



# Ghost-free $F(R, \mathcal{G})$ gravity

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## Abstract

In this work we shall address the ghost issue of  $F(R, \mathcal{G})$  gravity, which is known to be plagued with ghost degrees of freedom. These ghosts occur due to the presence of higher than two derivatives in the field equations, and can arise even when considering cosmological perturbations, where superluminal modes may arise in the theory. If we consider the quantum theory, the ghosts generate the negative norm states, which give the negative probabilities, and therefore the ghosts are physically inconsistent. Motivated by the importance of  $F(R, \mathcal{G})$  gravity for providing viable inflationary and dark energy phenomenologies, in this work we shall provide a technique that can render  $F(R, \mathcal{G})$  gravity theories free from ghost degrees of freedom. This will be done by introducing two auxiliary scalar fields, and by employing the Lagrange multiplier technique, the theory is ghost free in the Einstein frame. Also the framework can be viewed as a reconstruction technique and can be used as a method in order to realize several cosmological evolutions of interest. We demonstrate how we can realize several cosmologically interesting phenomenologies by using the reconstruction technique.

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## 1. Introduction

Modified gravity in its various forms [1–7] plays a prominent role towards the complete understanding of how the Universe evolves, due to the fact that Einstein-Hilbert gravity seems to be unable to describe the late-time acceleration era [8–22]. Apart from the dark energy description, modified gravity can provide a viable and perhaps necessary description for the early time acceleration dubbed inflation [23–25]. Single scalar field descriptions of inflation for the moment are quite popular, but these models provide an inflationary era with specific characteristics and there is not much freedom in model building in these theories. Specifically, single scalar field models lead to a Gaussian power spectrum which can be compatible with the latest Planck data [26], however the tensor spectral index is red-tilted and obeys the consistency relation, a fact that restricts too much the tensor spectrum. If in the upcoming observational data of the LISA mission [27,28] in about fifteen years from now, primordial gravitational waves signal will be found, single scalar field inflation will be instantly unable to provide a description for the early time era evolution of our Universe. This is due to the fact that currently the single scalar field prediction for the power spectrum of the primordial gravitational waves is way lower compared to the LISA sensitivity curve. Thus if a signal of primordial tensor spectrum is discovered, single scalar field cannot be the one generating this spectrum, unless the scalar field is a tachyon, which is quite problematic scenario. These issues render modified gravity a quite timely and appealing candidate for the unified description of inflation and dark energy simultaneously. The first work toward this direction was performed in the pioneer work [29]. Two of the most appealing characteristic modified gravities are  $f(R)$  and Gauss-Bonnet gravity [30–41] and higher order extensions of the above two [34,42–51]. In this work we shall consider  $F(R, \mathcal{G})$  gravity theories [34,46–51], and we shall consider the problem of having ghost degrees of freedom in these theories. It is known that  $F(R, \mathcal{G})$  gravity theories are plagued with ghost degrees of freedom which can occur at many levels in the theory, even as perturbative propagating modes when cosmological perturbations are considered. Furthermore if we consider the quantum theory, the ghosts generate the negative norm states, which give the negative probabilities, and therefore the ghosts are physically inconsistent. We aim to demonstrate how these ghost degrees of freedom are generated and we will provide a theoretical framework for  $F(R, \mathcal{G})$  which can yield a ghost-free framework. We also use the provided theoretical framework as a reconstruction technique in order to realize several cosmological evolutions of interest.

The paper is organized as follows: In section 2, we demonstrate how ghost degrees of freedom can arise in  $F(R, \mathcal{G})$  gravity. In section 3, we provide a ghost-free theoretical framework for  $F(R, \mathcal{G})$  gravity and in section 4, we describe in detail the cosmological framework of the ghost-free  $F(R, \mathcal{G})$  gravity, providing also the reconstruction technique for it. In section 5, we use the reconstruction technique for realizing several cosmological evolutions of interest, and finally the conclusions follow in the end of the article.

Throughout this paper we use the conventions for the curvature tensor  $R^\lambda_{\mu\rho\nu} = -\Gamma^\lambda_{\mu\rho,\nu} + \Gamma^\lambda_{\mu\nu,\rho} - \Gamma^\eta_{\mu\rho}\Gamma^\lambda_{\nu\eta} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\rho\eta}$  and for the Ricci tensor  $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$ .

## 2. Occurrence of ghost degrees of freedom in $F(R, \mathcal{G})$ gravity

In this section, we shall demonstrate that the  $F(R, \mathcal{G})$  has inherent ghost degrees of freedom. The  $F(R, \mathcal{G})$  gravity in vacuum has the following action,

$$S = \int d^4x \sqrt{-g} F(R, \mathcal{G}), \quad (1)$$

where  $F(R, \mathcal{G})$  is a function of the scalar curvature  $R$  and  $\mathcal{G}$  stands for the Gauss-Bonnet invariant given by,

$$\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \quad (2)$$

In general, the model (1) leads to ghost instabilities and ghost degrees of freedom, that eventually appear even at the level of cosmological perturbations. As an explicit example, we investigate the so-called  $f(\mathcal{G})$  gravity, which is a special model of  $F(R, \mathcal{G})$  gravity, as in [52], with action,

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa^2} R + f(\mathcal{G}) + \mathcal{L}_{\text{matter}} \right). \quad (3)$$

Upon variation of the action with respect to the metric, we obtain the following equation of motion,

$$\begin{aligned} 0 = & \frac{1}{2\kappa^2} \left( -R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R \right) + \frac{1}{2}T_{\text{matter}\mu\nu} + \frac{1}{2}g_{\mu\nu} (f(\mathcal{G}) - \mathcal{G}f'(\mathcal{G})) \\ & + D_{\mu\nu}{}^{\tau\eta} \nabla_\tau \nabla_\eta f'(\mathcal{G}), \\ D_{\mu\nu}{}^{\tau\eta} \equiv & (\delta_\mu{}^\tau \delta_\nu{}^\eta + \delta_\nu{}^\tau \delta_\mu{}^\eta - 2g_{\mu\nu}g^{\tau\eta}) R \\ & + (-4g^{\rho\tau} \delta_\mu{}^\eta \delta_\nu{}^\sigma - 4g^{\rho\tau} \delta_\nu{}^\eta \delta_\mu{}^\sigma + 4g_{\mu\nu}g^{\rho\tau}g^{\sigma\eta}) R_{\rho\sigma} \\ & + 4R_{\mu\nu}g^{\tau\eta} - 2R_{\rho\mu\sigma\nu} (g^{\rho\tau}g^{\sigma\eta} + g^{\rho\eta}g^{\sigma\tau}). \end{aligned} \quad (4)$$

