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Classical evolution of subspaces

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Abstract We study evolution of manifolds after their creation at high energies. Several kinds of gravitational Lagrangians with higher derivatives are considered. It is shown analytically and confirmed numerically that an asymptotic growth of the maximally symmetric manifolds depends strongly on their dimensionality. A number of final metrics describing our Universe is quite poor if we limit ourselves with a maximally symmetric extra space. We show that the initial conditions can be a reason of nontrivial solutions (funnels) and study their properties.

1 Introduction

The compact extra spaces is widely used idea. Their inclusion into physical theories helps to move forward on such issues as the grand unification [1,2], neutrino mass [3], the cosmological constant problem [4,5] and so on. Any multi-dimensional model has to lead to the effective 4-dim theory at low energies. This would imply relations between the observable four-dimensional physics and a metric of the higher dimensions.

One of the question remaining not clarified yet is: why specific number of dimensions are compactified and stable while others expand [6–8]? Which specific property of subspace leads to its quick growth? There are many attempts to clarify the problem, mostly related to introduction of fields other than gravity. It may be a scalar field [6,9] (most used case), gauge fields [10]. A static solutions can be obtained using the Casimir effect [11] or form fields [12,13]. Sometimes one of the subspace is assumed to be FRW space by definition [14]. Another possibility was discussed in [15,16]: it was shown that if the scale factor $a(t)$ of our 3D space is much larger than the growing scale factor $b(t)$ of the extra dimensions, a contradiction with observations can be avoided.

The origin of our Universe is usually related to its quantum creation from the space-time foam at high energies [17,18]. The probability of its creation is widely discussed, see e.g. [19]. Here we are interested in the subsequent classical evolution of the metrics rather than a calculation of this probability. Manifolds are nucleated having specific metrics. The set of such metrics is assumed to be very rich. After nucleation, these manifolds evolve classically forming a set of asymptotic manifolds, one of which could be our Universe. In this paper the asymptotic set of the maximally symmetric manifolds with positive curvature is studied in the framework of pure gravity with higher derivatives. We consider models of the $f(R)$ gravity and a more general model acting in 5 and 6 dimensions. No other fields are attracted to stabilize an extra space. We have found out that a number of asymptotic solutions is quite limited. This conclusion was confirmed both analytically and numerically. There is a set of initial conditions that lead to a common asymptote of classical solutions. In Sect. 3 we have elaborated a method for prediction the asymptotic behavior of metric judging on the specific form of the initial metric. We also study the funnel solution [20] as the result of an inhomogeneity of initial metric.

The gravity with higher derivatives is widely used in modern research despite the internal problems inherent in this approach [21]. Attempts to avoid the Ostrogradsky instabilities are made [22] and extensions of the Einstein–Hilbert action attract much attention. Promising branch of such models is based on the Gauss–Bonnet Lagrangian and its generalization to the Lovelock gravity. These models were adjusted to obtain differential equations of the second order so that the Ostrogradsky theorem is not dangerous for such models.

A lot of papers devoted to the $f(R)$-gravity – the simplest extension of the Einstein–Hilbert gravity. Reviews [23,24] contain description of the $f(R)$-theories including extension to the Gauss–Bonnet gravity. Examples of research with specific form of the function $f(R)$ can be found in [25,26]. Most of the research assume positive curvature of extra space metric, but as was shown in [27], hyperbolic manifolds can also be attracted to explain the observable acceleration of the Universe.

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In the framework of the gravity with higher derivatives, a variety of regimes with expanding three and contracting extra dimensions has been found in [28–32]. The power-law and the exponential analytical behavior of scale-factors are studied in [33,34]. Stability of specific extra space metrics is discussed in [5,34–36]. The conclusion is that stable metrics do exist but their fraction is quite small. Here we consider a wider class of metrics depending on the initial conditions. It was found out that different initial metrics can lead to one and the same asymptotic solution.

Throughout this paper we use the conventions for the curvature tensor $R^D_{ABC} = \partial_C\Gamma^D_{AB} - \partial_B\Gamma^D_{AC} + \Gamma^D_{EC}\Gamma^E_{BA} - \Gamma^D_{EB}\Gamma^E_{AC}$ and for the Ricci tensor $R_{MN} = R^F_{MFN}$.

2 Destiny of subspaces: exact results

2.1 Setup and classical equations

In this section we analyze the classical behavior of the extra space metrics. This can be done on the basis of two well known frames – the Jordan frame and the Einstein one, which are connected by the conformal transformation [37]. There are intensive debates on the selection of the frame that should be used for appropriate description of the Nature [38]. Our analysis is mainly based on the Jordan frame.

