

Iterative Method for Solving the Problem of Scattering of an Electromagnetic Wave by a Partially Shielded Conducting Sphere

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Abstract

The problem of diffraction of an electromagnetic wave of parallel polarization by an perfectly conducting sphere sitting on top of a conducting surface is considered. The wave is supposed to fall onto the surface at a right angle. The original problem is solved by an iterative method involving consecutive solutions of the problems of diffraction by the sphere and by the conducting screen. The criteria for terminating the iterative process is a small amount of energy reflected off the sphere.

Mathematics Subject Classification: 78A45, 39B12, 12E05

Keywords: Scattering of electromagnetic waves, Conducting sphere, Iterative method

1 Introduction

The problem of scattering of electromagnetic waves by the shielded objects is one of the main objectives of a radiolocation which arises at diffraction of waves on the objects which have been close located to a surface. Also similar problems arise at modeling antennas of a special type.

Diffraction analysis problem for perfect conducting sphere or another rotary body in the case of incident plane wave is one of fundamental [3]. The problem of scattering by the sphere located near a conducting surface is less studied problem. The method of solution to such problem is bringing a problem of diffraction by the shielded sphere to a problem of scattering by two mirror spheres [2]. In this case the scattering problem is reduced to the solution of the linear set of equations.

In this work the solution of an initial problem is looked for in the form of a so-called diffraction series (see, for example [4]), i.e. in the form of the infinite sum of results of reflection from the screen and scattering by the conducting sphere. It is supposed that the falling wave at first is reflected from the infinite conducting screen and after scatter by the sphere with subsequent reflection from the screen etc. Actually the iterative scheme where one iteration includes scattering by the sphere and reflection from the screen turns out.

The separate point is devoted to the problem of scattering of any wave by the conducting sphere. The received expressions for the scattered field are expressed through integrals. For the case of incidence of a plane wave along the normal vector, the integrals are calculated analytically and coincide with well-known values [1].

The directional pattern for the chosen geometry and distribution of electric intensity module is provided at sphere near field. The next graphics show the dependence of solution error from the relation of distance from sphere to the conducting screen. Variable parameters of analysis are sphere radius and number of iterations.

2 Statement of the problem

The perfectly conducting sphere located at distance L (from the sphere center) from the infinite conducting screen is considered (see Fig. 1). Let's the electromagnetic $||$ -polarized wave with the component $E_y^{(0)}(x, y, z) = A^{(0)}e^{ikz}$ fall on the sphere on a normal to the screen, where A^0 is amplitude of a falling wave, k is the wave number.

The mathematical problem statement consists in finding solution of the Maxwell equations:

$$\begin{cases} \operatorname{rot} \mathbf{E} = i\omega\mu\mu_0\mathbf{H}, \\ \operatorname{rot} \mathbf{H} = -i\omega\varepsilon\varepsilon_0\mathbf{E} \end{cases}$$

with respect to unknown \mathbf{E} and \mathbf{H} which correspond to boundary conditions on the sphere:

$$\begin{cases} E_\theta(R, \theta, \alpha) + E_\theta^{(0)}(R, \theta, \alpha) = 0, \\ E_\alpha(R, \theta, \alpha) + E_\alpha^{(0)}(R, \theta, \alpha) = 0 \end{cases}$$

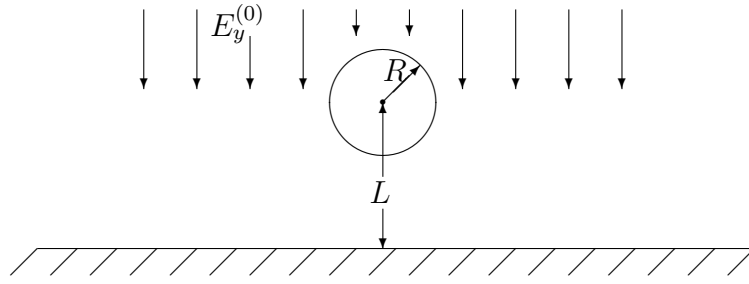


Figure 1: Geometry of the problem.

and to the following conditions on the screen:

$$\begin{cases} E_x(x, y, -L) + E_x^{(0)}(x, y, -L) = 0, \\ E_y(x, y, -L) + E_y^{(0)}(x, y, -L) = 0. \end{cases}$$

Also the field is limited in all space: $|\mathbf{E}| < \infty$ and the field satisfies the condition at infinity in the following manner:

$$\lim_{z \rightarrow \infty} [\mathbf{E} \times \mathbf{H}]_z \geq 0.$$

The given condition at the infinity means that at long distances from the screen and the sphere the energy flow along an axis z (the normal to the screen) is equal to zero or propagate from the screen.