Let a solution of (4) be  $g_{\mu\nu} = g_{\mu\nu}^{(0)}$  and we denote the curvatures and connections given by  $g_{\mu\nu}^{(0)}$  by using the indexes “(0)”. Then in order to investigate if any ghost could exist, we may consider the perturbation of (4) around the solution  $g_{\mu\nu}^{(0)}$  as follows  $g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}$ . For the variation of  $\delta g_{\mu\nu}$ , we may impose the transverse gauge condition  $0 = \nabla^\mu \delta g_{\mu\nu}$ , and further if we impose the condition  $\delta g^\mu{}_\mu = 0$ , we find

$$\begin{aligned} \delta\mathcal{G} = & -2RR^{\mu\nu}\delta g_{\mu\nu} + 8R^{\rho\sigma}R^\mu{}_\rho{}^\nu{}_\sigma\delta g_{\mu\nu} + 4R^{\mu\nu}\nabla^2\delta g_{\mu\nu} - 2R^{\mu\rho\sigma\tau}R^\nu{}_{\rho\sigma\tau}\delta g_{\mu\nu} \\ & - 4R^{\rho\mu\sigma\nu}\nabla_\rho\nabla_\sigma\delta g_{\mu\nu}, \end{aligned} \quad (5)$$

which also contains the second derivative of the metric  $g_{\mu\nu}$  with respect the cosmic time coordinate. Under the perturbation  $g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}$ , the term  $D_{\mu\nu}{}^{\tau\eta} \nabla_\tau \nabla_\eta f'(\mathcal{G})$  takes the following form,

$$D_{\mu\nu}{}^{\tau\eta} \nabla_\tau \nabla_\eta f'(\mathcal{G}) \rightarrow D_{\mu\nu}{}^{\tau\eta} \nabla_\tau \nabla_\eta f'(\mathcal{G}^{(0)}) + D_{\mu\nu}{}^{\tau\eta} \nabla_\tau \nabla_\eta (f''(\mathcal{G}^{(0)})\delta\mathcal{G}) + \dots, \quad (6)$$

which contains the fourth derivative of the metric  $g_{\mu\nu}$  with respect to the cosmic time coordinate, and therefore the perturbed equation (4) will have a ghost mode. Note that in Eq. (6), the “...” expresses the terms occurring from the variation of  $D_{\mu\nu}{}^{\tau\eta} \nabla_\tau \nabla_\eta$ . The propagating mode is a scalar expressed by the Gauss-Bonnet invariant as it is clear from Eq. (4).

### 3. Ghost-free $F(R, \mathcal{G})$ gravity

We review on the construction of the ghost-free  $F(R, \mathcal{G})$  theory of gravity based on [52]. By introducing two auxiliary fields  $\Phi$  and  $\Theta$ , the action of Eq. (1) can be rewritten as follows,

$$S = \int d^4x \sqrt{-g} \left\{ \frac{\Phi R}{2\kappa^2} + \Theta \mathcal{G} - V(\Phi, \Theta) \right\}, \tag{7}$$

where we have introduced the gravitational coupling  $\kappa$  in order to make  $\Phi$  and  $\Theta$  dimensionless. By varying the action (7) with respect to the scalar fields  $\Phi$  and  $\Theta$ , we obtain,

$$\frac{R}{2\kappa^2} = \frac{\partial V(\Phi, \Theta)}{\partial \Phi}, \quad \mathcal{G} = \frac{\partial V(\Phi, \Theta)}{\partial \Theta}, \tag{8}$$

which can be algebraically solved with respect  $\Phi$  and  $\Theta$ , that is,  $\Phi = \Phi(R, \mathcal{G})$  and  $\Theta = \Theta(R, \mathcal{G})$ . Then by substituting the obtained expressions for  $\Phi = \Phi(R, \mathcal{G})$  and  $\Theta = \Theta(R, \mathcal{G})$  in Eq. (7), we obtain the action, (1) with,

$$F(R, \mathcal{G}) = \frac{\Phi(R, \mathcal{G}) R}{2\kappa^2} + \Theta(R, \mathcal{G}) \mathcal{G} - V(\Phi(R, \mathcal{G}), \Theta(R, \mathcal{G})). \tag{9}$$

In order to investigate the properties of the action (7), we work in the Einstein frame, so under a conformal transformation of the form  $g_{\mu\nu} \rightarrow e^\phi g_{\mu\nu}$ , the curvatures are transformed as follows [29,53],

$$\begin{aligned} R_{\zeta\mu\rho\nu} &\rightarrow \left\{ R_{\zeta\mu\rho\nu} - \frac{1}{2} (g_{\zeta\rho} \nabla_\nu \nabla_\mu \phi + g_{\mu\nu} \nabla_\rho \nabla_\zeta \phi - g_{\mu\rho} \nabla_\nu \nabla_\zeta \phi - g_{\zeta\nu} \nabla_\rho \nabla_\mu \phi) \right. \\ &\quad + \frac{1}{4} (g_{\zeta\rho} \partial_\nu \phi \partial_\mu \phi + g_{\mu\nu} \partial_\rho \phi \partial_\zeta \phi - g_{\mu\rho} \partial_\nu \phi \partial_\zeta \phi - g_{\zeta\nu} \partial_\rho \phi \partial_\mu \phi) \\ &\quad \left. - \frac{1}{4} (g_{\zeta\rho} g_{\mu\nu} - g_{\zeta\nu} g_{\mu\rho}) \partial^\sigma \phi \partial_\sigma \phi \right\}, \\ R_{\mu\nu} &\rightarrow R_{\mu\nu} - \frac{1}{2} (2\nabla_\mu \nabla_\nu \phi + g_{\mu\nu} \square \phi) + \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\sigma \phi \partial_\sigma \phi, \\ R &\rightarrow \left( R - 3\square \phi - \frac{3}{2} \partial^\sigma \phi \partial_\sigma \phi \right) e^{-\phi}. \end{aligned} \tag{10}$$

Therefore the Gauss-Bonnet invariant  $\mathcal{G}$  is transformed in the following way,

$$\begin{aligned} \mathcal{G} &\rightarrow e^{-2\phi} \left[ \mathcal{G} + \nabla_\mu \left\{ 4 \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \partial_\nu \phi + 2 (\partial^\mu \phi \square \phi - (\nabla_\nu \nabla^\mu \phi) \partial^\nu \phi) \right. \right. \\ &\quad \left. \left. + \partial_\nu \phi \partial^\nu \phi \partial^\mu \phi \right\} \right]. \end{aligned} \tag{11}$$