Let a $D = 1 + d_1 + d_2$-dimensional space-time $T \times M_{d_1} \times M_{d_2}$ has been nucleated due to some quantum processes at high energies. The probability of this process is a subtle point and we do not discuss it in this paper. The entropy growth leads to evolution of subspaces, to those which are the maximally symmetric [39]. In this paper we study the classical evolution of subspaces M_{d_1} and M_{d_2} whose metric

$$ds^2 = dt^2 - e^{2\beta_1(t)}d\Omega_1^2 - e^{2\beta_2(t)}d\Omega_2^2 \tag{1}$$

is assumed to be maximally symmetric with a positive curvature. It is supposed that manifolds are born with accidental shape. Subsequently they acquire symmetries due to the entropy growth [39]. We start our study after the process of symmetrization is finished. In this section, we consider the following action

$$S = \frac{m_D^4}{2}\int d^D Z\sqrt{|g|}f(R), \tag{2}$$

where R is the scalar curvature of a D-dimensional space-time. This action appears to be an appropriate tool to study the behavior of the system just after its nucleation.

Einstein's equations of this theory are

$$-\frac{1}{2}f(R)\delta^B_A + \left(R^B_A + \nabla_A\nabla^B - \delta^B_A\Box\right)f_R = 0, \tag{3}$$

where $\Box = \nabla_A\nabla^A$ and $f_R \equiv df(R)/dR$. Using the results given in Appendix A, we can write the nontrivial equations of this system as

$$-\frac{1}{2}f(R) + f_R[e^{-2\beta_1(t)}(d_1-1) + \ddot\beta_1 + \dot\beta_1(d_1\dot\beta_1 + d_2\dot\beta_2)] + [(1-d_1)\dot\beta_1 - d_2\dot\beta_2]f_{RR}\dot R - f_{RRR}\dot R^2 - f_{RR}\ddot R = 0, \tag{4}$$

$$-\frac{1}{2}f(R) + f_R[e^{-2\beta_2(t)}(d_2-1) + \ddot\beta_2 + \dot\beta_2(d_1\dot\beta_1 + d_2\dot\beta_2)] + [(1-d_2)\dot\beta_2 - d_1\dot\beta_1]f_{RR}\dot R - f_{RRR}\dot R^2 - f_{RR}\ddot R = 0, \tag{5}$$

$$-\frac{1}{2}f(R) + \left[d_1\ddot\beta_1 + d_2\ddot\beta_2 + d_1\dot\beta_1^2 + d_2\dot\beta_2^2\right]f_R - \left(d_1\dot\beta_1 + d_2\dot\beta_2\right)f_{RR}\dot R = 0 \tag{6}$$

in terms of metric (1). Here we have kept in mind $\partial_t f_R = f_{RR}\dot R$ and $\partial_t^2 f_R = f_{RRR}\dot R^2 + f_{RR}\ddot R$. According to (A13), the Ricci scalar is

$$R = d_1\dot\beta_1^2 + d_2\dot\beta_2^2 + d_1\ddot\beta_1 + d_2\ddot\beta_2 + d_1\Big[e^{-2\beta_1(t)}(d_1-1) + \ddot\beta_1 + \dot\beta_1(d_1\dot\beta_1 + d_2\dot\beta_2)\Big] + d_2[e^{-2\beta_2(t)}(d_2-1) + \ddot\beta_2 + \dot\beta_2(d_1\dot\beta_1 + d_2\dot\beta_2)]. \tag{7}$$

For calculations it is convenient to consider the Ricci scalar $R(t)$ as additional unknown function and the definition (7) as the fourth equation. Three equations of this system [for example, (4), (5), (7)] can be solved with respect to the higher derivatives $\ddot\beta_1$, $\ddot\beta_2$, $\ddot R$. Then substituting $\ddot\beta_1$ and $\ddot\beta_2$ into equation (6), we obtain the equation

$$-5f(R) + \Big[5R - 5d_1(d_1-1)\dot\beta_1^2 - 10d_1d_2\dot\beta_1\dot\beta_2 - 5d_2(d_2-1)\dot\beta_2^2 - 5d_1(d_1-1)e^{-2\beta_1} - 5d_2(d_2-1)e^{-2\beta_2}\Big]f_R - 10\left[d_1\dot\beta_1 + d_2\dot\beta_2\right]f_{RR}\dot R = 0, \tag{8}$$

which plays the role of restricting the solutions of the coupled second order differential equations. This can be checked, for example, by writing the set of four Eqs. (4), (5), (7), (8) as an equivalent set of (six) coupled first-order equations plus one algebraic equation. The Eq. (8) reduces to the algebraic transcendental equation, i.e., it is a constraint. The complete set of initial conditions therefore requires specifying six pieces of information, namely $\beta_1(t_0)$, $\beta_2(t_0)$, $R(t_0)$, $\dot\beta_1(t_0)$, $\dot\beta_2(t_0)$, and $\dot R(t_0)$. These initial conditions are not independent due to Eq. (8). The latter will be used to derive an exact relation between these initial data.

2.2 Analysis of numerical solutions and their asymptotes

The system of differential equations is highly nonlinear so that one could expect a rich set of its solutions. In this section