty \le 1\}$;*

(ii) *$\operatorname{ext} K_B = \{T\sqrt{B} : T \in \mathcal{M}$ is partial isometry such that $(\mathbf{1} - T^*T)\mathcal{M}(\mathbf{1} - TT^*) = \{0\}\}$ for invertible B satisfying $B^{-1} \in S(\mathcal{M}, \tau)$;*

(iii) *the sets I_B and K_B are convex and t_τ-closed in $S(\mathcal{M}, \tau)$.*

Proof. (i). If $A \in K_B$, then it follows from Lemma 2.4 that there exists $Z \in \mathcal{M}$ with $\|Z\|_\infty \le 1$ such that $|A| = \sqrt{A^*A} = Z\sqrt{B}$. Standard polar decomposition yields $A = U|A| = UZ\sqrt{B} = T\sqrt{B}$ for $T = UZ$. Hence, $K_B \subset \{T\sqrt{B} : \ T \in \mathcal{M}$ *and* $\|T\|_\infty \le 1\}$. On the other hand, for every $T \in \mathcal{M}$ with $\|T\|_\infty \le 1$, we have $(T\sqrt{B})^*T\sqrt{B} \le B$. That is, $\{T\sqrt{B} : \ T \in \mathcal{M}$ *and* $\|T\|_\infty \le 1\} \subset K_B$.

(ii). It is well-known that $\operatorname{ext}\{T \in \mathcal{M} : \ \|T\|_\infty \le 1\}$ is the set of all partial isometries $U \in \mathcal{M}$ such that $(\mathbf{1} - U^*U)\mathcal{M}(\mathbf{1} - UU^*) = \{0\}$, see [35, Theorem 7.3.1].

(iii). The convexity of I_B is clear. Since the set $\{T \in \mathcal{M} : \ \|T\|_\infty \le 1\}$ is convex, it follows (i) that K_B is convex.

Consider a sequence $\{A_n\}_{n=1}^\infty \subset K_B$ such that $A_n \xrightarrow{\tau} A \in S(\mathcal{M}, \tau)$ as $n \to \infty$. One has $A_n^*A_n \xrightarrow{\tau} A^*A \in S(\mathcal{M}, \tau)$ as $n \to \infty$, since the involution and the multiplication operations are continuous in the measure topology [19]. Since $B - A_n^*A_n \ge 0$ for all $n \in \mathbb{N}$ and $B - A_n^*A_n \xrightarrow{\tau} B - A^*A$ as $n \to \infty$, we have $B - A^*A \ge 0$ by t_τ-closedness of the cone $S(\mathcal{M}, \tau)^+$ in $S(\mathcal{M}, \tau)$ [19].

It is well-known that $S(\mathcal{M}, \tau)_h$ and $S(\mathcal{M}, \tau)^+$ are closed with respect to the measure topology (see e.g. [25, Chapter II, Propositions 5.10 and 6.1]). Therefore, $A \in S(\mathcal{M}, \tau)_h$ with $B - A_n \xrightarrow{\tau} B - A \ge 0$ and $B + A_n \xrightarrow{\tau} B + A \ge 0$ for any sequence $\{A_n\} \subset I_B$. Hence, we obtain the t_τ-closedness of I_B. $\square$

Corollary 3.10. *If $B \in \mathcal{P}(\mathcal{M})$, then the set M_B is convex.*

Proof. Apply Theorem 3.9 and item (iii) of Theorem 3.1. $\square$

Proposition 3.11. *For every operator $B \in S(\mathcal{M}, \tau)^+$, the set M_B is t_τ-closed in $S(\mathcal{M}, \tau)$.*

Proof. Consider a sequence $\{A_n\}_{n=1}^\infty \subset M_B$ such that $A_n \xrightarrow{\tau} A \in S(\mathcal{M}, \tau)$ as $n \to \infty$. Since the involution and the multiplication operations are continuous in the measure topology, one has $A_n^*A_n \xrightarrow{\tau} A^*A \in S(\mathcal{M}, \tau)$

as $n \to \infty$. Then $|A_n| = \sqrt{A_n^* A_n} \xrightarrow{\tau} \sqrt{A^* A} = |A|$ as $n \to \infty$ via t_τ-continuity of the operator function $f(t) = \sqrt{t}$ $(t \geq 0)$, see [51] and [20]. Since $B - |A_n| \geq 0$ for all $n \in \mathbb{N}$ and $B - |A_n| \xrightarrow{\tau} B - |A|$ as $n \to \infty$, we have $B - |A| \geq 0$ by t_τ-closedness of the cone $S(\mathcal{M}, \tau)^+$ in $S(\mathcal{M}, \tau)$, which implies that M_B is t_τ-closed in $S(\mathcal{M}, \tau)$. □

Theorem 3.12. *For a von Neumann algebra $\mathcal{M}$, the following conditions are equivalent:*

(i) *the set M_B is convex for every operator $B \in \mathcal{M}^+$;*

(ii) *The equality $I_B = M_B \cap \mathcal{M}_h$ holds for every operator $B \in \mathcal{M}^+$;*

(iii) *$\mathcal{M}$ is abelian.*

Proof. (iii)⇒(i). By (i) of Theorem 3.1, $K_{B^2} = M_B$. Then, applying Theorem 3.9, we obtain that M_B is convex, which implies the validity of (i).

(iii)⇒(ii). By Theorem 3.1, we have $I_B \supset M_B \cap S_h(\mathcal{M}, \tau)$. Remark 3.7 implies that $I_B \subset \mathcal{M}$. Hence, $I_B \supset M_B \cap \mathcal{M}_h$.

On the other hand, assume that $\mathcal{M}$ is abelian and $-B \leq A \leq B$ (i.e. $A \in I_B$). Note that $AE^A(0, \infty) \leq BE^A(0, \infty)$ and $-BE^A(-\infty, 0) \leq AE^A(-\infty, 0)$. Hence, we have $|A| \leq B(E^A(0, \infty) + E^A(-\infty, 0)) \leq B$. That is, $A \in M_B$. Since $I_B \subset S(\mathcal{M}, \tau)_h \cap \mathcal{M}$, it follows that $I_B \subset M_B \cap \mathcal{M}_h$.

(i)⇒(iii). If $\mathcal{M}$ is noncommutative, then it contains a $*$-subalgebra $\mathcal{N}$ which $*$-isomorphic to $\mathbb{M}_2(\mathbb{C})$, see, for example, the proof of Theorem 1 in [52]. Put

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad P = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad Q = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

It is easy to check that P, Q are one-dimensional projections, and

$$A_1^* A_1 = 2P, \quad A_2^* A_2 = 2Q, \quad \frac{1}{2}A_1 + \frac{1}{2}A_2 = \operatorname{diag}(1, 0).$$

Then

$$|A_1| = \sqrt{2}P, \quad |A_2| = \sqrt{2}Q, \quad \left|\frac{1}{2}A_1 + \frac{1}{2}A_2\right| = \operatorname{diag}(1, 0).$$

For $B := \operatorname{diag}(7/8, 5)$, we have $|A_1| \leq B$, $|A_2| \leq B$, but the inequality $\left|\frac{1}{2}A_1 + \frac{1}{2}A_2\right| \leq B$ fails.