Then by writing  $\Phi = e^{-\phi}$ , the action of Eq. (7) can be rewritten by taking into account the conformal transformation  $g_{\mu\nu} \rightarrow e^\phi g_{\mu\nu}$  as follows,

$$\begin{aligned} S &= \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left( R - \frac{3}{2} \partial^\sigma \phi \partial_\sigma \phi \right) \right. \\ &\quad + \Theta \mathcal{G} - \partial_\mu \Theta \left\{ 4 \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \partial_\nu \phi + 2 (\partial^\mu \phi \square \phi - (\nabla_\nu \nabla^\mu \phi) \partial^\nu \phi) \right. \\ &\quad \left. \left. + \partial_\nu \phi \partial^\nu \phi \partial^\mu \phi \right\} - e^{2\phi} V(e^{-\phi}, \Theta) \right\}. \end{aligned} \tag{12}$$

This action (12) may have ghost degrees of freedom due to the existence of  $\Theta$ . We might eliminate the ghost degrees of freedom by writing  $\Theta$  as  $\Theta = e^\theta$  and add a constraint to the action (12) by using the Lagrange multiplier field  $\lambda$ , in the following way,

$$\begin{aligned}
 S = \int d^4x \sqrt{-g} & \left\{ \frac{1}{2\kappa^2} \left( R - \frac{3}{2} \partial^\sigma \phi \partial_\sigma \phi - \lambda \left( \partial_\mu \theta \partial^\mu \theta + \mu^2 \right) \right) \right. \\
 & + e^\theta \mathcal{G} - e^\theta \partial_\mu \theta \left\{ 4 \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \partial_\nu \phi + 2 \left( \partial^\mu \phi \square \phi - (\nabla_\nu \nabla^\mu \phi) \partial^\nu \phi \right) \right. \\
 & \left. \left. + \partial_\nu \phi \partial^\nu \phi \partial^\mu \phi \right\} - e^{2\phi} V \left( e^{-\phi}, e^\theta \right) \right\}. \tag{13}
 \end{aligned}$$

Then the scalar fields  $\theta$  and  $\lambda$  become non-dynamical degrees of freedom and the dynamical degrees of freedom are actually the metric and the scalar field  $\phi$ , as in the standard  $F(R)$  gravity, therefore no-ghost degrees of freedom occur in the theory.

We should note

$$\epsilon^\xi \eta^{\mu\nu} \epsilon_{\xi\eta}^{\rho\sigma} \partial_\mu \theta \partial_\xi \phi \nabla_\nu \nabla_\sigma \phi = 2\partial_\theta \left( \partial^\mu \phi \square \phi - (\nabla_\nu \nabla^\mu \phi) \partial^\nu \phi \right), \tag{14}$$

whose structure appears in the action (13) and also the Galileon model [54–56]. Therefore in the field equations, the term given by the variation of the structure (14) does not include the derivative with respect to time  $t$  higher than two, which tells that there could not appear any ghost coming from the higher derivative terms. As clear from Eq. (11), the structure (14) appears due to the combination in the Gauss-Bonnet invariant (2). Therefore if we consider the action including the combination  $R^2 + bR_{\mu\nu}R^{\mu\nu} + cR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$  different from the combination in the Gauss-Bonnet invariant (2) ( $b \neq -4c$ ), or in more general, the Lagrangian density  $F(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$ , there appears the ghost.

#### 4. Field equations and formalism in the ghost-free $F(R, \mathcal{G})$ gravity

Upon varying the action (13) with respect to  $\lambda$ ,  $\theta$ ,  $\phi$ , and metric  $g_{\mu\nu}$ , we obtain the following field equations,

$$0 = \partial_\mu \theta \partial^\mu \theta + \mu^2, \tag{15}$$

$$\begin{aligned}
 0 = \frac{1}{\kappa^2} \nabla^\mu \left( \lambda \partial_\mu \theta \right) + e^\theta \mathcal{G} + e^\theta & \left\{ 4 \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \nabla_\mu \nabla_\nu \phi \right. \\
 & + 2 \left( (\square \phi)^2 + \partial^\mu \phi \partial_\mu \square \phi - (\nabla_\mu \nabla_\nu \nabla^\mu \phi) \partial^\nu \phi - (\nabla_\nu \nabla^\mu \phi) (\nabla_\mu \nabla^\nu \phi) \right) \\
 & \left. + 2 (\nabla_\mu \nabla_\nu \phi) \partial^\nu \phi \partial^\mu \phi + \partial_\nu \phi \partial^\nu \phi \square \phi \right\}, \tag{16}
 \end{aligned}$$

$$\begin{aligned}
 0 = \frac{3}{2\kappa^2} \square \phi + 4 \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \nabla_\nu \nabla_\mu e^\theta + 2 (\square e^\theta) \square \phi + 2 (\partial_\mu e^\theta) \partial^\mu \square \phi \\
 - 2 \square \left( (\partial_\mu e^\theta) \partial^\mu \phi \right) + 2 \nabla^\mu \nabla_\nu \left( (\partial_\mu e^\theta) \partial^\nu \phi \right) \\
 - 2 \nabla^\nu \left( (\partial_\mu e^\theta) \nabla_\nu \nabla^\mu \phi \right) + 2 \nabla_\nu \left( (\partial_\mu e^\theta) \partial^\nu \phi \partial^\mu \phi \right) + \nabla^\mu \left( (\partial_\mu e^\theta) \partial_\nu \phi \partial^\nu \phi \right) \\
 - 2e^{2\phi} V \left( e^{-\phi}, e^\theta \right) + e^\phi \frac{\partial V \left( e^{-\phi}, e^\theta \right)}{\partial e^{-\phi}}, \tag{17}
 \end{aligned}$$

