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Invariant Subspaces of Operators on a Hilbert Space

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Abstract—In year 2006 the author proposed an approach to the invariant subspace problem for an operator on a Hilbert space, based on projection-convex combinations in C^*-algebras with the unitary factorization property. In this paper, we present an operator inequality characterizing the invariant subspace of such an operator. Eight corollaries are obtained. For an operator C^*-algebra $\mathcal{A}$ with a faithful trace, we give a sufficient condition of commutation for a partial isometry from $\mathcal{A}$ with a projection onto its invariant subspace.

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1. INTRODUCTION

In [1, 2] we proposed an approach to the invariant subspace problem for an operator on a Hilbert space, based on projection-convex combinations in C^*-algebras with the unitary factorization property. In this paper, we present an operator inequality characterizing the invariant subspace of such an operator (Theorem 1). From Theorem 1, eight corollaries are obtained.

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$, and let $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all bounded linear operators on $\mathcal{H}$. Let φ be a faithful trace on a C^*-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, and let a partial isometry U and a projection P of $\mathcal{A}$ be such that $PU^*UP \geq PUU^*P$, $\varphi(P) < +\infty$ and $P\mathcal{H}$ is an invariant subspace of the operator U. Then $UP = PU$ (Theorem 2). It is a generalization of Theorem 4.1 of [3]. Let a Hermitian operator $A \in \mathcal{B}(\mathcal{H})$ and a projection $P \in \mathcal{B}(\mathcal{H})$ be such that $i[A, P] \geq \lambda|A| + \mu P$ for some numbers $\lambda \in \mathbb{R}^+$, $\mu \in \mathbb{R}$, moreover, $\lambda = 0 \Leftrightarrow \mu = 0$. Then $[A, P] = 0$ (Theorem 3).

2. NOTATION AND DEFINITIONS

A C^*-algebra is a complex Banach $*$-algebra $\mathcal{A}$ such that $||A^*A|| = ||A||^2$ for all $A \in \mathcal{A}$. For a C^*-algebra $\mathcal{A}$, by $\mathcal{A}^{\mathrm{pr}}$, $\mathcal{A}^{\mathrm{id}}$, $\mathcal{A}^{\mathrm{sa}}$, and $\mathcal{A}^+$ we denote the subsets of projections ($A = A^2 = A^*$), idempotents ($A = A^2$), Hermitian elements ($A^* = A$), and positive elements of $\mathcal{A}$, respectively. If $A \in \mathcal{A}$, then $|A| = \sqrt{A^*A} \in \mathcal{A}^+$. If $A \in \mathcal{A}^{\mathrm{sa}}$, then $A_+ = (|A| + A)/2$ and $A_- = (|A| - A)/2$ lie in $\mathcal{A}^+$ and $A = A_+ - A_-$, $A_+A_- = 0$. If I is the unit of an algebra $\mathcal{A}$ and $P \in \mathcal{A}^{\mathrm{id}}$, then $P^\perp = I - P \in \mathcal{A}^{\mathrm{id}}$. By $[A, B]$ we denote the commutator of elements A and B of $\mathcal{A}$, i.e. the element $AB - BA$.

By *trace* on a C^*-algebra we mean a mapping $\varphi : \mathcal{A}^+ \to [0, +\infty]$ such that

$$\varphi(X + Y) = \varphi(X) + \varphi(Y), \quad \varphi(\lambda X) = \lambda\varphi(X) \quad \text{for all} \quad X, Y \in \mathcal{A}^+, \quad \lambda \geq 0$$

(here $0 \cdot (+\infty) \equiv 0$), and

$$\varphi(Z^*Z) = \varphi(ZZ^*) \quad \text{for all} \quad Z \in \mathcal{A}.$$

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A trace φ is said to be *faithful*, if $\varphi(X) = 0$ $(X \in \mathcal{A}^+) \Rightarrow X = 0$.

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$, and let $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all bounded linear operators on $\mathcal{H}$. Any C^*-algebra can be realized as a C^*-subalgebra in $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (Gelfand–Naimark theorem; see Theorem 3.4.1 in [4]). An operator $A \in \mathcal{B}(\mathcal{H})$ is called *hyponormal* if $A^*A \geq AA^*$; an *isometry* if $A^*A = I$; a *partial isometry* if A^*A is a projection.

Definition. The subspace $\mathcal{K} \subset \mathcal{H}$ is *invariant* under the operator $A \in \mathcal{B}(\mathcal{H})$ if $A\xi \in \mathcal{K}$ for every $\xi \in \mathcal{K}$.