The solution of the diffraction problem is found as a sum of infinite solutions sequence of diffraction problems by the sphere and by the conductive screen. It is supposed that the incidence wave firstly diffracts by the conductive screen and after this by the sphere, subsequently scatters by the conductive screen etc. As a result of the diffraction by one of objects the wave reflected onto the second object transmit less energy. Let's fix the stopping condition for solution of diffraction problem is the amount of energy are reflected from the sphere.

3 Diffraction by perfectly conducting sphere

The problem of scattering of a plane electromagnetic wave by the single conducting sphere was considered repeatedly [3], [1]. In this case, this problem has the analytical solution.

The scattering of any electromagnetic wave on the conducting sphere is considered here. The solution is also constructed analytically, in the form of the integral contained values of a incidence field on the sphere.

Components of the vector of electric intensity are sought in the form

$$E_\theta(r, \theta, \alpha) = \sum_{n=1}^{+\infty} B_{0n} \chi_n(r) \frac{d\varphi_{0n}}{d\theta} + \sum_{m \neq 0} E_\theta^m(r, \theta) e^{im\alpha}, \quad (1)$$

$$E_\alpha(r, \theta, \alpha) = i\omega\mu_0\mu \sum_{n=1}^{+\infty} A_{0n} \zeta_n^{(2)}(kr) \frac{d\varphi_{0n}}{d\theta} + \sum_{m \neq 0} E_\alpha^m(r, \theta) e^{im\alpha}, \quad (2)$$

$$E_r(r, \theta, \alpha) = \frac{1}{r} \sum_{n=1}^{+\infty} B_{0n} n(n+1) \zeta_n^{(2)}(kr) \varphi_{0n} + \sum_{m \neq 0} E_r^m(r, \theta) e^{im\alpha}, \quad (3)$$

where

$$E_\theta^m(r, \theta) = -\frac{i\omega\mu_0\mu}{\sin\theta} \sum_{n=|m|}^{+\infty} A_{mn} \zeta_n^{(2)}(kr) \varphi_{mn} + \frac{1}{im} \sum_{n=|m|}^{+\infty} B_{mn} \chi_n(r) \frac{d\varphi_{mn}}{d\theta},$$

$$E_\alpha^m(r, \theta) = \frac{\omega\mu_0\mu}{m} \sum_{n=|m|}^{+\infty} A_{mn} \zeta_n^{(2)}(kr) \frac{d\varphi_{mn}}{d\theta} + \frac{1}{\sin\theta} \sum_{n=|m|}^{+\infty} B_{mn} \chi_n(r) \varphi_{mn},$$

$$E_r^m(r, \theta) = \frac{1}{imr} \sum_{n=|m|}^{+\infty} B_{mn} n(n+1) \zeta_n^{(2)}(kr) \varphi_{mn},$$

with designations

$$\chi_n(r) = \frac{1}{r} \frac{d(r\zeta_n^{(2)}(kr))}{dr}, \quad \varphi_{mn} = P_n^m(\cos\theta).$$

Let's set

$$E_{\theta m}^{(0)}(R, \theta) = \frac{1}{2\pi} \int_0^{2\pi} E_\theta^{(0)}(R, \theta, \alpha) e^{-im\alpha} d\alpha,$$

$$E_{\alpha m}^{(0)}(R, \theta) = \frac{1}{2\pi} \int_0^{2\pi} E_\alpha^{(0)}(R, \theta, \alpha) e^{-im\alpha} d\alpha$$