$$\begin{aligned}
0 = & -\frac{1}{2\kappa^2} \left( R_{\mu\nu} - \frac{3}{2} \partial_\mu \phi \partial_\nu \phi - \lambda \partial_\mu \theta \partial_\nu \theta \right) \\
& + \frac{1}{2} g_{\mu\nu} \left\{ \frac{1}{2\kappa^2} \left( R - \frac{3}{2} \partial^\sigma \phi \partial_\sigma \phi - \lambda \left( \partial_\mu \theta \partial^\mu \theta + \mu^2 \right) \right) \right. \\
& - e^\theta \partial_\rho \theta \left\{ 4 \left( R^{\rho\sigma} - \frac{1}{2} g^{\rho\sigma} R \right) \partial_\sigma \phi + 2 \left( \partial^\rho \phi \square \phi - (\nabla_\sigma \nabla^\rho \phi) \partial^\sigma \phi \right) + \partial_\sigma \phi \partial^\sigma \phi \partial^\rho \phi \right\} \\
& \left. - e^{2\phi} V \left( e^{-\phi}, e^\theta \right) \right\} \\
& + 2 \left( \nabla_\mu \nabla_\nu e^\theta \right) R - 2 g_{\mu\nu} \left( \square e^\theta \right) R - 4 \left( \nabla^\rho \nabla_\mu e^\theta \right) R_{\nu\rho} - 4 \left( \nabla^\rho \nabla_\nu e^\theta \right) R_{\mu\rho} \\
& + 4 \left( \square e^\theta \right) R_{\mu\nu} + 4 g_{\mu\nu} \left( \nabla_\rho \nabla_\sigma e^\theta \right) R^{\rho\sigma} - 4 \left( \nabla^\rho \nabla^\sigma e^\theta \right) R_{\mu\rho\nu\sigma} \\
& - \frac{1}{2} e^\theta \partial_\mu \theta \left\{ 4 R_\nu{}^\rho \partial_\rho \phi - 2 R \partial_\nu \phi + 2 \left( \partial_\nu \phi \square \phi - (\nabla_\rho \nabla_\nu \phi) \partial^\rho \phi \right) + \partial_\rho \phi \partial^\rho \phi \partial_\nu \phi \right\} \\
& - \frac{1}{2} e^\theta \partial_\nu \theta \left\{ 4 R_\mu{}^\rho \partial_\rho \phi - 2 R \partial_\mu \phi + 2 \left( \partial_\mu \phi \square \phi - (\nabla_\rho \nabla_\mu \phi) \partial^\rho \phi \right) + \partial_\rho \phi \partial^\rho \phi \partial_\mu \phi \right\} \\
& + 2 \left( \partial^\rho e^\theta \right) R_{\rho\mu} \partial_\nu \phi + 2 \left( \partial^\rho e^\theta \right) R_{\rho\nu} \partial_\mu \phi - 2 \left( \partial^\rho e^\theta \right) \partial_\rho \phi R_{\mu\nu} - 2 \nabla_\rho \nabla_\nu \left( \left( \partial^\rho e^\theta \right) \partial_\mu \phi \right) \\
& - 2 \nabla_\rho \nabla_\mu \left( \left( \partial^\rho e^\theta \right) \partial_\nu \phi \right) + \square \left( \left( \partial_\mu e^\theta \right) \partial_\nu \phi + \left( \partial_\nu e^\theta \right) \partial_\mu \phi \right) \\
& + g_{\mu\nu} \left( \nabla_\rho \nabla_\sigma + \nabla_\sigma \nabla_\rho \right) \left( \left( \partial^\rho e^\theta \right) \partial^\sigma \phi \right) \\
& + 2 \left( \nabla_\mu \nabla_\nu - g_{\mu\nu} \square \right) \left( \left( \partial^\rho e^\theta \right) \partial_\rho \phi \right) \\
& + \nabla_\mu \left( \left( \partial_\rho e^\theta \right) \partial^\rho \phi \partial_\nu \phi \right) + \nabla_\nu \left( \left( \partial_\rho e^\theta \right) \partial^\rho \phi \partial_\mu \phi \right) - g_{\mu\nu} \nabla_\rho \left( \left( \partial_\sigma e^\theta \right) \partial^\sigma \phi \partial^\rho \phi \right) \\
& - \left( \partial_\rho e^\theta \right) \left( \left( \nabla_\mu \nabla^\rho \phi \right) \partial_\nu \phi + \left( \nabla_\nu \nabla^\rho \phi \right) \partial_\mu \phi \right) \\
& - \nabla^\rho \left( \left( \partial_\mu e^\theta \right) \partial_\nu \phi \partial_\rho \phi \right) - \nabla^\rho \left( \left( \partial_\nu e^\theta \right) \partial_\mu \phi \partial_\rho \phi \right) + \frac{1}{2} \nabla^\rho \left( \left( \partial_\mu e^\theta \right) \partial_\rho \phi \partial_\nu \phi \right) \\
& + \frac{1}{2} \nabla^\rho \left( \left( \partial_\nu e^\theta \right) \partial_\rho \phi \partial_\mu \phi \right) + \left( \partial_\rho e^\theta \right) \partial^\rho \phi \partial_\mu \phi \partial_\nu \phi. \tag{18}
\end{aligned}$$

We now assume that the background metric is the Friedmann-Robertson-Walker (FRW) space-time with a flat spatial part,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2, \tag{19}$$

and we also assume  $\theta$  and  $\phi$  only depend on the cosmic time  $t$ . Then Eq. (15) yields,

$$\theta = \pm \mu t. \tag{20}$$

We absorb the signature  $\pm$  into the redefinition of  $\mu$  by assuming that  $\mu$  can be negative, hence  $\theta = \mu t$ . Then Eqs. (16), and (17) take the following forms,

$$\begin{aligned}
0 = & -\frac{\mu}{\kappa^2} (\dot{\lambda} + 3H\lambda) + e^{\mu t} \left[ 24H^2 \left( \dot{H} + H^2 \right) - 12H^2 \ddot{\phi} - 12 \left( 2\dot{H} + 3H^2 \right) H\dot{\phi} \right. \\
& \left. + 18H^2 \dot{\phi}^2 + 12H\ddot{\phi}\dot{\phi} + 6\dot{H}\dot{\phi}^2 + 3 \left( \ddot{\phi}\dot{\phi}^2 + H\dot{\phi}^3 \right) \right] - e^{2\phi + \mu t} \left. \frac{\partial V \left( e^{-\phi}, e^\theta \right)}{\partial e^\theta} \right|_{\theta = \mu t}, \tag{21}
\end{aligned}$$

$$\begin{aligned}
 0 = & -\frac{3}{2\kappa^2} (\ddot{\phi} + 3H\dot{\phi}) \\
 & + \left[ \mu \left( 9H\dot{\phi}^2 + 36H^2\dot{\phi} - 36H^3 + 6\dot{\phi}\ddot{\phi} + 12\dot{H}\dot{\phi} + 12H\ddot{\phi} - 24H\dot{H} \right) \right. \\
 & \left. + \mu^2 \left( 3\dot{\phi}^2 + 12H\dot{\phi} - 12H^2 \right) \right] e^{\mu t} - 2e^{2\phi} V(e^{-\phi}, e^{\mu t}) + e^{\phi} \frac{\partial V(e^{-\phi}, e^{\theta})}{\partial e^{-\phi}} \Bigg|_{\theta=\mu t}, \tag{22}
 \end{aligned}$$

where  $H = \dot{a}/a$  and the  $(\mu, \nu) = (t, t)$  and  $(\mu, \nu) = (i, j)$  components in Eq. (18) yield,