(ii)⇒(iii). It suffices to show that there exists $B \in \mathbb{M}_2(\mathbb{C})^+$ such that $I_B \neq M_B \cap \mathbb{M}_2(\mathbb{C})_h$. Assume that $I_B = M_B \cap \mathbb{M}_2(\mathbb{C})_h$ for all $B \in \mathbb{M}_2(\mathbb{C})^+$. Since $-|X| \leq X \leq |X|$ and $-|Y| \leq Y \leq |Y|$ for all $X, Y \in \mathbb{M}_2(\mathbb{C})_h$, we have $-|X| - |Y| \leq X + Y \leq |X| + |Y|$. By the assumption, we obtain the inequality $|X + Y| \leq |X| + |Y|$ for all $X, Y \in \mathbb{M}_2(\mathbb{C})_h$, which is not true when we take

$$X := \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y := 2^{-1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

see [2, p. 310]. The theorem is proved. □

Lemma 3.13 *(cf. [9, Lemma 1]). Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$ and $B \in S(\mathcal{M}, \tau)^+$. If $A \in I_B$, then there exist $X \in S(\mathcal{M}, \tau)_h$ and $Y \in S(\mathcal{M}, \tau)^+$ such that $A = XY + YX$, $B = X^2 + Y^2$. If, in addition, $A \in S(\mathcal{M}, \tau)^+$, then $X \in S(\mathcal{M}, \tau)^+$.*

Proof. Put

$$X := \frac{1}{2}(\sqrt{A + B} - \sqrt{B - A}), \quad Y := \frac{1}{2}(\sqrt{A + B} + \sqrt{B - A}). \tag{1}$$

Please cite this article in press as: A. Bikchentaev, F. Sukochev, When weak and local measure convergence implies norm convergence, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.01.028

It is clear that $X \in S(\mathcal{M},\tau)_h$ and $Y \in S(\mathcal{M},\tau)$ with $A = XY + YX$, $B = X^2 + Y^2$. Assume that $A \geq 0$. Then, $A + B \geq B - A$. Applying Lemma 2.5, we have $\sqrt{A+B} \geq \sqrt{B-A}$. Relation (1) implies X belongs to $S(\mathcal{M},\tau)^+$. $\square$

Theorem 3.14. *For all $X, Y \in S(\mathcal{M},\tau)_h$, there exists $T \in \mathcal{M}_h$ with $\|T\|_\infty \leq 1$ such that $XY + YX = \sqrt{X^2+Y^2}T\sqrt{X^2+Y^2}$.*

Proof. Let $X = U_1|X|$ and $Y = V_1|Y|$ be the polar decompositions of $X, Y \in S(\mathcal{M},\tau)_h$. We have

$$X^2,\ Y^2 \leq X^2 + Y^2.$$

By Lemma 2.4, there exist $U_2, V_2 \in \mathcal{M}$ with $\|U_2\|_\infty \leq 1$, $\|V_2\|_\infty \leq 1$ such that

$$|X| = \sqrt{X^2} = U_2\sqrt{X^2+Y^2}, \quad |Y| = \sqrt{Y^2} = V_2\sqrt{X^2+Y^2}.$$

Let $P = \mathrm{supp}(X^2+Y^2)$. Without loss of generality, we may assume that $\mathrm{supp}(U_2), \mathrm{supp}(V_2) \leq P$. Moreover,

$$|X|^2 + |Y|^2 = \sqrt{X^2+Y^2}(U_2^*U_2 + V_2^*V_2)\sqrt{X^2+Y^2}$$

implies that $\sqrt{X^2+Y^2} = (U_2^*U_2 + V_2^*V_2)^{\frac{1}{2}}\sqrt{X^2+Y^2}$. Hence, $P = (U_2^*U_2 + V_2^*V_2)^{\frac{1}{2}}$. In particular, $U_2^*U_2 + V_2^*V_2 = P$.

Letting $U = U_1U_2$, $V = V_1V_2$, we have $X = U\sqrt{X^2+Y^2}$, $Y = V\sqrt{X^2+Y^2}$. It is clear that

$$U^*U + V^*V = U_2^*U_1^*U_1U_2 + V_2^*V_1^*V_1V_2 \leq U_2^*U_2 + V_2^*V_2 = P \leq \mathbf{1}.$$

Letting $T := U^*V + V^*U$, it follows from the relation $(U \pm V)^*(U \pm V) \geq 0$ that $T \in \mathcal{M}_h$ with $\|T\|_\infty \leq 1$. Note that

$$(X \pm Y)^2 = \sqrt{X^2+Y^2}(U \pm V)^*(U \pm V)\sqrt{X^2+Y^2}. \tag{2}$$

Hence,

$$X^2 + Y^2 = \frac{(X+Y)^2 + (X-Y)^2}{2} \overset{(2)}{=} \sqrt{X^2+Y^2}(U^*U + V^*V)\sqrt{X^2+Y^2}. \tag{3}$$

Now, subtracting (3) from (2), we have

$$XY + YX = \sqrt{X^2+Y^2}(U^*V + V^*U)\sqrt{X^2+Y^2} = \sqrt{X^2+Y^2}T\sqrt{X^2+Y^2}. \quad \square$$

Corollary 3.15. *If $B \in S(\mathcal{M},\tau)^+$ then*
(i) $I_B = \{\sqrt{B}T\sqrt{B} :\ T \in \mathcal{M}_h$ *and* $\|T\|_\infty \leq 1\}$;
(ii) $\mathrm{ext}\, I_B = \{\sqrt{B}T\sqrt{B} :\ T \in \mathcal{M}_h \cap \mathcal{U}(\mathcal{M})\}$ *for B with $B^{-1} \in S(\mathcal{M},\tau)$.*

Proof. (i). Let $T \in \mathcal{M}_h$ and $-\mathbf{1} \leq T \leq \mathbf{1}$. We multiply both sides of the inequality by the operator $\sqrt{B}$ on the left and the right, and achieve $\sqrt{B}T\sqrt{B} \in I_B$. That is, $\{\sqrt{B}T\sqrt{B} :\ T \in \mathcal{M}_h$ *and* $\|T\|_\infty \leq 1\} \subset I_B$.

Applying Lemma 3.13 and Theorem 3.14, we obtain that $I_B \subset \{\sqrt{B}T\sqrt{B} :\ T \in \mathcal{M}_h$ *and* $\|T\|_\infty \leq 1\}$.

(ii). It is well-known that $\mathrm{ext}\{T \in \mathcal{M}^{\mathrm{sa}} :\ \|T\|_\infty \leq 1\} = \mathcal{M}_h \cap \mathcal{U}(\mathcal{M})$ (see e.g. [35, Proposition 7.4.6]). Since

$$\begin{aligned}\operatorname{ext} I_B &= \operatorname{ext}\{A \in S(\mathcal{M},\tau)_h : -B \le A \le B\}\\ &= \operatorname{ext}\{A \in S(\mathcal{M},\tau)_h : -\mathbf{1} \le B^{-\frac{1}{2}} A B^{-\frac{1}{2}} \le \mathbf{1}\},\end{aligned}$$

it follows that $\operatorname{ext} I_B = \{A \in I_B : B^{-\frac{1}{2}} A B^{-\frac{1}{2}} \in \mathcal{M}_h \cap \mathcal{U}(\mathcal{M})\} = \{\sqrt{B}T\sqrt{B} : T \in \mathcal{M}_h \cap \mathcal{U}(\mathcal{M})\}$. □

4. Operator intervals and uniformly absolutely continuous norms

In this section, we study the compactness of operator intervals. In particular, we demonstrate the connection of the compactness of operator interval and the order continuity of symmetric (quasi- or Δ-)norms. Before proceeding to the main result, we present some well known facts.