Fourier coefficients of the tangential components of the vector \mathbf{E} of the incidence wave on the sphere wave. \mathbf{E} . Then

$$\left\{ \begin{array}{l} \sum_{n=1}^{+\infty} B_{0n} \chi_n(R) \varphi'_{0n} = -E_{\theta 0}^{(0)}, \\ i\omega\mu_0\mu \sum_{n=1}^{+\infty} A_{0n} \zeta_n^{(2)}(kR) \varphi'_{0n} = -E_{\alpha 0}^{(0)}. \end{array} \right.$$

Here and elsewhere right parts are functions in R and θ . Let's take the orthogonality of arbitrary Lagrange polynomials:

$$\int_0^\pi P'_n(\cos \theta) P'_m(\cos \theta) \sin \theta d\theta = \frac{2n(n+1)}{2n+1} \delta_{nm}.$$

Obtain for $n = 1, 2, \dots$

$$A_{0n} = -\frac{1}{i\omega\mu_0\mu\zeta_n^{(2)}(kR)} \frac{2n+1}{2n(n+1)} \int_0^\pi E_{\alpha 0}^{(0)} \varphi'_{0n} \sin \theta d\theta,$$

$$B_{0n} = -\frac{1}{\chi_n(R)} \frac{2n+1}{2n(n+1)} \int_0^\pi E_{\theta 0}^{(0)} \varphi'_{0n} \sin \theta d\theta.$$

For each m systems of equations with respect to the coefficients A_{mn} B_{mn} are obtained

$$\left\{ \begin{array}{l} -\frac{i\omega\mu_0\mu}{\sin \theta} \sum_{n=|m|}^{+\infty} A_{mn} \zeta_n^{(2)}(kR) \varphi_{mn} + \frac{1}{im} \sum_{n=|m|}^{+\infty} B_{mn} \chi_n(R) \varphi'_{mn} = -E_{\theta m}^{(0)}, \\ \frac{\omega\mu_0\mu}{m} \sum_{n=|m|}^{+\infty} A_{mn} \zeta_n^{(2)}(kR) \varphi'_{mn} + \frac{1}{\sin \theta} \sum_{n=|m|}^{+\infty} B_{mn} \chi_n(R) \varphi_{mn} = -E_{\alpha m}^{(0)}. \end{array} \right.$$

Let's set $C_n = i\omega\mu_0\mu A_{mn} \zeta_n^{(2)}(kR)$, $D_n = B_{mn} \chi_n(R)$. The following expression for unknown coefficients for $m = 1, 2, \dots$, $n = |m|, |m| = 1, \dots$ is obtained:

$$A_{mn} = \frac{1}{i\omega\mu_0\mu\zeta_n^{(2)}(kR)} \frac{(2n+1)m(n-m)!}{2n(n+1)(n+m)!} \int_0^\pi \left(i \frac{\partial(\sin \theta E_{\alpha m}^{(0)})}{\partial \theta} + m E_{\theta m}^{(0)} \right) \varphi_{mn} d\theta,$$

$$B_{mn} = \frac{1}{\chi_n(R)} \frac{(2n+1)m(n-m)!}{2n(n+1)(n+m)!} \int_0^\pi \left(i \frac{\partial(\sin \theta E_{\theta m}^{(0)})}{\partial \theta} - m E_{\alpha m}^{(0)} \right) \varphi_{mn} d\theta.$$

Let's use the integrating by parts

$$\int_0^\pi i \frac{\partial(\sin \theta E)}{\partial \theta} \varphi_{mn} d\theta = - \int_0^\pi i E \varphi'_{mn} \sin \theta d\theta,$$

and transform the expression for A_{mn} and B_{mn} to the form

$$A_{mn} = \frac{1}{i\omega\mu_0\mu\zeta_n^{(2)}(kR)} \frac{(2n+1)m(n-m)!}{2n(n+1)(n+m)!}$$

$$\times \left[m \int_0^\pi \varphi_{mn} E_{\theta m}^{(0)} d\theta - i \int_0^\pi \varphi'_{mn} E_{\alpha m}^{(0)} \sin \theta d\theta \right],$$

$$B_{mn} = -\frac{1}{\chi_n(R)} \frac{(2n+1)m(n-m)!}{2n(n+1)(n+m)!} \left[i \int_0^\pi \varphi'_{mn} E_{\theta m}^{(0)} \sin \theta d\theta + m \int_0^\pi \varphi_{mn} E_{\alpha m}^{(0)} d\theta \right].$$