3. MAIN RESULTS

Theorem 1. *Let $P \in \mathcal{B}(\mathcal{H})^{id}$ and $A \in \mathcal{B}(\mathcal{H})$.*

(i) *If $P\mathcal{H}$ is an invariant subspace of an operator A then $APP^*A^* \leq ||AP||^2 PP^* \leq ||A||^2||P||^2 PP^*$.*

(ii) *If $APP^*A^* \leq cPP^*$ for some number $c > 0$ then $P\mathcal{H}$ is an invariant subspace of an operator A.*

Proof. It is well known that the subspace $P\mathcal{H}$ is invariant under the operator $A \in \mathcal{B}(\mathcal{H})$ if and only if $AP = PAP$ [5, Chap. 0, Theorem 0.1].

(i). Since $P^*A^* = (AP)^* = (PAP)^* = P^*A^*P^*$ and $APP^*A^* \leq ||AP||^2 I$, $||AP|| \leq ||A||||P||$, we have

$$APP^*A^* = PAP \cdot P^*A^*P^* = P \cdot APP^*A^* \cdot P^* \leq P \cdot ||AP||^2 I \cdot P^*$$
$$= ||AP||^2 PP^* \leq ||A||^2||P||^2 PP^*.$$

(ii). Multiply both sides of the relation $APP^*A^* \leq cPP^*$ by the idempotent $P^\perp$ from the left and by the idempotent $P^{\perp *}$ from the right. Then $0 = P^\perp APP^*A^*P^{\perp *} = |P^*A^*P^{*\perp}|^2$. Hence $|P^*A^*P^{*\perp}| = 0$ and $P^*A^*P^{*\perp} = 0$. Thus $P^*A^* = P^*A^*P^*$ and $AP = (P^*A^*)^* = (P^*A^*P^*)^* = PAP$, i.e. $P\mathcal{H}$ is an invariant subspace of the operator A. □

Corollary 1. *For $A \in \mathcal{B}(\mathcal{H})$ and $P \in \mathcal{B}(\mathcal{H})^{pr}$ the following conditions are equivalent:*

(i) *$AP = PAP$, i.e. $P\mathcal{H}$ is an invariant subspace of the operator A;*

(ii) *$APA^* \leq ||A||^2 P$.*

Corollary 2. *For $A \in \mathcal{B}(\mathcal{H})^{sa}$ and $P \in \mathcal{B}(\mathcal{H})^{pr}$ the following conditions are equivalent:*

(i) $[A, P] = 0$;

(ii) $APA \leq ||A||^2 P$.

Corollary 3 ([6, Chap. 2, item 217]). *If $B \in \mathcal{B}(\mathcal{H})^+$, $P \in \mathcal{B}(\mathcal{H})^{pr}$ and $B \leq P$, then $[B, P] = 0$.*

Proof. If $A \in \mathcal{B}(\mathcal{H})^+$ and $\dfrac{A}{||A||} = \sqrt{B}$, then $A^2 \leq ||A||^2 P$. Since $APA \leq AIA = A^2$, we have $APA \leq ||A||^2 P$. □

Corollary 4. *Let $A \in \mathcal{B}(\mathcal{H})$ be such that $||A|| \leq 1$, $P \in \mathcal{B}(\mathcal{H})^{pr}$ and $AP^\perp A^* \geq P^\perp$. Then $P\mathcal{H}$ is an invariant subspace of the operator A.*

Proof. We have

$$I - P = P^\perp \leq AP^\perp A^* = AA^* - APA^* \leq I - APA^*,$$

i.e. $APA^* \leq P$. □

Corollary 5. *Let $A \in \mathcal{B}(\mathcal{H})$ be such that $||A|| = 1$, $P \in \mathcal{B}(\mathcal{H})^{pr}$ and $APA^* = P$. Then $P\mathcal{H}$ and $P^\perp\mathcal{H}$ are invariant subspaces of the operator A, i.e. $AP = PA$.*

Corollary 6. *Let $P, Q \in \mathcal{B}(\mathcal{H})^{pr}$ be such that $P\mathcal{H}$ is an invariant subspace of an operator $A \in \mathcal{B}(\mathcal{H})$ and $QAPA^*Q \geq \lambda AQA^*$ for some number $\lambda > 0$. Then $Q\mathcal{H}$ is also an invariant subspace of the operator A.*