Assume that $\mathcal{M} = \ell_\infty$ and $\tau(X) = \sum_{k=1}^{\infty} x_k$ for $X = \{x_k\}_{k=1}^{\infty} \in \mathcal{M}^+$. In this case, $S_0(\mathcal{M},\tau) = c_0$ is the space of complex sequences converging to zero. It is well-known that a $\|\cdot\|_\infty$-closed set $\mathcal{A} \subset c_0$ is $\|\cdot\|_\infty$-compact if and only if there exists $B \in c_0^+$ such that $|A| \le B$ for all $A \in \mathcal{A}$ [13, Ch. 5, exercise 5.6.47].

Proposition 4.1. *Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$ such that $\tau(\mathbf{1}) = \infty$. If $B \in S(\mathcal{M},\tau)^+$ is such that any of the sets I_B, K_B or M_B is compact (in measure topology), then B is τ-compact.*

Proof. Let $B \in S(\mathcal{M},\tau)^+$ be non τ-compact. For such an operator B, we have $b := \lim_{t\to\infty} \mu_t(B) > 0$. Since the trace τ is semifinite, there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of pairwise orthogonal projections in $\mathcal{M}$ and a number $a > 0$ such that $\tau(P_n) \ge a > 0$ and $bP_n \le B$ for all $n \in \mathbb{N}$. Clearly no subsequence $\{bP_{n_k}\}_{k=1}^{\infty}$ of $\{bP_n\}_{n=1}^{\infty}$ t_τ-converges. The assertion is proved. □

Recall that a von Neumann algebra is of type $\mathrm{I}_{\mathrm{fin}}$ if it is finite and of type I.

Theorem 4.2. *Let $\mathcal{M}$ be a semifinite von Neumann algebra and let $0 \le B \in S(\mathcal{M},\tau)$. If either K_B or M_B is compact (in measure topology), then B is affiliated with the $\mathrm{I}_{\mathrm{fin}}$ direct summand of $\mathcal{M}$.*

Proof. Combining [50, Theorem V.1.19] with [50, Theorem V.1.27], we infer that there exists unique centrally orthogonal decomposition

$$\mathbf{1} = z_{\mathrm{I}_{\mathrm{fin}}} + z_{\mathrm{I}_\infty} + z_{\mathrm{II}_1} + z_{\mathrm{II}_\infty}$$

so that

$$\mathcal{M} = \mathcal{M}_{\mathrm{I}_{\mathrm{fin}}} \oplus \mathcal{M}_{\mathrm{I}_\infty} \oplus \mathcal{M}_{\mathrm{II}_1} \oplus \mathcal{M}_{\mathrm{II}_\infty},$$

where $\mathcal{M}_{\mathrm{I}_{\mathrm{fin}}}$ has type $\mathrm{I}_{\mathrm{fin}}$, $\mathcal{M}_{\mathrm{I}_\infty}$ has homogeneous type I_∞, $\mathcal{M}_{\mathrm{II}_1}$ has type II_1 and $\mathcal{M}_{\mathrm{II}_\infty}$ has type II_∞.

Step 1. We shall show that $z_{\mathrm{I}_\infty} B \in S(\mathcal{M}_{\mathrm{I}_\infty},\tau)$ vanishes. For brevity, let us simply assume that $\mathcal{M}$ has homogeneous type I_∞ and show that in this case $B = 0$.

Without loss of generality, we may assume that $\mathcal{M}$ acts on separable Hilbert space. Indeed, the argument for the general case is exactly the same.

By [50, Theorem V.1.27], we may write

$$\mathcal{M} = \sum_{\alpha}^{\oplus} \mathcal{A}_\alpha \bar{\otimes} B(H_\alpha)$$

where $\mathcal{A}_\alpha$ is abelian von Neumann algebra and H_α is infinite dimensional separable Hilbert space. To prove our claim, it is sufficient to assume that $\mathcal{M} = \mathcal{A}_\alpha \bar{\otimes} B(H_\alpha)$, or, equivalently that $\mathcal{M} = L_\infty(\Omega,\mu)\bar{\otimes}\mathcal{B}(\mathcal{H})$, where (Ω, μ) is a σ-finite measure space. It is sufficient to show that $B = 0$ in two cases: when (Ω, μ) is a discrete measure space or else if (Ω, μ) is atomless. Firstly, we consider the case when (Ω, μ) is a discrete measure space, in other words, we could assume that

$$\mathcal{M} = \mathcal{B}(\mathcal{H}) \bar{\otimes} \ell_\infty.$$

Assume that $(e_k)_{k \geq 0}$ is the standard basic sequence in ℓ_∞. Fix $k \geq 0$ such that $B \cdot (\mathbf{1} \otimes e_k) \neq 0$. By taking $\mathcal{M} \cdot (\mathbf{1} \otimes e_k)$ instead of $\mathcal{M}$ and $B \cdot (\mathbf{1} \otimes e_k)$ instead of B, we may assume without loss of generality that $\mathcal{M} = \mathcal{B}(\mathcal{H})$. The measure topology induced on $\mathcal{B}(\mathcal{H})$ is simply the uniform norm topology [19].

Fix rank one projection p and a constant $c > 0$ such that $B \geq cp$. Obviously, $K_{cp} \subset K_B$ and $M_{cp} \subset M_B$. By Theorem 3.9 and Proposition 3.11, K_p and M_p are closed in measure. Therefore, by the assumption that K_B (or M_B) is compact, K_p (or M_p) is compact. If q is another rank one projection and if U_q is a partial isometry such that

$$U_q^* U_q = p, \quad U_q U_q^* = q,$$

then $U_q \in K_p$ and $U_q \in M_p$. Therefore,

$$\{U_q : \ q \text{ is rank one projection}\}$$

is compact (in uniform norm topology). If q and r are orthogonal rank one projections, then

$$q \cdot (U_q - U_r) = q \cdot (qU_q - rU_r) = qU_q = U_q.$$

Therefore,

$$\|U_q - U_r\|_\infty \geq \|q(U_q - U_r)\|_\infty = \|U_q\|_\infty = 1.$$

This contradicts with the compactness assumption.

Now, we assume that $\mathcal{M} = L_\infty(\Omega, \mu) \bar{\otimes} \mathcal{B}(\mathcal{H})$, where (Ω, μ) is an atomless measure space. In fact, since we work with separable Hilbert spaces, we can assume that (Ω, μ) is a standard measure space, and for the sake of clarity and brevity, we shall assume further that (Ω, μ) coincides with the interval $[0, 1]$ equipped with Lebesgue measure. Fix a τ-finite projection $p \in \mathcal{M}$ and a constant $c > 0$ such that $B \geq cp$. By [5, Lemma 5.1] the projection p may be viewed as a function defined a.e. on $[0, 1]$ taking values in finite projections in $\mathcal{B}(\mathcal{H})$. Without loss of generality, we assume that the function $\omega \to p(\omega)$ has full support on $[0, 1]$. Consider now the sequence $\{r_n(\cdot)\}_{n=1}^\infty$ of Rademacher functions on $[0, 1]$, that is a concrete sequence of independent (Bernoulli) random variable taking values ± 1 with probability $1/2$. Again referring to [5] we may define a sequence of partial isometries $U_n \in \mathcal{M}$ as vector valued functions

$$\phi_n(\omega) := r_n(\omega) p(\omega), \quad n \geq 1.$$

We trivially have $U_n^* U_n = p$ and $U_n \in K_p$ and $U_n \in M_p$ for all $n \geq 1$. However, it is a well known fact that the Rademacher system does not contain any subnet converging in measure. Combining this observation with [5, Remark 5.4], we infer that the sequence $\{U_n\}_{n \geq 1}$ does not contain any subnet converging in measure as well which again contradicts with the compactness assumption. This completes the proof of the claim that $B = 0$ in the setting when $\mathcal{M}$ is of type I_∞.