$$\begin{aligned}
 0 = & \frac{1}{\kappa^2} \left( 3H^2 - \frac{3}{4}\dot{\phi}^2 - \lambda\mu^2 \right) + \left( 3\dot{\phi}^3 - 36H^2\dot{\phi} + 18H\dot{\phi}^2 + 24H^3 \right) \mu e^{\mu t} \\
 & - e^{2\phi} V(e^{-\phi}, e^{\theta}) \Big|_{\theta=\mu t}, \tag{23}
 \end{aligned}$$

$$\begin{aligned}
 0 = & \frac{1}{\kappa^2} \left( 2\dot{H} + \frac{3}{4}\dot{\phi}^2 + 3H^2 \right) + \left( 8H^2 + 2\dot{\phi}^2 - 8H\dot{\phi} \right) \mu^2 e^{\mu t} \\
 & + \left( -8\dot{H}\dot{\phi} + 4\dot{\phi}\ddot{\phi} + 16H^3 - 8H\ddot{\phi} - \dot{\phi}^3 + 16H\dot{H} - 12H^2\dot{\phi} \right) \mu e^{\mu t} \\
 & - e^{2\phi} V(e^{-\phi}, e^{\theta}) \Big|_{\theta=\mu t}. \tag{24}
 \end{aligned}$$

We should note that all the equations, (21), (22), (23), and (24) do not contain higher order derivatives and the maximum order of derivatives contained is two, a fact that indicates that the theory is ghost-free.

By combining Eqs. (23) and (24), we may delete  $V(e^{-\phi}, e^{\theta})$  and we can solve the obtained equation with respect to  $\lambda$ ,

$$\begin{aligned}
 \frac{\mu^2}{\kappa^2} \lambda = & -\frac{1}{\kappa^2} \left( 2\dot{H} + \frac{3}{2}\dot{\phi}^2 \right) - \left( 8H^2 + 2\dot{\phi}^2 - 8H\dot{\phi} \right) \mu^2 e^{\mu t} \\
 & - \left( -8\dot{H}\dot{\phi} + 4\dot{\phi}\ddot{\phi} - 8H^3 - 8H\ddot{\phi} - 18H\dot{\phi}^2 - 4\dot{\phi}^3 + 16H\dot{H} + 24H^2\dot{\phi} \right) \mu e^{\mu t}, \tag{25}
 \end{aligned}$$

which determines the  $t$  dependence of  $\lambda$ . Eq. (23) also determines the  $t$  dependence of  $V(e^{-\phi}, e^{\theta})$ ,

$$\begin{aligned}
 & e^{2\phi} V(e^{-\phi}, e^{\theta}) \Big|_{\theta=\mu t} \\
 = & \frac{1}{\kappa^2} \left( 2\dot{H} + \frac{3}{4}\dot{\phi}^2 + 3H^2 \right) + \left( 8H^2 + 2\dot{\phi}^2 - 8H\dot{\phi} \right) \mu^2 e^{\mu t} \\
 & + \left( -8\dot{H}\dot{\phi} + 4\dot{\phi}\ddot{\phi} + 16H^3 - 8H\ddot{\phi} - \dot{\phi}^3 + 16H\dot{H} - 12H^2\dot{\phi} \right) \mu e^{\mu t}. \tag{26}
 \end{aligned}$$

By eliminating  $\lambda$  in Eq. (21) by using Eq. (25), we get,

$$\begin{aligned}
 0 = & \frac{1}{\kappa^2} \left( 2\ddot{H} + 6H\dot{H} + \frac{9}{2}H\dot{\phi}^2 + 3\dot{\phi}\ddot{\phi} \right) \\
 & + \left( 48H^2\dot{H} + 16\dot{H}^2 + 16H\ddot{H} + 36H^3\dot{\phi} - 8\ddot{H}\dot{\phi} - 12H^2\ddot{\phi} - 16\dot{H}\ddot{\phi} - 8H\ddot{\phi} \right. \\
 & \left. - 36H^2\dot{\phi}^2 - 12\dot{H}\dot{\phi}^2 - 12H\dot{\phi}\ddot{\phi} + 4\ddot{\phi}^2 + 4\dot{\phi}\ddot{\phi} - 9H\dot{\phi}^3 - 9\dot{\phi}^2\ddot{\phi} \right) \mu e^{\mu t}
 \end{aligned}$$

$$\begin{aligned}
& + \left( 16H^3 + 32H\dot{H} - 16\dot{H}\dot{\phi} - 16H\ddot{\phi} + 8\dot{\phi}\ddot{\phi} - 12H\dot{\phi}^2 - 4\dot{\phi}^3 \right) \mu^2 e^{\mu t} \\
& + \left( 8H^2 + 2\dot{\phi}^2 - 8H\dot{\phi} \right) \mu^3 e^{\mu t} \\
& - \mu e^{2\phi + \mu t} \left. \frac{\partial V(e^{-\phi}, e^{\theta})}{\partial e^{\theta}} \right|_{\theta = \mu t}. \tag{27}
\end{aligned}$$

We should note that all the equations (21), (22), (23), and (24) are not independent equations. For example, we consider the derivative of Eq. (24) with respect to  $t$ ,

$$\begin{aligned}
0 = & \frac{1}{\kappa^2} \left( 2\ddot{H} + \frac{3}{2}\dot{\phi}\ddot{\phi} + 6H\dot{H} \right) + (16H\dot{H} + 4\dot{\phi}\ddot{\phi} - 8\dot{H}\dot{\phi} - 8H\ddot{\phi}) \mu^2 e^{\mu t} \\
& + \left( 8H^2 + 2\dot{\phi}^2 - 8H\dot{\phi} \right) \mu^3 e^{\mu t} \\
& + \left( -8\ddot{H}\dot{\phi} - 8\dot{H}\ddot{\phi} + 4\ddot{\phi}^2 + 4\dot{\phi}\ddot{\phi} + 48H^2\dot{H} - 8\dot{H}\ddot{\phi} - 8H\ddot{\phi} - 3\dot{\phi}^2\ddot{\phi} \right. \\
& \left. + 16\dot{H}^2 + 16H\ddot{H} - 24H\dot{H}\dot{\phi} - 12H^2\ddot{\phi} \right) \mu e^{\mu t} \\
& + \left( -8\dot{H}\dot{\phi} + 4\dot{\phi}\ddot{\phi} + 16H^3 - 8H\ddot{\phi} - \dot{\phi}^3 + 16H\dot{H} - 12H^2\dot{\phi} \right) \mu^2 e^{\mu t} \\
& + 192H\dot{H} \left( \dot{H} + H^2 \right) e^{\mu t} + 96H^2 \left( \ddot{H} + 2H\dot{H} \right) e^{\mu t} + 96\mu H^2 \left( \dot{H} + H^2 \right) e^{\mu t} \\
& - 2\dot{\phi} e^{2\phi} V(e^{-\phi}, e^{\theta}) \Big|_{\theta = \mu t} + \dot{\phi} e^{2\phi} \left. \frac{\partial V(e^{-\phi}, e^{\theta})}{\partial e^{-\phi}} \right|_{\theta = \mu t} - \mu e^{2\phi + \mu t} \left. \frac{\partial V(e^{-\phi}, e^{\theta})}{\partial e^{\theta}} \right|_{\theta = \mu t}. \tag{28}
\end{aligned}$$

If we eliminate  $V(e^{-\phi}, e^{\theta})$  in Eq. (28) by using Eqs. (22) and (27), we find that the right hand side vanishes identically. Therefore all the equations (21), (22), (23), and (24) are not independent equations.