So, the problem of diffraction of the electromagnetic wave by perfect conducting sphere with intensity vector $\mathbf{E}^{(0)} = (E_r^{(0)}, E_\theta^{(0)}, E_\alpha^{(0)})$ has solution in the form:

$$A_{0n} = -\frac{1}{i\omega\mu_0\mu\zeta_n^{(2)}(kR)} \frac{2n+1}{2n(n+1)} \int_0^\pi E_{\alpha 0}^{(0)} \varphi'_{0n} \sin \theta d\theta,$$

$$B_{0n} = -\frac{1}{\chi_n(R)} \frac{2n+1}{2n(n+1)} \int_0^\pi E_{\theta 0}^{(0)} \varphi'_{0n} \sin \theta d\theta,$$

for $n = 1, 2, \dots$ and

$$A_{mn} = \frac{1}{i\omega\mu_0\mu\zeta_n^{(2)}(kR)} \frac{(2n+1)m(n-m)!}{2n(n+1)(n+m)!}$$

$$\times \left[m \int_0^\pi \varphi_{mn} E_{\theta m}^{(0)} d\theta - i \int_0^\pi \varphi'_{mn} E_{\alpha m}^{(0)} \sin \theta d\theta \right],$$

$$B_{mn} = -\frac{1}{\chi_n(R)} \frac{(2n+1)m(n-m)!}{2n(n+1)(n+m)!} \left[i \int_0^\pi \varphi'_{mn} E_{\theta m}^{(0)} \sin \theta d\theta + m \int_0^\pi \varphi_{mn} E_{\alpha m}^{(0)} d\theta \right],$$

for $m = 1, 2, \dots, n = |m|, |m| + 1, \dots$

Let's consider the case when the plane wave falls on the sphere along axis z . Thus integrals for A_{mn} and B_{mn} are calculated obviously. The received analytical expressions for coefficients coincide with expressions received other authors [1].

4 Communication between spherical systems of coordinates

Let's consider the relations between two spherical systems, placed on the axis z . The result is formulated in the form of predication:

Lemma 4.1 *The transformations between pair of vectors components of the spherical systems (r, θ, α) and (r', θ', α') with centerpoint, placed on the axis z at a distance $2L$ from each other may be represented as:*

$$\begin{pmatrix} u_r \\ u_\theta \\ u_\alpha \end{pmatrix} = \mathbf{M} \begin{pmatrix} u_{r'} \\ u_{\theta'} \\ u_{\alpha'} \end{pmatrix}, \quad \begin{pmatrix} u_{r'} \\ u_{\theta'} \\ u_{\alpha'} \end{pmatrix} = \mathbf{M}^T \begin{pmatrix} u_r \\ u_\theta \\ u_\alpha \end{pmatrix}$$

with matrix

$$\mathbf{M} = \begin{pmatrix} \frac{r \cos^2 \theta + t \sin \theta}{q} & \cos \theta \frac{t - r \sin \theta}{q} & 0 \\ -\cos \theta \frac{t - r \sin \theta}{q} & \frac{r \cos^2 \theta + t \sin \theta}{q} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$q = \sqrt{4L^2 + r^2 + 4Lr \cos \theta}, \quad t = \sqrt{4L^2 + r^2 \sin^2 \theta + 4Lr \cos \theta}.$$

Moreover

$$r' = q, \quad \theta' = \arccos \frac{2L + r \cos \theta}{q}, \quad \alpha' = \alpha.$$

Proof. Let's make following sequence in order to make the transformation from one spherical system to another. At the beginning, spherical system (r', θ', α') changes to Cartesian system (x', y', z') with the same centerpoint:

$$x' = r' \sin \theta' \cos \alpha', \quad y' = r' \sin \theta' \sin \alpha', \quad z' = r' \cos \theta'.$$