Step 2: Suppose that $\mathcal{M} = \mathcal{M}_{\mathrm{II}_1}$, or else that $\mathcal{M} = \mathcal{M}_{\mathrm{II}_\infty}$. Fix a τ-finite projection p and a constant $c > 0$ such that $B \geq cp$. Obviously, $K_{cp} \subset K_B$ and $M_{cp} \subset M_B$. Therefore, K_p (or M_p) must be compact. In other words, the unit ball of the algebra $p\mathcal{M}p$ is compact (in measure topology). Without loss of generality, let $\tau(p) = 1$ and let $i : L_\infty(0,1) \to p\mathcal{M}p$ be a trace preserving unital $*$-isomorphism (see e.g. [14]). It follows that the unit ball of $L_\infty(0,\tau(p))$ is compact (in measure topology). Again appealing to the example of the Rademacher sequence $\{r_n\}_{n\geq 1}$ we arrive at the contradiction. This completes the proof that in either case $B = 0$.

Combining steps 1 and 2, we arrive at the assertion that B must be affiliated with $\mathcal{M}_{\mathrm{I_{fin}}}$. □

A nonzero projection $P \in \mathcal{M}$ is called an atom if $0 \neq Q \leq P$, $Q \in \mathcal{P}(\mathcal{M})$, implies that $Q = P$. A von Neumann algebra $\mathcal{M}$ is atomic if every nonzero projection in $\mathcal{M}$ majorizes some atom.

Lemma 4.3. *Let $(\mathcal{M},\tau)$ be a semifinite von Neumann algebra. If $0 \leq B \in \mathcal{M} \cap S_0(\mathcal{M},\tau)$ and if a uniformly bounded net $T_i \to 0$ in local measure topology, then $B^{\frac12}T_iB^{\frac12} \to 0$ in measure.*

Proof. Let $e = E^B(0,\varepsilon)$, $\varepsilon > 0$. In particular, $\tau(e^\perp) < \infty$. Using (i), (iii) and (iv) of Lemma 2.3, for any $s,t > 0$, we have

$$\begin{aligned}\mu(s+3t; B^{\frac12}T_iB^{\frac12}) &\leq \mu(s; e^\perp B^{\frac12}T_iB^{\frac12}e^\perp) + \mu(t; e^\perp B^{\frac12}T_iB^{\frac12}e)\\ &\quad + \mu(t; eB^{\frac12}T_iB^{\frac12}e^\perp) + \mu(t; eB^{\frac12}T_iB^{\frac12}e)\\ &\leq \mu(s; e^\perp B^{\frac12}T_iB^{\frac12}e^\perp) + 3\|B\|_\infty^{\frac12}\sup_i \|T_i\|_\infty \varepsilon^{\frac12}.\end{aligned}$$

Since ε is arbitrarily chosen and $\mu(s; e^\perp B^{\frac12}T_iB^{\frac12}e^\perp) \to_i 0$ [19, Section 2.5], it follows that $\mu(s + 3t; B^{\frac12}T_iB^{\frac12}) \to_i 0$, which completes the proof. □

Lemma 4.4. *Let $(\mathcal{M},\tau)$ be a semifinite atomic von Neumann algebra. If $0 \leq B \in \mathcal{M}$ is τ-compact and if a uniformly bounded net $T_i \to 0$ ultraweakly, then $B^{\frac12}T_iB^{\frac12} \to 0$ in measure.*

Proof. Assume $T_i \to 0$ ultraweakly, that $\|T_i\|_\infty \leq 1$ and that $\mathcal{M}$ is atomic. By Lemma 4.3, it suffices to show that $T_i \to_i 0$ locally in measure. By the definition of local convergence in measure, it may be assumed that $\tau(\mathbf{1}) < \infty$ and it then suffices to show that $T_i \to 0$ for the measure topology. Since $\mathcal{M}$ is atomic, it follows that there exists a sequence $\{Q_n\}_{n=1}^\infty$ of finite rank projections (see [50], Definition III 5.9) such that $Q_n \uparrow_n \mathbf{1}$. Let $t > 0$ and choose n_0 such that

$$\tau(\mathbf{1} - Q_{n_0}) < t$$

and note that

$$\mu(t; \mathbf{1} - Q_{n_0}) = \chi_{[0,\tau(\mathbf{1}-Q_{n_0}))}(t) = 0.$$

This implies that, for all i,

$$\begin{aligned}\mu(2t; \mathbf{1} - Q_{n_0}) &\leq \mu(t; Q_{n_0}T_i(\mathbf{1} - Q_{n_0})) + \mu(t; (\mathbf{1} - Q_{n_0})T_i)\\ &\leq 2\mu(t; \mathbf{1} - Q_{n_0}) = 0.\end{aligned}$$

Consequently,

$$\begin{aligned}\mu(3t; T_i) &\leq \mu(t; Q_{n_0}T_iQ_{n_0}) + \mu(2t; T_i - Q_{n_0}T_iQ_{n_0})\\ &\leq \mu(t; Q_{n_0}T_iQ_{n_0}) + \mu(t; T_i - Q_{n_0}T_i) + \mu(t; Q_{n_0}T_i - Q_{n_0}T_iQ_{n_0})\\ &\leq \mu(t; Q_{n_0}T_iQ_{n_0}).\end{aligned}$$

Since $T_i \to 0$ ultraweakly and Q_{n_0} is finite-dimensional, it follows that $\tau(Q_{n_0}T_iQ_{n_0}) \to_i 0$, which implies that $Q_{n_0}T_iQ_{n_0} \to_i 0$ in measure (see e.g. [36,39]). It follows that $\mu(3t; T_i) \to_i 0$ and so T_i converges to 0 for the measure topology. This completes the proof. □

Theorem 4.5. *Let τ be a faithful normal semifinite trace on an atomic von Neumann algebra $\mathcal{M}$. An operator $B \in S(\mathcal{M},\tau)^+$ is τ-compact if and only if the set I_B is t_τ-compact.*

Proof. Let $(A_n)_n \subset I_B$ be an arbitrary sequence. We first prove the case when $B \in \mathcal{M}^+ \cap S_0(\mathcal{M},\tau)$. By Lemma 2.4, we can write $A_i + B = (2B)^{\frac{1}{2}} S_i (2B)^{\frac{1}{2}}$, where $\|S_i\|_\infty \le 1$, $i \in I$. By Banach–Alaoglu Theorem, there exists a subnet $(S_{\psi(j)})_{j\in J}$ such that $S_{\psi(j)} \to S \in \mathcal{M}$ ultraweakly. Applying the preceding lemma to the net $T_j = S_{\psi(j)} - S$, we obtain the proof of sufficiency for $B \in \mathcal{M}^+$.