## 5. Reconstruction of several cosmologies

In this section, we shall use the framework we developed in the previous section as a reconstruction technique in order to realize several models of cosmological interest. This reconstruction technique is based simply on fixing the behaviors or time dependencies of  $H$  and  $\phi$  as follows,

$$H = H(t), \quad \phi = \phi(t), \tag{29}$$

and eventually we determine which model can realize the given cosmological evolutions. As discussed in the last section, all the equations (21), (22), (23), and (24) are not independent equations. Since we solve the equations with respect to  $\lambda$  in (25), we consider Eq. (26), which is equivalent to Eq. (23), and Eq. (27), which are independent equations which do not include  $\lambda$ . As a working hypothesis, we assume that  $V(e^{-\phi}, e^{\theta})$  is given by a sum of a  $\phi$ -dependent part and of a  $\theta$ -dependent part as follows,

$$V(e^{-\phi}, e^{\theta}) = V_{\Phi}(e^{-\phi}) + V_{\Theta}(e^{\theta}). \tag{30}$$



Then Eq. (27) indicates,

$$\begin{aligned}
 V_{\Theta}(e^{\theta}) = & \int^{\frac{\theta}{\mu}} dt e^{-2\phi} \left[ \frac{1}{\kappa^2} \left( 2\ddot{H} + 6H\dot{H} + \frac{9}{2}H\dot{\phi}^2 + 3\dot{\phi}\ddot{\phi} \right) \right. \\
 & + \left( 48H^2\dot{H} + 16\dot{H}^2 + 16H\ddot{H} + 36H^3\dot{\phi} - 8\ddot{H}\dot{\phi} - 12H^2\ddot{\phi} - 16\dot{H}\ddot{\phi} - 8H\ddot{\phi} \right. \\
 & \left. \left. - 36H^2\dot{\phi}^2 - 12\dot{H}\dot{\phi}^2 - 12H\dot{\phi}\ddot{\phi} + 4\ddot{\phi}^2 + 4\dot{\phi}\ddot{\phi} - 9H\dot{\phi}^3 - 9\dot{\phi}^2\ddot{\phi} \right) \mu e^{\mu t} \right. \\
 & + \left( 16H^3 + 32H\dot{H} - 16\dot{H}\dot{\phi} - 16H\ddot{\phi} + 8\dot{\phi}\ddot{\phi} - 12H\dot{\phi}^2 - 4\dot{\phi}^3 \right) \mu^2 e^{\mu t} \\
 & \left. + \left( 8H^2 + 2\dot{\phi}^2 - 8H\dot{\phi} \right) \mu^3 e^{\mu t} \right], \tag{31}
 \end{aligned}$$

and Eq. (26) implies,

$$\begin{aligned}
 V_{\Phi}(e^{-\phi}) = & e^{-2\phi} \left[ \frac{1}{\kappa^2} \left( 2\dot{H} + \frac{3}{4}\dot{\phi}^2 + 3H^2 \right) + \left( 8H^2 + 2\dot{\phi}^2 - 8H\dot{\phi} \right) \mu^2 e^{\mu t} \right. \\
 & \left. + \left( -8\dot{H}\dot{\phi} + 4\dot{\phi}\ddot{\phi} + 16H^3 - 8H\ddot{\phi} - \dot{\phi}^3 + 16H\dot{H} - 12H^2\dot{\phi} \right) \mu e^{\mu t} \right] \Big|_{t=t(\phi)} \\
 & - V_{\Theta}(e^{\mu t(\phi)}). \tag{32}
 \end{aligned}$$

Here we have assumed  $\phi = \phi(t)$  can be solved with respect to  $t$  as  $t = t(\phi)$ . Then for an arbitrary cosmological evolution given by  $H = H(t)$  and  $\phi = \phi(t)$ , if we construct the potential as in (30), (31), and (32), the evolution of  $H$  and  $\phi$  can be realized. Note that by making the inverse conformal transformation we performed below Eq. (9), one may obtain the above potentials in the original frame, but the resulting theory is too complicated to be analyzed, so we omit the details.

If we are only interested in the in realizing a specific cosmology with a given Hubble rate  $H$ , we may choose  $\phi = \phi(t)$  to have a specific but simple form like,

$$\phi = \phi_0 t, \tag{33}$$

with a constant  $\phi_0$ . Then Eqs. (31) and (32) are simplified and take the following form,

$$\begin{aligned}
 V_{\Theta}(e^{\theta}) = & \int^{\frac{\theta}{\mu}} dt e^{-2\phi_0 t} \left[ \frac{1}{\kappa^2} \left( 2\ddot{H} + 6H\dot{H} + \frac{9}{2}H\phi_0^2 \right) \right. \\
 & + \left( 48H^2\dot{H} + 16\dot{H}^2 + 16H\ddot{H} + 36H^3\phi_0 - 8\ddot{H}\phi_0 - 36H^2\phi_0^2 - 12\dot{H}\phi_0^2 \right. \\
 & \left. \left. - 9H\phi_0^3 \right) \mu e^{\mu t} + \left( 16H^3 + 32H\dot{H} - 16\dot{H}\phi_0 - 12H\phi_0^2 - 4\phi_0^3 \right) \mu^2 e^{\mu t} \right. \\
 & \left. + \left( 8H^2 + 2\phi_0^2 - 8H\phi_0 \right) \mu^3 e^{\mu t} \right], \tag{34}
 \end{aligned}$$

$$\begin{aligned}
 V_{\Phi}(e^{-\phi}) = & e^{-2\phi_0 t} \left[ \frac{1}{\kappa^2} \left( 2\dot{H} + \frac{3}{4}\phi_0^2 + 3H^2 \right) + \left( 8H^2 + 2\phi_0^2 - 8H\phi_0 \right) \mu^2 e^{\mu t} \right. \\
 & \left. + \left( -8\dot{H}\phi_0 + 16H^3 - \phi_0^3 + 16H\dot{H} - 12H^2\phi_0 \right) \mu e^{\mu t} \right] \Big|_{t=\frac{\phi}{\phi_0}} - V_{\Theta}\left(e^{\frac{\mu\phi}{\phi_0}}\right). \tag{35}
 \end{aligned}$$