The transformation from spherical vectors to the Cartesian be realized according to the rule

$$\begin{pmatrix} u_{x'} \\ u_{y'} \\ u_{z'} \end{pmatrix} = \mathbf{Q}' \begin{pmatrix} u_{r'} \\ u_{\theta'} \\ u_{\alpha'} \end{pmatrix}$$

with matrix

$$\mathbf{Q}' = \begin{pmatrix} \sin \theta' \cos \alpha' & \cos \theta' \cos \alpha' & -\sin \alpha' \\ \sin \theta' \sin \alpha' & \cos \theta' \sin \alpha' & \cos \alpha' \\ \cos \theta' & -\sin \theta' & 0 \end{pmatrix}.$$

The next step let change new Cartesian coordinates (x, y, z) with the centerpoint, which is displaced on $z = z' + 2L$ ($x = x'$, $y = y'$). The last step change to the spherical system (r, θ, α) :

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arccos \frac{z}{r}, \quad \alpha = \arctan \frac{y}{x}$$

with the transform matrix between vectors \mathbf{Q} .

As a result, the transform matrix between two spherical systems will be

$$\mathbf{M} = \mathbf{Q}^T \mathbf{Q}'.$$

Let's express coordinates (r', θ', α') in terms of (r, θ, α) and take into account the equation $\mathbf{M}^{-1} = \mathbf{M}^T$. This proposes the Lemma. As well notice, that $|\mathbf{M}| = 1$. \square

5 Reflection from perfectly conducting screen

Let's consider the problem of reflection of plane wave at conducting screen. The overdetermined boundary value problem is applied for this purpose. The reflected wave in the case of the normal incidence be as follows: $E_{y1} = -A^{(0)}e^{-ik(z+L)}$.

The reflected from conducting screen electromagnetic wave radiated by sphere (arbitrary object) is equal to the wave, which placed inverse manner on the other side of screen. The second "mirror" sphere radiating the same spherical waves. Angles θ' and α' are varying as $\theta' = \pi - \theta$ and $\alpha' = -\alpha$. Results of the Lemma 1 and the fact that $\cos(\pi - \theta') = -\cos \theta'$ are took into account. Than the field representation for the second sphere in the form (1)–(3) with range coefficients is obtained:

$$A'_{mn} = (-1)^{n-m} A_{mn}, \quad B'_{mn} = -(-1)^{n-m} B_{mn},$$

which are expressed through Fourier coefficients A_{mn} and B_{mn} for the initial sphere.

6 Numerical results

Non-zero components of the incident parallel polarized wave with $E_y^{(0)} = A^{(0)}e^{ikz}$ can be found from the Maxwell equations:

$$H_x^{(0)} = \frac{1}{i\omega\mu_0\mu} \frac{\partial E_y}{\partial z} = A^{(0)} \sqrt{\frac{\varepsilon_0\varepsilon}{\mu_0\mu}} e^{ikz},$$

$$H_z^{(0)} = -\frac{1}{i\omega\mu_0\mu} \frac{\partial E_y}{\partial x} = 0.$$

For proceeding to vectors of spherical system, it is sufficient to multiply by matrix \mathbf{Q} :

$$E_\theta = E_x \cos \theta \cos \alpha + E_y \cos \theta \sin \alpha - E_z \sin \theta,$$

$$E_\alpha = -E_x \sin \alpha + E_y \cos \alpha.$$

Let's consider the field reflection from screen without sphere. In this case, component E_y of the summarized field of the reflected from the screen wave and incident wave be as follows

$$E_y^{(1)} = A^{(0)}(e^{ikr \cos \theta} - e^{-ik(r \cos \theta + 2L)}) = 2iA^{(0)}e^{-ikL} \sin(kr \cos \theta + L).$$

Than

$$E_\theta^{(1)} = E_y^{(1)} \cos \theta \sin \alpha, \quad E_\alpha^{(1)} = E_y^{(1)} \cos \alpha.$$

Let's find coefficients of Fourier transform in α :

$$E_{\theta,m}^{(1)}(r, \theta) = \frac{E_y^{(1)} \cos \theta}{2\pi} \int_0^{2\pi} e^{-im\alpha} \sin \alpha d\alpha = \frac{i}{2} E_y^{(1)} \cos \theta \begin{cases} 1, m = -1, \\ -1, m = 1, \\ 0, |m| \neq 1, \end{cases}$$

$$E_{\alpha,m}^{(1)}(r, \theta) = \frac{E_y^{(1)} \cos \theta}{2\pi} \int_0^{2\pi} e^{-im\alpha} \cos \alpha d\alpha = \frac{1}{2} E_y^{(1)} \cos \theta \begin{cases} 1, |m| = 1, \\ 0, |m| \neq 1. \end{cases}$$