Since measure topology is complete metrizable [19], we can endow $S(\mathcal{M},\tau)$ with a metric d. Now, assume that $B \in S_0(\mathcal{M},\tau)^+$. Let $e_n = E^B[0,n]$. By the latter result, there exists a subsequence $\{A_i^{(1)}\}$ of $\{A_i\}$ such that $e_1 A_i^{(1)} e_1 \xrightarrow{t_\tau} e_1 A^{(1)} e_1$ for some $A^{(1)} \in I_{Be_1} \subset I_B$. In particular, we can find an $A_{n(1)}$ from $\{A_n\}$ such that

$$d(e_1 A_{n(1)} e_1, A^{(1)}) \le 1.$$

Similarly, there exists a subsequence $\{A_i^{(2)}\}$ of $\{A_i^{(1)}\}$ such that $e_2 A_i^{(2)} e_2 \xrightarrow{t_\tau} e_2 A^{(2)} e_2$ for some $A^{(2)} \in I_{Be_2} \subset I_B$. In particular, $e_1 A^{(2)} e_1 = A^{(1)}$. We can find an $A_{n(2)}$ from $\{A_i^{(2)}\}$ such that $n(2) > n(1)$ with

$$d(e_2 A_{n(2)} e_2, A^{(2)}) \le \frac{1}{2}.$$

Argument inductively, we obtain a sequence $\{A^{(n)}\} \subset I_B$ such that $A^{(n)} = e_n A^{(m)} e_n$ for any $m \ge n$, and a subsequence $\{A_{n(j)}\}$ of $\{A_n\}$ such that

$$d(e_j A_{n(j)} e_j, A^{(j)}) \le \frac{1}{j}.$$

Since $\tau(\mathbf{1} - e_m) \to 0$ and $e_n A^{(m)} e_n = A^{(n)}$ for any $m \ge n$, it follows that $\{A^{(m)}\}$ converges in measure. We denote $C := t_\tau - \lim A^{(n)}$. That is, $d(A^{(m)}, C) \to_m 0$. Since I_B is closed in measure topology (see Theorem 3.9), it follows that $C \in I_B$. Note that

$$\begin{aligned} d(A_{n(j)}, C) &\le d(A_{n(j)}, e_j A_{n(j)} e_j) + d(e_j A_{n(j)} e_j, A^{(j)}) + d(A^{(j)}, C) \\ &\le d(A_{n(j)}, e_j A_{n(j)} e_j) + \frac{1}{j} + d(A^{(j)}, C). \end{aligned} \tag{4}$$

Since $\tau(\mathbf{1} - e_m) \to_m 0$, it follows from [39, Corollary 2.3.16.] that

$$\begin{aligned} \mu(t; A_{n(j)} - e_j A_{n(j)} e_j) &\le \mu(\frac{t}{2}; A_{n(j)} - e_j A_{n(j)}) + \mu(\frac{t}{2}; e_j A_{n(j)} - e_j A_{n(j)} e_j) \\ &\le 2\mu(t/4; 1 - e_j)\mu(t/4; A_{n(j)}) \to 0. \end{aligned}$$

That is, $d(A_{n(j)}, e_j A_{n(j)} e_j) \to_j 0$. Hence, (4) converges to 0 as $j \to \infty$. We obtain the compactness of I_B.

The necessity follows from Proposition 4.1. □

Corollary 4.6. *Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$. The following conditions are equivalent:*

(i) *$\mathcal{M}$ is atomic;*

(ii) *the set I_B is t_τ-compact for every τ-compact operator $B \in S(\mathcal{M},\tau)^+$.*

Proof. Assume that $\mathcal{M}$ is not atomic. Then, it is a direct sum of atomic von Neumann algebra and a non-trivial von Neumann algebra without minimal projections [16, p. 325]. Without loss of generality we can assume that there exists an isometric embedding of $L_\infty([0,1],\nu)$ into $\mathcal{M}$, where ν is the Lebesgue measure on $[0,1]$, which preserves the trace (on $L_\infty([0,1],\nu)$ the trace is $\tau(f) = \int_{[0,1]} f\,d\nu$), see [16, p. 325]. In the algebra $L_\infty([0,1],\nu)$, consider the sequence of Rademacher functions $r_n(t) = \operatorname{sign}\sin 2^n\pi t$ with $0 \le t \le 1$. The sequence $\{r_n\}_{n=1}^\infty$ contains no t_τ-converging subsequences:

$$\nu\{t \in [0,1] :\ |r_n(t) - r_k(t)| \ge 1\} = \nu\{t \in [0,1] :\ r_n(t) \ne r_k(t)\} = \frac{1}{2},\ n \ne k.$$

Thus, taking a τ-compact operator $B = \chi_{[0,1]}$, the sets I_B and $K_B = M_B$ are not t_τ-compact. The assertion is proved. □

Now, by Krein–Milman Theorem (see also Corollary 3.15), we have

Corollary 4.7. *Let τ be a faithful normal semifinite trace on an atomic von Neumann algebra $\mathcal{M}$ and let there exist a constant $a > 0$ such that $\tau(P) \ge a$ for any atom P of $\mathcal{M}$. If an operator $B \in S(\mathcal{M},\tau)^+$ is τ-compact, then the set I_B is the t_τ-closure of the convex hull of its extreme points.*

Proof. We have $S(\mathcal{M},\tau) = \mathcal{M}$. Moreover, the topology t_τ coincides with the $\|\cdot\|_\infty$-topology on $\mathcal{M}$. □

A function $\|\cdot\|$ from Ω to $\mathbb{R}$ is a Δ-norm, if for all $x, y \in \Omega$ the following properties hold:

1. $\|x\| \geqslant 0$, $\|x\| = 0 \Leftrightarrow x = 0$;
2. $\|\alpha x\| \leqslant \|x\|$, $\forall\ |\alpha| \le 1$;
3. $\lim_{\alpha\to 0} \|\alpha x\| = 0$;
4. $\|x + y\| \le C_\Omega \cdot (\|x\| + \|y\|)$

for a constant $C_\Omega \ge 1$ independent of x, y. Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ. Let $\mathcal{E}$ be a linear subset in $S(\mathcal{M},\tau)$ equipped with a Δ-norm $\|\cdot\|_{\mathcal{E}}$. We say that $\mathcal{E}$ is a *symmetrically Δ-normed space* if for $x \in \mathcal{E}$, $y \in S(\mathcal{M},\tau)$ and $\mu(y) \le \mu(x)$ imply that $y \in \mathcal{E}$ and $\|y\|_{\mathcal{E}} \le \|x\|_{\mathcal{E}}$ [32,33].

Let $E(\mathcal{M},\tau)$ be a symmetrically Δ-normed spaces affiliated with $\mathcal{M}$. Let $E^{oc}(\mathcal{M},\tau)$ be the set of all elements of order continuous Δ-norm [33], i.e.,

$$E^{oc}(\mathcal{M},\tau) = \{X \in E(\mathcal{M},\tau) : |X| \ge X_n \downarrow 0 \Rightarrow \|X_n\|_E \downarrow 0\}.$$

We note that for some symmetric spaces $E(\mathcal{M},\tau)$, $E^{oc}(\mathcal{M},\tau)$ is trivial, that is $E^{oc}(\mathcal{M},\tau) = \{0\}$ [19, p. 246].

If $X \in E(\mathcal{M},\tau)$, then X is said to have absolutely continuous Δ-norm if and only if for all decreasing sequences $\{P_n\}$ in $\mathcal{P}(\mathcal{M})$ with $P_n \downarrow_n 0$, we have $\|P_nXP_n\|_E \to 0$ [17]. A subset $\mathcal{A}$ of $E(\mathcal{M},\tau)$ is called of uniformly absolutely continuous Δ-norm if

$$\sup_{X\in\mathcal{A}} \|P_nXP_n\|_E \to 0,\ \forall P_n \downarrow 0 \subset \mathcal{P}(\mathcal{M}).$$

Lemma 4.8. *Let $E(\mathcal{M},\tau)$ be a symmetrically Δ-normed operator space. Assume that $T \in E^{oc}(\mathcal{M},\tau)$. Then, every decreasing sequence $\{P_n\}_n$ of projections in $\mathcal{M}$ such that $P_n \downarrow 0$, we have $\|TP_n\|_E \downarrow 0$ and $\|P_nT\|_E \downarrow 0$. In particular, T has uniformly absolutely continuous Δ-norm.*