Proof. Since $QPQ \leq Q$ and $APA^* \leq ||A||^2 P$, we have

$$||A||^2 Q \geq ||A||^2 QPQ \geq QAPA^*Q \geq \lambda AQA^*,$$

i.e. $AQA^* \leq \lambda^{-1}||A||^2 Q$. □

Corollary 7. *Let $P, Q \in \mathcal{B}(\mathcal{H})^{pr}$ and $P\mathcal{H}$ be an invariant subspace of an operator $A \in \mathcal{B}(\mathcal{H})$.*

(i) *We have $APA^* \leq A^*PA + ||A||^2\, I - A^*A$.*

(ii) *If the operator $T = PA^*Q$ is hyponormal then $Q\mathcal{H}$ is an invariant subspace of the operator AP.*

Proof. (i). We have $APA^* \leq ||A||^2 P$ by Theorem 1. Since $||A^*|| = ||A||$ and $P^\perp\mathcal{H}$ is an invariant subspace of the operator A^*, we have $A^*P^\perp A \leq ||A||^2 P^\perp$ by Theorem 1. Summing up these two inequalities term by term, we obtain

$$APA^* + A^*P^\perp A = APA^* + A^*A - A^*PA \leq ||A||^2\, I.$$

(ii). Both sides of the relation $APA^* \leq ||A||^2 P$ multiplication by the projection Q from the left and the right, given inequalities $TT^* \leq T^*T$ and $QPQ \leq Q$, allows us to obtain

$$PA^*QAP = TT^* \leq T^*T = QAPA^*Q \leq ||A||^2 QPQ \leq ||A||^2 Q.$$

The assertion is proved. □

Corollary 8. *Let $P \in \mathcal{B}(\mathcal{H})^{pr}$ and $P\mathcal{H}$ be an invariant subspace of an isometry $A \in \mathcal{B}(\mathcal{H})$. Then $APA^* \leq P \leq A^*PA$.*

Theorem 2 (cf. [3, Theorem 4.1]). *Let φ be a faithful trace on a C^*-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, and a partial isometry U and a projection P of $\mathcal{A}$ be such that $PU^*UP \geq PUU^*P$, $\varphi(P) < +\infty$ and $P\mathcal{H}$ be an invariant subspace of an operator U. Then $UP = PU$.*

Proof. Since $||U|| = 1$, we have $UPU^* \leq P$ by Theorem 1. Both sides of these relation multiplication by the projection UU^* from the left and the right, given equality $UU^*U = U$ [7, Problem 98], allows us to obtain

$$UPU^* \leq UU^*\, P\, UU^*. \tag{1}$$

Since $PU^*UP \geq PUU^*P$ and UU^* is a projection, from (1) we have

$$\varphi(PU^*UP) = \varphi(UPU^*) \leq \varphi(UU^*PUU^*) = \varphi(P(UU^*)^2P) = \varphi(PUU^*P) \leq \varphi(PU^*UP)$$

by the monotonocity of the trace φ on $\mathcal{A}^+$. Thus, $\varphi(PUU^*P) = \varphi(PU^*UP)$ and since the trace φ is faithful, we have

$$PU^*UP = PUU^*P. \tag{2}$$

Both sides of the relation $UPU^* \leq P$ multiplication by U^* from the left and and by U from the right, leads us to the inequality $U^*UPU^*U \leq U^*PU$. Since $Q = U^*U$ is a projection, by the monotonocity of the trace φ on $\mathcal{A}^+$ and by (2) we obtain

$$\varphi(PU^*UP) = \varphi(U^*UPU^*U) \leq \varphi(U^*PU) = \varphi(PUU^*P) = \varphi(PU^*UP).$$

Therefore, $\varphi(U^*UPU^*U) = \varphi(U^*PU)$ and

$$U^*UPU^*U = U^*PU, \tag{3}$$

since the trace φ is faithful. Relation (3) multiplication by projection P from the left and and the right-hand sides, given equality $UP = PUP$, provides us with the identity $(PU^*UP)^2 = PU^*PUP = PU^*UP$, i.e. the operator $PU^*UP = PUU^*P$ is a projection. Therefore, the operator PU is a partial isometry, hence the operator U^*PU is a projection. Now from (3) we infer that the product QPQ of the projections P and Q is a projection. Thus, $PQ = QP$ by [8, Proposition 2]. (The equality $PQ = QP$ follows also from Corollary 5.) From (3) we have $PU^*U = U^*UPU^*U = U^*PU$, and given equality $PU^* = PU^*P$ we obtain $PU^*PU = U^*PU$. The last equality is equivalent to the inequality $U^*PU \leq P$. So, since $||U^*|| = 1$, we have $U^*P = PU^*P$ by Theorem 1. Passing to adjoints here, we obtain $PU = (U^*P)^* = (PU^*P)^* = PUP = UP$. □

In particular, if an isometry $U \in \mathcal{B}(\mathcal{H})$ and $P\mathcal{H}$ is a finite-dimensional invariant subspace of an operator U, then $PU = UP$.

