Linear Algebra and its Applications 534 (2017) 97–101

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The Hankel matrix rank theorem revisited

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ARTICLE INFO

Article history:
Received 25 July 2017
Accepted 12 August 2017
Available online 18 August 2017
Submitted by A. Böttcher

MSC:
15B05

Keywords:
Hankel matrix
Rank
Submatrix

ABSTRACT

We give a new short proof of a version of a Hankel matrix rank theorem. That version expresses the rank of H by the smallest possible rank of an infinite Hankel matrix containing H. The new approach is based on application of the Kronecker theorem.

A Hankel matrix is a rectangular matrix of form

$$H = \begin{pmatrix} s_1 & s_2 & s_3 & \dots & s_q \\ s_2 & s_3 & s_4 & \dots & s_{q+1} \\ s_3 & s_4 & s_5 & \dots & s_{q+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_p & s_{p+1} & s_{p+2} & \dots & s_l \end{pmatrix} \quad (l = p + q - 1). \tag{1}$$

Thus, a Hankel matrix is characterized by the property that the (i, j) entry depends only on the sum $i + j$. Infinite Hankel matrices

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http://dx.doi.org/10.1016/j.laa.2017.08.010

$$H_\infty = \begin{pmatrix} s_1 & s_2 & s_3 & \dots \\ s_2 & s_3 & . \;\; . & . \;\; . \\ s_3 & . & . & . \\ . & . & . & . \end{pmatrix} \tag{2}$$

are also considered in the literature. The intense study of Hankel matrices traces back to the second half of the 19th century and it is still continuing due to many applications in algebra, functional analysis, random processes, etc. (see [1–4] and references therein). Algebraic aspects of the theory of Hankel matrices are analyzed in the monograph of Iohvidov [2], where the classical results of Frobenius are comprehended and developed. One of the central results of [2] is the theorem on the rank of a complex square Hankel matrix. That theorem is based on the concept of (r, k)-characteristic. Then by a significant complication of this notion the rank theorem was generalized to rectangular matrices [5].

Later [6] we developed a different version of the Hankel matrix rank theorem, where, in contrast to Iohvidov's theorem [5], the key parameter is not the (r, k)-characteristic but the smallest rank of an infinite Hankel matrix containing a given matrix as a corner submatrix. That theorem gave another view on the rank problem of Hankel matrices. Its shortcoming, however, was a quite artificial and technical proof based on some results on generalized Hankel matrices arising in the theory of linear automata [7].

In this note we fill this gap and give a new proof based on a straightforward application of the Kronecker theorem on the basic minor of an infinite Hankel matrix. The new proof is shorter, more natural, and gives a new way to understand the essence of the problem.

The structure of this note is the following. First we give a new proof of Kronecker's theorem by an approach different from that suggested in [1] and [2]. Let us remark that Lemma 1 exploited in this proof is then used to prove the main result. Then we give a proof of the main result.

We treat the rows of matrix H_∞ as vectors of the linear space of infinite rows over a field F, and the same with columns. In the sequel both H and H_∞ are supposed to be nonzero matrices.

Lemma 1. *A Hankel matrix H_∞ has a finite rank r if and only if the first r rows of H_∞ are linearly independent and generate the $(r+1)$st row as a linear combination.*

Proof. Define a linear operator φ in the space of infinite rows as follows: if $s = (s_1, s_2, \dots)$, then $\varphi(s) = (s_2, s_3, \dots)$. Then the rows of the Hankel matrix H_∞ can be considered as a sequence

$$s, \varphi(s), \varphi^2(s), \dots \tag{3}$$

The statement of the lemma becomes equivalent to the following obvious fact: either for every r, the vectors

$$s, \varphi(s), \varphi^2(s), \dots, \varphi^{r-1}(s) \tag{4}$$

are independent, or they are independent for some r and the vector $\varphi^r(s)$ is a linear combination of vectors (4). In the latter case all further vectors (3) are linear combinations of vectors (4). □

Let H_k be the principal corner submatrix of size k of the matrix H_∞.

Theorem 2 *(Kronecker). If a Hankel matrix H_∞ has a finite rank r, then the submatrix H_r is nondegenerate.*

Proof. Due to Lemma 1, the first r rows of the matrix H_∞ are linearly independent. Since the matrix H_∞ is symmetric, it follows that the first r columns are also linearly independent and thus form a basis in the column space of H_∞. Denote the columns of H_∞ by $h_1, h_2, \ldots$, and given a column h_j, let $h_j^{[r]}$ be the column of the first r entries of h_j. For $k \geq 1$, the column h_{r+k} is a linear combination of $h_1, \ldots, h_r$. Consequently, $h_{r+k}^{[r]}$ is a linear combination of $h_1^{[r]}, \ldots, h_r^{[r]}$. Now assume $h_1^{[r]}, \ldots, h_r^{[r]}$ are linearly dependent. Then the first r rows of H_∞ do not contain a nonzero minor of order r. As the theorem on the equality of the rank (= maximal order of a nonzero minor), the row rank, and the column rank holds for both finite and infinite matrices, it follows that the first r rows of H_∞ must be linearly dependent, which is a contradiction. □

Obviously, an arbitrary extension H_∞ of the matrix H contains the triangle table

$$\tau(s_1, s_2, \ldots, s_l) = \begin{pmatrix} s_1 & \ldots & s_q & \ldots & s_l \\ \cdot & \cdot & \cdot & \cdot & \\ s_p & \ldots & s_l & & \\ & \ldots & & & \\ s_l & & & & \end{pmatrix}. \tag{5}$$

A row $s_i, s_{i+1}, \ldots, s_l$ of the table (5) is said to be a prefix linear combination of the rows located above if it is a linear combination of the initial subrows of length $l - i + 1$ of the mentioned rows.