Note, that

$$E_{\theta,-1}^{(1)}(r, \theta) = -E_{\theta,1}^{(1)}(r, \theta), \quad E_{\alpha,-1}^{(1)}(r, \theta) = E_{\alpha,1}^{(1)}(r, \theta).$$

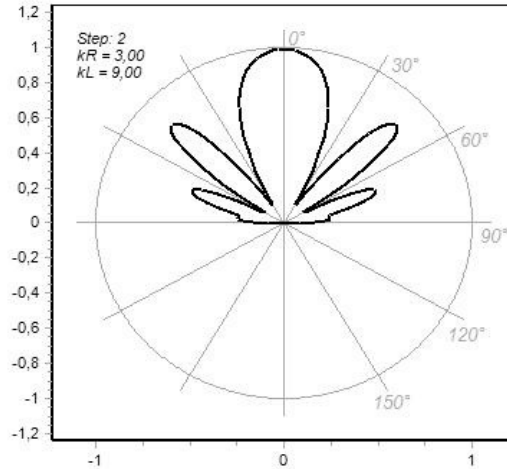


Figure 2: Radiation pattern for the shielded conducting spherical antenna ($kR = 3$, $kL = 9$). The result was converged after two iterations.

Next, let consequently solve the problem of diffraction by the conductive screen and the problem of diffraction by the sphere. The terminating criteria will be ration smallness (variable ε) the energy flow of the wave scattering by the sphere at the current iteration to the energy flow on the primary diffraction by the sphere.

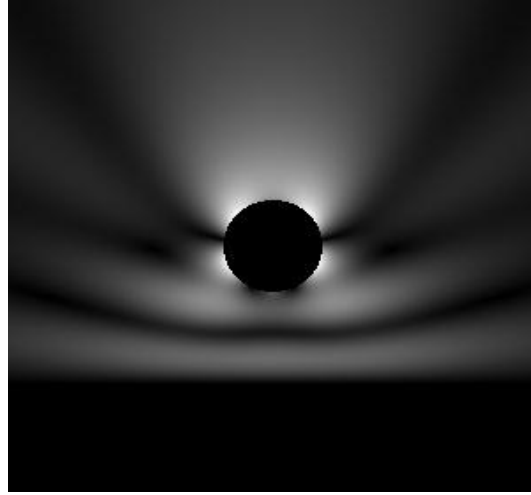


Figure 3: Distribution of $|\mathbf{E}|$ for the shielded conducting spherical antenna ($kR = 3$, $kL = 9$). The result was converged after two iterations. Lighter shades correspond to great values of $|\mathbf{E}|$.

Let's consider as an example the electromagnetic wave scattering by the shielded sphere with $kR = 3$, $kL = 9$. The accuracy after two iterations is $\varepsilon \approx 7.6 \cdot 10^{-3}$. For this case, the radiation pattern is shown in Fig. 2. The distribution of module of electromagnetic intensity near the sphere is shown in the Fig. 3.

The calculated radiation patterns are in good agreement with [2].

Let's consider the dependance of the accuracy ε via the relation of distance between the sphere and the conductive screen L to the radius R . It follows that this method converge quickly for bigger values of L/R .

Fig. 4 shows the dependence of values of $\lg \varepsilon$ from L/R . The solid line corresponds to three iterations. When the value of L/R is more then one, than the accuracy becomes 10^{-2} . The dashed graph corresponds to five iterations. Here for the same value of L/R the accuracy comes close to 10^{-4} . The dotted graph corresponds to nine iterations. Here accuracy becomes $\varepsilon = 10^{-7}$.

Next graph (Fig. 5) shows the dependence $\lg \varepsilon$ from number of iterations for fixed L/R . Solid line corresponds to the case, when sphere lays on the screen. There is the most inconvenient case for given algorithm. The dashed line correspond to the case, when $L/R = 3$. The dotted line corresponds to the case, when $L/R = 5$. The dependence $\lg \varepsilon$ via number of iterations is closed to linear.