Theorem 3. *Let $A \in \mathcal{B}(\mathcal{H})^{sa}$ and $P \in \mathcal{B}(\mathcal{H})^{pr}$ be such that*

$$i[A, P] \geq \lambda|A| + \mu P \tag{4}$$

for some numbers $\lambda \in \mathbb{R}^+$, $\mu \in \mathbb{R}$, *moreover* $\lambda = 0 \Leftrightarrow \mu = 0$. *Then* $[A, P] = 0$ *and for* $\lambda > 0$ *we have* $A_+ = A_+P = PA_+$ *and* $|A| \leq -\frac{\mu}{\lambda}P$.

Proof. If $\lambda = \mu = 0$, then we have $AP = PA$ by Proposition 4.2 of [3]. For $\lambda > 0$ relation (4) multiplication by projection $P^\perp$ from the left and and the right-hand sides, leads us to

$$0 = P^\perp \cdot i(AP - PA) \cdot P^\perp \geq \lambda P^\perp |A| P^\perp \geq 0.$$

Therefore, $\lambda P^\perp |A| P^\perp = 0$ and $P^\perp |A| P^\perp = P^\perp A_+ P^\perp + P^\perp A_- P^\perp = 0$. Since $P^\perp A_+ P^\perp \geq 0$, we have $P^\perp A_+ P^\perp = P^\perp A_- P^\perp = 0$. We have

$$P^\perp A_+ P^\perp = |\sqrt{A_+} P^\perp|^2 = 0$$

and $\sqrt{A_+} P^\perp = 0$, hence $A_+ P^\perp = \sqrt{A_+} \cdot \sqrt{A_+} P^\perp = 0$. Now $A_+ = A_+P = PA_+$, $A = AP = PA$ и $|A| = |A|P = P|A|$. From (4) we obtain $0 \geq \lambda|A| + \mu P$, i.e. $\mu < 0$ and $|A| \leq -\frac{\mu}{\lambda}P$.

For an operator $A \geq 0$ we have $[A, P] = 0 \Leftrightarrow i[A, P] \leq a(AP + PA)$ for some $a \in \mathbb{R}^+$. Indeed, if $A \geq 0$ and $AP = PA$, then $AP + PA = 2PAP \geq 0$. Conversely, if $a = 0$, then by Proposition 4.2 of [3] we have $AP = PA$. For $a > 0$ we apply the proof of the implication (iv)⇒(i) of Proposition 2 of [8], that featured the projector Q instead of the operator A. □

Let $\mathcal{A}$ be a C^*-algebra. For any $P \in \mathcal{A}^{\mathrm{id}}$ there exists a unique decomposition $P = \widetilde{P} + Z$, where $\widetilde{P} \in \mathcal{A}^{\mathrm{pr}}$ and $Z \in \mathcal{A}$ is a nilpotent with $Z^2 = 0$, moreover,

$$Z\widetilde{P} = 0, \quad \widetilde{P}Z = Z \tag{5}$$

[9, Theorem 1.3].

Theorem 4. *Let a* C^**-algebra* $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ *and* $P \in \mathcal{A}^{id}$ *be so that* $P\mathcal{H}$ *is an invariant subspace of an operator* $A \in \mathcal{A}$. *Let* $P = \tilde{P} + Z$ *be the above mentioned decomposition. Then* $AZ = \tilde{P}AZ$ *and* $ZA\tilde{P} = ZAZ = 0$.

Proof. Note that $\tilde{P}\mathcal{H} = P\mathcal{H}$, hence $A\tilde{P} = \tilde{P}A\tilde{P}$ and $AP = PAP$ by [5, Chap. 0, Theorem 0.1]. The equality $AP = PAP$ can be rewritten as

$$AZ = \tilde{P}AZ + ZA\tilde{P} + ZAZ. \tag{6}$$

We multiply relation (6) by the projection $\tilde{P}$ from the right, apply (5), and obtain $ZA\tilde{P} = 0$. This relation multiplication by the operator Z from the right and application of the equality $\tilde{P}Z = Z$ lead us to the equality $ZAZ = 0$. Now from (6) we have $AZ = \tilde{P}AZ$. □

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