Please cite this article in press as: A. Bikchentaev, F. Sukochev, When weak and local measure convergence implies norm convergence, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.01.028

Proof. Note that $T \in S_0(\mathcal{M},\tau)$ (see e.g. [33, Remark 2.9]). Since $\mu(P_nT) = \mu(|P_nT|)$, it follows that $\|P_nT\|_E = \||P_nT|\|_E$. Since $\{T^*P_nT\}_n$ is decreasing and $T^*P_nT \to 0$ in measure topology, it follows from [25, Chapter II, Remark 5.9] that $(T^*P_nT)^{1/2} \downarrow 0$ (see also [51]). Noting that

$$\||P_nT|\|_E = \|((P_nT)^*(P_nT))^{1/2}\|_E = \|(T^*P_nT)^{1/2}\|_E$$

and $(T^*P_nT)^{1/2} \le |T|$, order continuity of T implies that $\|P_nT\|_E \downarrow 0$. Similarly, $\|TP_n\|_E \downarrow 0$. □

Corollary 4.9. *Let τ be a faithful normal semifinite trace on an atomic von Neumann algebra $\mathcal{M}$. If $E(\mathcal{M},\tau)$ is a symmetrically Δ-normed operator space, then I_B is compact (in the Δ-norm topology) for every $B \in E^{oc}(\mathcal{M},\tau)^+$.*

Proof. Let $(A_n)_{n\ge 0} \subset I_B$. Since I_B is compact in measure (see Corollary 4.6), there exists a subsequence $(A_{n_k})_{k\ge 0}$ such that $A_{n_k} \to A$ in measure. Since I_B is closed in measure, it follows that $A \in I_B$. By Lemma 4.8, the sequence $(B + A_{n_k})_{k\ge 0}$ is of uniformly absolutely continuous Δ-norm.[1]

For the sake of convenience, we denote $C_k = A_{n_k} + B - A - B$. It suffices to prove that $C_k \to 0$ in E. By passing to a subsequence of $\{C_k\}$, we may assume that there exists a sequence $\{p_j\}$ such that $p_j \to \mathbf{1}$, $\tau(p_j^\perp) \to 0$ as $j \to \infty$, $\|C_kp_j\|_\infty \to 0$ as $k \to \infty$ for any j [18, Lemma 2.3].

Since $2B \in E^{oc}(\mathcal{M},\tau)$, it follows that $C_k \in E^{oc}(\mathcal{M},\tau)$. By Lemma 4.8, $\|C_ke_n^\perp\|_E \to 0$ as $n \to \infty$. On the other hand,

$$\begin{aligned}
\|C_ke_n\|_E &\le C_E\|C_kp_je_n\|_E + C_E^2\|p_jC_kp_j^\perp e_n\|_E + C_E^2\|p_j^\perp C_kp_j^\perp e_n\|_E \\
&\le (C_E + C_E^2)\big\|\|C_kp_j\|_\infty e_n\big\|_E + C_E^2\|p_j^\perp C_kp_j^\perp e_n\|_E \\
&\le (C_E + C_E^2)\big\|\|C_kp_j\|_\infty e_n\big\|_E + C_E^3(\|p_j^\perp(A_{n_k} + B)p_j^\perp\|_E + \|p_j^\perp(A + B)p_j^\perp\|_E) \\
&\le (C_E + C_E^2)\big\|\|C_kp_j\|_\infty e_n\big\|_E + C_E^3\|p_j^\perp \cdot 2B \cdot p_j^\perp\|_E.
\end{aligned}$$

Since $\tau(p_j^\perp) \to 0$, it follows from Lemma 4.8 that $\|p_j^\perp Bp_j^\perp\|_E \to_j 0$. For every fixed j, $\|C_kp_j\|_\infty \to_k 0$. Hence, $\big\|C_ke_n\big\|_E \to_k 0$, which completes the proof. □

Noting that $S_0(\mathcal{M},\tau)$ can be equipped with an order continuous symmetric Δ-norm,[2] the following corollary is an extension of Theorem 4.5.

Corollary 4.10. *Let τ be a faithful normal semifinite trace on an atomic von Neumann algebra $\mathcal{M}$. If $E(\mathcal{M},\tau)$ is a symmetrically Δ-normed operator space, then the Δ-norm on $E(\mathcal{M},\tau)$ is order continuous if and only if the set I_B is $\|\cdot\|_E$-compact for every $B \in E(\mathcal{M},\tau)^+$.*

Proof. "⇒" If the Δ-norm is order continuous, then Corollary 4.9 implies that I_B is compact in the Δ-norm topology for every $B \in E(\mathcal{M},\tau)^+$.

"⇐" If I_B is $\|\cdot\|_E$-compact, then I_B is compact in measure. By Proposition 4.1, we have $B \in S_0(\mathcal{M},\tau)$. If $B \notin E(\mathcal{M},\tau)^{oc}$, then there exists a sequence $\{b_n\}$ with $B \ge b_n \downarrow 0$ but with $\|b_n\|_E \nrightarrow 0$ as $n \to \infty$. Since $B \in S_0(\mathcal{M},\tau)$, it follows from [25, Chapter II, Theorem 6.3] (see also [19]) that $b_n \to 0$ in measure. In particular, any subsequence of $\{b_n\}$ converges to 0 in measure. Since I_B is compact in the Δ-norm topology, it follows that there is a subsequence $\{b_{n_k}\}$ of $\{b_n\}$ with $b_{n_k} \to_k b$ for some $b \in I_B$ in $\|\cdot\|_E$-topology, and

[1] For the case of strongly symmetrically normed spaces, the assertion of this corollary follows immediately from Theorem 6.11 of [18].

[2] For example, one can take the Δ-norm $\|\cdot\|_E$ by setting $\|X\|_E = \inf_{t>0}\{t + \mu(t;X)\}$ [34,31]. It is clear that $\|\cdot\|_E$ is an order continuous Δ-norm on $S_0(0,\infty)$. Hence, $\|\cdot\|_E$ is an order continuous Δ-norm on the noncommutative counterpart $S_0(\mathcal{M},\tau)$ [33].

therefore, in measure. Hence, $b = 0$, which implies that $\|b_{n_k}\|_E \to 0$. Therefore, $\|b_n\|_E \to 0$, which is a contradiction. □

Corollary 4.10 might be considered as a non-commutative analogue of the Banach lattice specialization of Theorem 6.56 of the monograph [1], which goes back to a much older theorem of I. Kawai (loc. cit., 1957).

Corollary 4.11. *Let τ be a faithful normal semifinite trace on an atomic von Neumann algebra $\mathcal{M}$ and let $B \in E^{oc}(\mathcal{M}, \tau)$. If $(A_i)_{i\in I} \subset I_B$ and if $A_i \to A$ locally in measure, then $A_i \to A$ in Δ-norm topology.*

Proof. Assume the contrary. Choose a subnet $(A_{\psi_1(j)})_{j\in J}$ such that $\|A_{\psi_1(j)} - A\|_E \geq \varepsilon$ for every $j \in J$. Since I_B is compact in Δ-norm topology, one can extract a further subnet $(A_{\psi_1(\psi_2(k))})_{k\in K}$ such that $A_{\psi_1(\psi_2(k))} \to C \in I_B$ in the Δ-norm topology. In particular, $A_{\psi_1(\psi_2(k))} \to C$ in measure [49], and therefore, $A_{\psi_1(\psi_2(k))} \to C$ locally in measure. However, $A_i \to A$ locally in measure and, passing to a subnet, $A_{\psi_1(\psi_2(k))} \to A$ locally in measure. By the uniqueness of the limit, $C = A$. Therefore, $A_{\psi_1(\psi_2(k))} \to A$ in the Δ-norm topology and, simultaneously, $\|A_{\psi_1(\psi_2(k))} - A\|_E \geq \varepsilon$ for every $k \in K$. This contradiction completes the proof. □