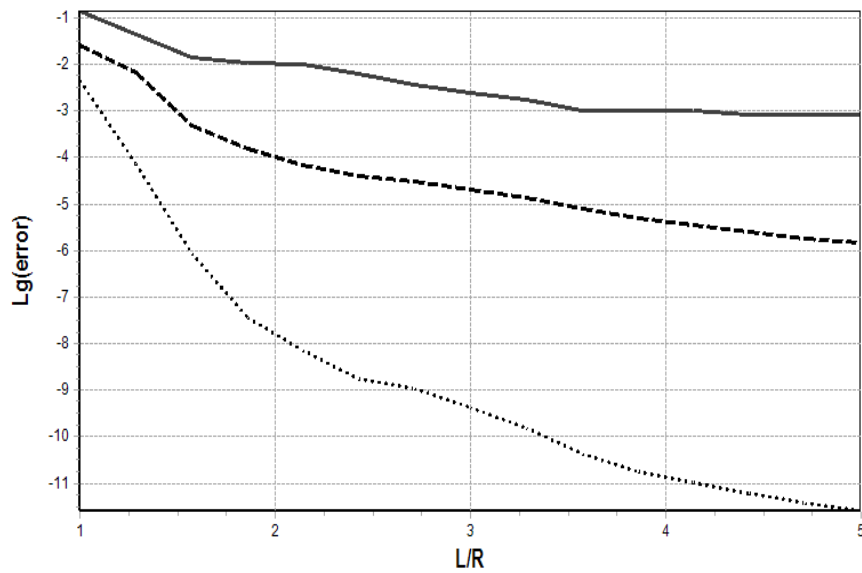


Figure 4: The dependence of the solution error on ratio L to R for different iterations. 3 iterations corresponds to the solid line, 5 iterations corresponds to the dashed line and 9 iterations corresponds to the dotted line.

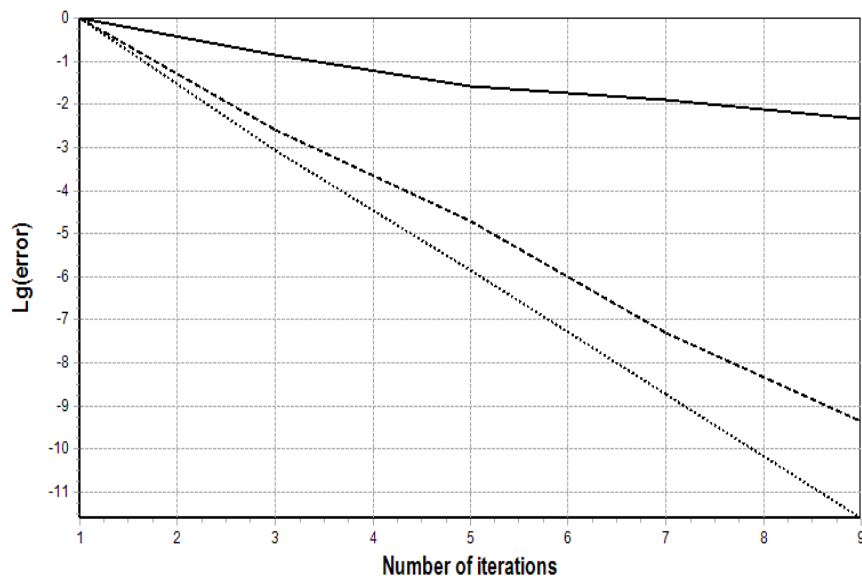


Figure 5: The dependence of solution error on number of iterations for different ratios of L to R . Ratio of 1 corresponds to the solid line, ratio of 3 corresponds dashed line, ratio of 5 corresponds dotted line.

7 Conclusion

The iteration algorithm presented in the paper quite fast converges to the exact solution. It is sufficient three iterations for archiving the accuracy of 10^{-3} . The accuracy of the method improves with the rising distance between sphere and screen. At sufficient high distances number of iterations may be limited to one or two iterations.

Acknowledgements. The work was performed according to the Russian Government Program of Competitive Growth of Kazan Federal University. Thanks to Dmitry Chikrin from the KFU's ITIS Higher Institute for proof reading and essential corrections.

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Received: August 11, 2014