Furthermore by choosing,

$$2\phi_0 = \mu, \tag{36}$$

we obtain,

$$V_{\Theta}(e^{\theta}) = \int^{\frac{\theta}{\mu}} dt \left[ \frac{e^{-\mu t}}{\kappa^2} \left( -2\mu\dot{H} - 3\mu H^2 + \frac{9}{8}H\mu^2 \right) + 34\mu^2 H^3 - \mu^3 H^2 - \frac{65}{8}\mu^4 H \right] + \frac{e^{-\mu t}}{\kappa^2} \left( 2\dot{H} + 3H^2 \right) + \left( 16H^3 + 16H\dot{H} \right) \mu + \left( -4\dot{H} + 16H^2 \right) \mu^2 - 11H\mu^3 \Big|_{t=\frac{\theta}{\mu}}, \tag{37}$$

$$V_{\Phi}(e^{-\phi}) = \left[ \frac{3\mu^2 e^{-\mu t}}{16\kappa^2} - 14\mu^2 H^2 + 7H\mu^3 - \frac{3\mu^4}{8} \right] \Big|_{t=\frac{2\phi}{\mu}} - \int^{\frac{2\phi}{\mu}} dt \left[ \frac{e^{-\mu t}}{\kappa^2} \left( -2\mu\dot{H} - 3\mu H^2 + \frac{9}{8}H\mu^2 \right) + 34\mu^2 H^3 - \mu^3 H^2 - \frac{65}{8}\mu^4 H \right]. \tag{38}$$

By using Eqs. (37) and (38), we may consider the realization of several cosmological evolution. As a first example, we may consider,

$$H = \frac{H_0 e^{-\mu t}}{1 + e^{-\mu t}} = H_0 \left( 1 - \frac{1}{1 + e^{-\mu t}} \right). \tag{39}$$

When  $t \rightarrow -\infty$ ,  $H$  goes to a constant  $H \rightarrow H_0$ , which may describe the de Sitter inflationary evolution. On the other hand, we find  $H \rightarrow 0$  when  $t \rightarrow +\infty$ . Therefore, inflation ends when  $t \sim 0$ . We now ignore the contribution from the perfect matter fluids that may be present, for the behavior of  $H$  given in (39). Then if the matter fluids are coupled with the scalar fields  $\theta$  and/or  $\phi$ , the matter fluids may produce non-trivial effects at  $t \sim 0$  and the behavior of  $H$  could be changed so that the matter or radiation dominated phase could be generated. Then Eqs. (37) and (38) yield,

$$V_{\Theta}(e^{\theta}) = \left( \frac{6H_0^2}{\kappa^2} - \frac{9\mu H_0}{8\kappa^2} \right) e^{-\theta} + \left( \frac{2H_0\mu}{\kappa^2} - \frac{6H_0^2}{\kappa^2} + \frac{9\mu H_0}{8\kappa^2} - 34\mu H_0^3 + \mu^2 H_0^2 + \frac{65}{8}\mu^3 H_0 \right) \ln(1 + e^{-\theta}) + \left( \frac{2H_0\mu}{\kappa^2} - \frac{6H_0^2}{\kappa^2} + \frac{9\mu H_0}{8\kappa^2} - 102\mu H_0^3 - 2\mu^2 H_0^2 - \frac{65}{8}\mu^3 H_0 \right) \ln(1 + e^{-\theta}) + \frac{2H_0\mu}{\kappa^2} - \frac{6H_0^2}{\kappa^2} + 16H_0^3\mu + 16H_0^2\mu^2 - 11H_0\mu^3 + \left( -\frac{2H_0\mu}{\kappa^2} + \frac{6H_0^2}{\kappa^2} - 116\mu H_0^3 + 33\mu^2 H_0^2 + 7H_0\mu^3 \right) \frac{1}{1 + e^{-\theta}}$$

$$\begin{aligned}
 & + \left( \frac{2H_0\mu}{\kappa^2} - \frac{3H_0^2}{\kappa^2} + 65H_0^3\mu - 16H_0^2\mu^2 + 4H_0\mu^3 \right) \frac{1}{(1 + e^{-\mu t})^2} \\
 & + \left( 16H_0^3\mu + 16H_0^2\mu^2 \right) \frac{1}{(1 + e^{-\theta})^3}, \tag{40}
 \end{aligned}$$

$$\begin{aligned}
 V_\Phi(e^{-\phi}) = & \left( -\frac{3H_0^2}{\kappa^2} + \frac{9\mu H_0}{8\kappa^2} + \frac{3\mu^2}{16\kappa^2} \right) e^{-2\phi} \\
 & - \left( \frac{2H_0\mu}{\kappa^2} - \frac{6H_0^2}{\kappa^2} + \frac{9\mu H_0}{8\kappa^2} - 34\mu H_0^3 + \mu^2 H_0^2 + \frac{65}{8}\mu^3 H_0 \right) \ln(1 + e^{-2\phi}) \\
 & - \left( \frac{2H_0\mu}{\kappa^2} - \frac{3H_0^2}{\kappa^2} - 68\mu H_0^3 - 27\mu^2 H_0^2 - 7\mu^3 H_0 \right) \frac{1}{1 + e^{-2\phi}} \\
 & - \frac{17\mu H_0^3 - 14\mu^2 H_0^2}{(1 + e^{-2\phi})^2}. \tag{41}
 \end{aligned}$$

Hence the above functional forms for the model at hand, realize the cosmological evolution of Eq. (39).

We may consider a more complicated cosmological evolution than that in Eq. (39). Since observationally it seems that there is a tension between the value of the Hubble constant inferred from small redshifts, as in the observation of Type Ia supernova (SNIa) calibrated by Cepheid observations  $H = H_1 \sim 73 \text{ km s}^{-1} \text{ Mpc}^{-1}$  [57] and that from large redshifts, as the cosmic microwave background (CMB)  $H = H_e \sim 67 \text{ km s}^{-1} \text{ Mpc}^{-1}$  [58]. Although the tension might come from the uncertainties of the Cepheid calibration (see [59,60] for example), a solution to solve the tension is to introduce an early dark energy occurring after the inflationary era [61,62]. In our model, instead of (39), we may consider,

$$H = \frac{H_0 e^{-\mu t}}{1 + e^{-\mu t}} + \frac{H_e - H_1 e^{-\mu(t-t_0)}}{1 + e^{-\mu(t-t_0)}} + H_1. \tag{42}$$

Here  $t_0$  is a positive constant and  $H_0$  is much larger than  $H_e$  and  $H_1$ . We also assume  $t_0 \gg \frac{1}{\mu}$ . Then during the era when  $1/\mu \ll t \ll t_0$ ,  $H$  behaves as a constant  $H \sim H_e - H_1 + H_1 = H_e$ , which could correspond to the Hubble rate at the clear up of the Universe, when the CMB was generated. In this epoch, our model plays the role of the early dark energy. On the other hand, when  $t \gg t_0 (\gg 1/\mu)$ , we find  $H \sim H_1$ , which correspond to the present Hubble rate. Therefore the model (42) can describe all of the inflation, the early dark energy, and the dark energy in the present Universe.