Theorem 4.12. *Assume that $(\mathcal{M}, \tau)$ is a semifinite von Neumann algebra. Let $E(0, \infty)$ be a symmetrically normed operator space with order continuous norm.*[3] *If $x_n \in I_B$, $B \in E(\mathcal{M}, \tau)$ and $x_n \to x$ locally in measure, then $\|x_n - x\|_E \to 0$.*

Proof. Let $\{e_i\}$ be a net of τ-finite projections increasing to $\mathbf{1}$. Note that $-B \leq x_n \leq B$. Since $x_n \to x$ locally in measure, it follows that $x \in I_B$ (see e.g. [25, Chapter II, Proposition 7.6]). On the other hand, since

$$0 \leq x_n + B \leq 2B,\ 0 \leq x + B \leq 2B,$$

it follows that $\|x + B\|_E, \|x_n + B\| \leq \|2B\|_E$. For every $t \in \mathbb{R}$, we have

$$\begin{aligned} &\mu(t; x_n - x) \\ &\leq \mu(\frac{t}{4}; e_i(x_n - x)e_i) + \mu(\frac{t}{4}; e_i(x_n - x)e_i^\perp) + \mu(\frac{t}{4}; e_i^\perp(x_n - x)e_i) + \mu(\frac{t}{4}; e_i^\perp(x_n - x)e_i^\perp) \\ &\leq \mu(\frac{t}{4}; e_i(x_n - x)e_i) + 3\mu(\frac{t}{4}; (x_n - x)e_i^\perp). \end{aligned}$$

By Lemma 2.4, we have

$$\begin{aligned} x_n + B &= \sqrt{2B} Z_n \sqrt{2B} \\ x + B &= \sqrt{2B} Z \sqrt{2B}, \end{aligned}$$

where $\|Z_n\|_\infty, \|Z\|_\infty \leq 1$. Hence, $(x_n - x)e_i^\perp = \sqrt{2B}(Z_n - Z)\sqrt{2B}e_i^\perp$. Note that $\sqrt{2B}(Z_n - Z)$ is bounded in $S(\mathcal{M}, \tau)$ with $\mu(\sqrt{2B}(Z_n - Z)) \leq 2\mu(\sqrt{2B})$. Moreover, since $\|\cdot\|_E$ is absolutely continuous, it follows from [17, Theorem 3.1] that

$$\left\|\sqrt{2B}e_i^\perp\right\|_{E^{(2)}}^{1/2} = \left\||\sqrt{2B}e_i^\perp|^2\right\|_E = \left\|e_i^\perp 2Be_i^\perp\right\|_E \to 0,$$

[3] In particular, $E(\mathcal{M}, \tau)$ is a strongly symmetric operator space having order continuous norm (see e.g. [25, Chapter IV, Corollary 5.6, Theorem 14.3 and Theorem 14.6]). Lemma 4.8 implies that every element in $E(\mathcal{M}, \tau)$ has uniformly absolutely continuous norm.

where $\|\cdot\|_{E^{(2)}}$ is the norm of the 2-convexification of $E(\mathcal{M},\tau)$ [24], that is, $\sqrt{2B}e_i^\perp \to 0$ in measure topology [49]. By [39, corollary 2.3.16], we have

$$\mu(\frac{t}{4};(x_n - x)e_i^\perp) = \mu(\frac{t}{4};\sqrt{2B}(Z_n - Z)\sqrt{2B}e_i^\perp) \le 2\mu(\frac{t}{8};\sqrt{2B})\mu(\frac{t}{8};\sqrt{2B}e_i^\perp),$$

that is, $\mu(\frac{t}{4};(x_n - x)e_i^\perp) \to 0$ as $e_i^\perp \downarrow 0$. By the assumption that $x_n \to x$ locally in measure, we have $\mu(e_i(x_n - x)e_i) \to_n 0$ for every fixed e_i, and therefore, $\mu(t; x_n - x) \to_n 0$.

Note that for any net of projections $\{p_i\}$ decreasing to 0, we have

$$\sup_n \|p_i(x_n - x)p_i\|_E = \sup_n \|p_i(x_n + B - x - B)p_i\|_E \le 2\|p_i 2Bp_i\|_E \to 0,$$

which implies that $\{x_n - x\}$ is a bounded set of uniformly absolutely continuous norm [17, Definition 3.3]. Hence, by [17, Corollary 3.5], we have $\|x_n - x\|_E \to 0$. □

We call reader's attention to the connection between order continuous norms and weak compactness of the interval $\{y : 0 \le y \le x\}$. In the classical (commutative) case this connection can be found in [41, Theorem 2.4.2]. A noncommutative analogue of the latter result is contained in [17, Proposition 4.3]. The following proposition is a direct corollary.

Proposition 4.13. *Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$. If $E(\mathcal{M},\tau)$ is a Banach symmetric operator space, then the norm on $E(\mathcal{M},\tau)$ is order continuous if and only if the set I_B is weakly compact for every $B \in E(\mathcal{M},\tau)^+$.*

Acknowledgments

Research of the first author was supported by the subsidies allocated to Kazan Federal University for the state assignment in the sphere of scientific activities (1.9773.2017/8.9, 1.12878.2018/12.1). Research of the second author was partially supported by the Australian Research Council (FL170100052).

We are very grateful to the Referee for really helpful remarks and suggestions.

References

[1] C.D. Aliprantis, O. Burkinshaw, Locally Solid Riesz Spaces with Applications to Economics, second edition, Mathematical Surveys and Monographs, vol. 105, American Mathematical Society, Providence, RI, 2003.

[2] Jonathan Arazy, Pei-Kee Lin, On p-convexity and q-concavity of unitary matrix spaces, Integral Equations Operator Theory 8 (3) (1985) 295–313.

[3] S.V. Astashkin, N. Kalton, F.A. Sukochev, Cesaro mean convergence of martingale differences in rearrangement invariant spaces, Positivity 12 (3) (2008) 387–406.

[4] T.N. Bekjan, D. Dauitbek, Submajorization inequalities of τ-measurable operators for concave and convex functions, Positivity 19 (2) (2015) 341–345.

[5] A.F. Ber, B. de Pagter, F.A. Sukochev, Derivations in algebras of operator-valued functions, J. Operator Theory 66 (2) (2011) 261–300.

[6] A.M. Bikchentaev, On a lemma of F.A. Berezin, Math. Notes 87 (5–6) (2010) 768–773.

[7] A.M. Bikchentaev, Block projection operator in normed solid spaces of measurable operators, Russian Math. (Izv. VUZ) 56 (2) (2012) 75–79.

[8] A.M. Bikchentaev, On Hermitian operators X and Y meeting the condition $-Y \le X \le Y$, Lobachevskii J. Math. 34 (3) (2013) 227–233.

[9] A.M. Bikchentaev, On operator monotone and operator convex functions, Russian Math. (Izv. VUZ) 60 (5) (2016) 61–65.

[10] A.M. Bikchentaev, Ideal spaces of measurable operators affiliated to a semifinite von Neumann algebra, Sib. Math. J. 59 (2) (2018) 243–251.

[11] A.M. Bikchentaev, On sets of measurable operators convex and closed in topology of convergence in measure, Dokl. Math. 98 (3) (2018) 545–548.

[12] A.M. Bikchentaev, A.A. Sabirova, Dominated convergence in measure on semifinite von Neumann algebras and arithmetic averages of measurable operators, Sib. Math. J. 53 (2) (2012) 207–216.