Let $m(s_1, s_2, \ldots, s_l)$ be the maximal number k such that none of the rows of the table (5) with index $i \leq k$ is a prefix linear combination of rows located above.

Lemma 3. *The smallest possible rank of an infinite Hankel extension of the matrix H is equal to $m = m(s_1, s_2, \ldots, s_l)$.*

Proof. 1. The rank of an arbitrary extension H_∞ is at least m. Otherwise some ith row of that extension ($i \leq m$) is a linear combination of the previous rows. Therefore, its prefix $s_i, \ldots, s_l$ is a prefix linear combination of the rows located above in $\tau(s_1, s_2, \ldots, s_l)$, which contradicts the definition of m.

2. There exists an extension H_∞ of rank m. Define an element s_{l+1} as follows. If $m < l$, then the row $s_{m+1}, \ldots, s_l$ of the table $\tau(s_1, s_2, \ldots, s_l)$ is a prefix linear combination of

the rows located above. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the coefficients of that combination. In this case, we set $s_{l+1} = \alpha_1 s_{l-m+1} + \alpha_2 s_{l-m+2} + \ldots + \alpha_m s_l$. If $m = l$, then s_{l+1} can be an arbitrary element of the field F. Clearly, for every element s_{l+1}, the row with the number $m+1$ of the table $\tau(s_1, s_2, \ldots, s_l, s_{l+1})$ is a prefix linear combination of the first m rows. Similarly one defines the element, s_{l+2}, etc. This way one constructs an infinite sequence

$$s_1, s_2, \ldots, s_l, s_{l+1}, s_{l+2}, \ldots$$

for which the corresponding Hankel matrix H_∞ has its first m rows independent and the $(m+1)$st row is a combination of the first m rows. Lemma 1 yields that the rank of this matrix is m. □

Now all the preparations are done and we can attack the main theorem.

Theorem 4. *For every rectangular Hankel matrix, we have* $\operatorname{rk} H = \min(p, q, m, p+q-m)$.

Proof. Let H_∞ be the matrix of the minimal rank m, H_m be its corner submatrix of order m. There are three possible cases.

1. $m \leq \min(p, q)$, and hence, $m \leq p + q - m$. In this case, H_m is a submatrix of H. This matrix in nondegenerate by Theorem 2, hence $\operatorname{rk} H \geq m$. On the other hand, H is a submatrix of the matrix H_∞, which is also of rank m. Consequently, $\operatorname{rk} H = m = \min(p, q, m, p + q - m)$.

2. $\min(p, q) < m < \max(p, q)$. Suppose, for example, $p < m < q$. Then, of course, $p < p+q-m$. The first p rows of the nondegenerate matrix H_m are linearly independent and are actually subrows of the rows H. Hence, the rows H are linearly independent. Taking into account the inequality $p < p+q-m$, we obtain $\operatorname{rk} H = p = \min(p, q, m, p+q-m)$. In the case $q < m < p$ we similarly conclude that $\operatorname{rk} H = q = \min(p, q, m, p + q - m)$.

3. $m \geq \max(p, q)$, or, equivalently, $p + q - m = \min(p, q, m, p + q - m)$. Denote by Q the matrix composed with the first q columns of H_m. Arguing as in the first part of the proof of Lemma 3, we see that none of the rows of Q with index $i > p$ is a linear combination of rows of Q located above the ith row. There are in total $m-p$ such rows, therefore $\operatorname{rk} Q = \operatorname{rk} H + (m-p)$. On the other hand, $\operatorname{rk} Q = q$, because the columns of Q being columns of a nondegenerate matrix are independent. Therefore, $\operatorname{rk} H + (m-p) = q$, i.e. $\operatorname{rk} H = p + q - m$. □

Let us remark that in case 3, the nondegenerate Hankel matrix H_m of order $m = p + q - \operatorname{rk} H$ contains the matrix H as a corner submatrix. As shown in [8], for an arbitrary $(p \times q)$ matrix A, not necessarily Hankel, the number $p + q - \operatorname{rk} A$ is equal to the smallest order of a nondegenerate matrix containing A as a submatrix. Consequently, in case 3 it is impossible to reduce the order of the comprising nondegenerate matrix by relaxing the Hankel property.

Acknowledgements

I am grateful to Prof. V. Protasov for very useful discussions and for his help in preparing this note. I am also expressing my thanks to the handling editor Prof. A. Böttcher and to anonymous referee for their attentive reading, useful comments, and kind suggestions.

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