As another example, we may consider a model mimicking the  $\Lambda$ CDM model,

$$H = \frac{1}{l} \coth\left(\frac{3t}{2l}\right). \tag{43}$$

Here  $l$  is the length of the effective de Sitter radius. Specifically if we choose,

$$\mu = \frac{3}{l}, \tag{44}$$

we can perform the integrations in (37) and (38) and we obtain,

$$V_{\Theta}(e^{\theta}) = \frac{\mu^2}{\kappa^2} \left( -\frac{3}{4}\theta + \frac{25}{24e^{\theta}} + \frac{3}{4} \ln(e^{\theta} - 1) \right) + \mu^4 \left( \frac{289}{216}\theta - \frac{313}{108} \ln(e^{\theta} - 1) + \frac{-35e^{3\theta} + 47e^{2\theta} + 43e^{\theta} - 119}{27(e^{\theta} - 1)^3} \right), \quad (45)$$

$$V_{\Phi}(e^{-\phi}) = \frac{\mu^2}{\kappa^2} \left( -\frac{483}{16}e^{-2\phi} - \frac{411}{4} \ln(1 - e^{-2\phi}) - \frac{96}{1 - e^{-2\phi}} \right) + \mu^4 \left( \frac{313}{108} \ln(e^{2\phi} - 1) - \frac{289}{108}\phi + \frac{87e^{4\phi} + 482e^{2\phi} - 825}{216(e^{2\phi} - 1)^2} \right). \quad (46)$$

Therefore the model can mimic the  $\Lambda$ CDM model without the presence of a cold dark matter perfect fluid.

We can restore the form of the function  $F(R, \mathcal{G})$  for this model in the limit  $t \rightarrow \infty$ . The equations (8) can be rewritten as

$$\frac{R}{2\kappa^2} = -\frac{\mu^2}{\kappa^2} \frac{\Phi(25\Phi^2 + 11)}{24(\Phi^2 - 1)} + \mu^4 \left( \frac{289\Phi^6 + 927\Phi^4 - 1041\Phi^2 + 337}{108\Phi(\Phi^2 - 1)^3} \right), \quad 1 > \Phi > 0, \quad (47)$$

$$\mathcal{G} = -\frac{\mu^2}{\kappa^2\Theta^2} \frac{(7\Theta - 25)}{24(\Theta - 1)} - \frac{\mu^4(337\Theta^4 - 1186\Theta^3 + 1584\Theta^2 - 1982\Theta - 289)}{216\Theta(\Theta - 1)^4}, \quad \Theta > 1. \quad (48)$$

In the limit  $\Phi \rightarrow 0, \Theta \rightarrow \infty$  (this corresponds to  $\mu > 0, t \rightarrow \infty$ ), we get

$$\Phi = -\frac{337\mu^4\kappa^2}{54} \frac{1}{R} + \frac{113569\mu^{10}\kappa^4(20\kappa^2\mu^2 + 33)}{104976} \frac{1}{R^3} + \mathcal{O}\left(\frac{1}{R^5}\right), \quad (R \rightarrow -\infty), \quad (49)$$

$$\Theta = -\frac{337\mu^4\kappa^2}{216} \frac{1}{\mathcal{G}} + \frac{9(18\mu^2\kappa^2 + 7)}{337\mu^2\kappa^2} + \mathcal{O}(\mathcal{G}), \quad (\mathcal{G} \rightarrow 0). \quad (50)$$

Substituting these expressions into equation (9), we obtain in the limit  $\Phi \rightarrow 0, \Theta \rightarrow \infty$

$$F(R, \mathcal{G}) = -\frac{337\mu^4}{432} \ln\left(\frac{R^4}{(2\kappa^2)^4\mathcal{G}^2}\right) + \frac{\mu^4}{216} (4044 \ln(\mu) - 2359 \ln(2) - 3033 \ln(3) + 1011 \ln(337) + 818) - \frac{113569}{419904} \mu^{10}\kappa^2 (20\mu^2\kappa^2 + 33) \frac{1}{R^2} + \left(\frac{162}{337} + \frac{63}{337\mu^2\kappa^2}\right) \mathcal{G} + \mathcal{O}\left(\frac{1}{R^4}\right) + \mathcal{O}(\mathcal{G}^2). \quad (51)$$

## 6. Conclusions

In this work we addressed the ghost issue of  $F(R, \mathcal{G})$  gravity, which is plagued with ghost degrees of freedom. These ghost degrees of freedom make the presence at any level that the theory is considered, and may arise even at the level of cosmological perturbations, where superluminal modes may occur. If we consider the quantum theory, the ghosts generate the negative norm states, which give the negative probabilities, and therefore the ghosts are physically inconsistent. The ghosts are due to the presence of higher derivatives eventually in the theory, higher

than two at the equations of motion level. Thus, due to the fact that these theories often provide viable descriptions of inflation and dark energy, we provided a theoretical framework in which the ghost degrees of freedom disappear. Specifically we introduced two auxiliary scalar fields, and by using the Lagrange multiplier technique, we generated a ghost free  $F(R, \mathcal{G})$  theory in the Einstein frame, with the only dynamical field being one scalar field. Accordingly, we derived the field equations of the ghost free  $F(R, \mathcal{G})$  gravity for a general metric, and for the flat FRW metric. The field equations can be viewed as a general reconstruction technique, in which one may specify the Hubble rate, the scalar field evolution or even both the previous two, and by using the field equations one may discover which model can generate such an evolution. We used the reconstruction technique to realize several cosmological evolutions of interest, such as a de Sitter, an early dark energy evolution and an evolution mimicking the  $\Lambda$ CDM evolution. In a future work, we shall develop further the formalism in order to study in a quantitative way the inflationary era and the dark energy era, in this work we aimed to present the essential features of a ghost-free  $F(R, \mathcal{G})$  gravity in the Einstein frame.

### CRedit authorship contribution statement

All authors contributed equally.

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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