[13] V.I. Bogachev, O.G. Smolyanov, Real and Functional Analysis: A University Course, Regular and Chaotic Dynamics, Moscow, 2009 (Russian).
[14] V.I. Chilin, F.A. Sukochev, Weak convergence in non-commutative symmetric spaces, J. Operator Theory 31 (1) (1994) 35–65.
[15] V.I. Chilin, P.G. Dodds, A.A. Sedaev, F.A. Sukochev, Characterisations of Kadec–Klee properties in symmetric spaces of measurable functions, Trans. Amer. Math. Soc. 348 (12) (1996) 4895–4918.
[16] J. Długosz, A connection between spectral radius and trace, Colloq. Math. 44 (2) (1988) 323–326.
[17] P.G. Dodds, Sets of uniformly absolutely continuous norm in symmetric spaces of measurable operators, Trans. Amer. Math. Soc. 368 (6) (2016) 4315–4355.
[18] P.G. Dodds, de Ben Pagter, The non-commutative Yosida–Hewitt decomposition revisited, Trans. Amer. Math. Soc. 364 (12) (2012) 6425–6457.
[19] P. Dodds, B. de Pagter, Normed Köthe spaces: a non-commutative viewpoint, Indag. Math. (N.S.) 25 (2) (2014) 206–249.
[20] P.G. Dodds, T.K.-Y. Dodds, Lipschitz continuity of the absolute value and Riesz projections in symmetric operator spaces, J. Funct. Anal. 148 (1) (1997) 28–69.
[21] P.G. Dodds, T.K.-Y. Dodds, Ben de Pagter, Noncommutative Köthe duality, Trans. Amer. Math. Soc. 339 (2) (1993) 717–750.
[22] P. Dodds, T. Dodds, P. Dowling, C. Lennard, F. Sukochev, A uniform Kadec–Klee property for symmetric operator spaces, Math. Proc. Cambridge Philos. Soc. 118 (3) (1995) 487–502.
[23] P.G. Dodds, T.K. Dodds, F.A. Sukochev, O.Ye. Tikhonov, A non-commutative Yosida–Hewitt theorem and convex sets of measurable operators closed locally in measure, Positivity 9 (3) (2005) 457–484.
[24] P. Dodds, T. Dodds, F. Sukochev, On p-convexity and q-concavity in non-commutative symmetric spaces, Integral Equations Operator Theory 78 (2014) 91–114.
[25] P. Dodds, B. de Pagter, F. Sukochev, Theory of noncommutative integration, unpublished manuscript.
[26] N. Dunford, J.T. Schwartz, Linear Operators. I. General Theory, With the assistance of W.G. Bade and R.G. Bartle, Pure and Applied Mathematics, vol. 7, Interscience Publishers, Inc., 1958.
[27] T. Fack, H. Kosaki, Generalized s-numbers of τ-measurable operators, Pacific J. Math. 123 (2) (1986) 269–300.
[28] I.C. Gohberg, M.G. Kreĭn, Introduction to the Theory of Linear Nonselfadjoint Operators, Transl. Math. Monographs, vol. 18, Amer. Math. Soc., Providence, RI, 1969.
[29] W.L. Green, T.D. Morley, The extreme points of order intervals of positive operators, Adv. in Appl. Math. 15 (3) (1994) 360–370.
[30] F. Hansen, An operator inequality, Math. Ann. 246 (3) (1980) 249–250.
[31] J. Huang, F. Sukochev, Interpolation between $L_0(\mathcal{M},\tau)$ and $L_\infty(\mathcal{M},\tau)$, submitted for publication.
[32] J. Huang, G. Levitina, F. Sukochev, Completeness of symmetric Δ-normed spaces of τ-measurable operators, Studia Math. 237 (3) (2017) 201–219.
[33] J. Huang, F. Sukochev, D. Zanin, Logarithmic submajorization and order-preserving isometries, arXiv:1808.10557v1, submitted for publication.
[34] H. Hudzik, L. Maligranda, An interpolation theorem in symmetric function F-spaces, Proc. Amer. Math. Soc. 110 (1) (1990) 89–96.
[35] R.V. Kadison, J.R. Ringrose, Fundamentals of the Theory of Operator Algebras. Vol. II. Advanced Theory, Pure Appl. Math., vol. 100, Academic Press, New York and London, 1986.
[36] N.J. Kalton, F.A. Sukochev, Symmetric norms and spaces of operators, J. Reine Angew. Math. 621 (2008) 81–121.
[37] A.V. Krygin, E.M. Sheremet'ev, F. Sukochev, Conjugation of weak and measure convergence in noncommutative symmetric spaces, Dokl. Akad. Nauk UzSSR (2) (1993) 8–9 (in Russian).
[38] A.V. Krygin, E.M. Sheremet'ev, F. Sukochev, Convergence in measure, weak convergence and structure of subspaces in symmetric spaces of measurable operators, 1993, unpublished manuscript.
[39] S. Lord, F. Sukochev, D. Zanin, Singular Traces. Theory and Applications, de Gruyter Studies in Mathematics, vol. 46, 2013.
[40] G.Ya. Lozanovsky, Lozanovsky's Notebooks. Part II. Problems 610–1469 (Notebooks IV–XII), Translated from the Russian and edited by Marek Wójtowicz, Wydawnictwo Uniwersytetu Kazimierza Wielkiego, Bydgoszcz, 2012.
[41] P. Meyr-Nieberg, Banach Lattices, Springer-Verlag, Berlin–Heidelberg–New York, 1991.
[42] F.J. Narcowich, R-operators. II. On the approximation of certain operator-valued analytic functions and the Hermitian moment problem, Indiana Univ. Math. J. 26 (3) (1977) 483–513.
[43] F.J. Narcowich, On the extreme points of the interval between two operators, Proc. Amer. Math. Soc. 67 (1) (1977) 84–86.
[44] E. Nelson, Notes on non-commutative integration, J. Funct. Anal. 15 (2) (1974) 103–116.
[45] Andrej Novikov, L_1-space for a positive operator affiliated with von Neumann algebra, Positivity 21 (1) (2017) 359–375.
[46] M. Rosenblum, J. Rovnyak, Hardy Classes and Operator Theory, Oxford University Press, New York, 1985.
[47] I.E. Segal, A non-commutative extension of abstract integration, Ann. Math. 57 (3) (1953) 401–457.
[48] Ş. Strătilă, L. Zsidó, Lectures on von Neumann Algebras, Abacus Press, England, 1979.
[49] F. Sukochev, Completeness of quasi-normed symmetric operator spaces, Indag. Math. (N.S.) 25 (2) (2014) 376–388.
[50] M. Takesaki, Theory of Operator Algebras I, Springer-Verlag, New York, 1979.
[51] O.E. Tikhonov, Continuity of operator functions in topologies connected with a trace on a von Neumann algebra, Sov. Mat. (Izv. VUZ) 31 (1) (1987) 110–114.
[52] O.E. Tikhonov, Subadditivity inequalities in von Neumann algebras and characterization of tracial functionals, Positivity 9 (2) (2005) 259–264.
[53] F.J. Yeadon, Convergence of measurable operators, Proc. Camb. Philos. Soc. 74 (2) (1973) 257–268.
[54] K. Yosida, Functional Analysis, sixth edition, Fundamental Principles of Mathematical Sciences, vol. 123, Springer-Verlag, Berlin–New York, 1980.

Please cite this article in press as: A. Bikchentaev, F. Sukochev, When weak and local measure convergence implies norm convergence, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.01.028