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Generalized Reduced Module of a Domain Over the Unit Disc with Circular and Radial Slits

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Abstract—For $(n+1)$-ly connected planar domain D with analytic boundary we construct the function $F(w, w_0) = (w - w_0)f(w, w_0)$ which maps D conformally onto the unit disk with circular and radial slits. We show that if $n \geq 2$, then Mityuk's function, $M(w) = -(2\pi)^{-1} \ln |f(w, w)|$, representing the generalized reduced module of the domain D has at least one stationary point in D.

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1. INTRODUCTION

Classical task in the complex function theory concerns with the construction of the conformal mappings

$$F(w, w_0) = (w - w_0)f(w, w_0), \quad f(w_0, w_0) \neq 0, \tag{1}$$

from the planar finitely connected regions onto the canonical domains such as the unit disk centered at the origin with cuts along the arcs of prescribed form, namely, circular concentric arcs, radial slits, or their various disjoint combinations.

I. P. Mityuk [1] has proposed a way to define a generalized reduced modules connected with function $F(w, w_0)$. The generalized reduced module,

$$M(w) = -\frac{1}{2\pi} \ln |f'(w, w)| \tag{2}$$

of a multiply connected domain D at a point w will be called *Mityuk's function* with respect to the distinguished canonical domain.

Connection of the functions (2) with the exterior inverse boundary value problems goes back to Gakhov [2]. As it has appeared, the non-emptiness of the critical points set of the function $M(w)$ is equivalent to the suitable exterior problem. The existence of critical points of Mityuk's function in the case of circular concentric slits has been proved by Kinder [3]. The case of circular and radial slits is studied in the present report (see also [4] and [5]).

Let D be $(n+1)$-ly connected domain with the boundary ∂D, consisting of disjoint analytic curves L_k, $k = \overline{0, n}$; the contour L_0 encircles the others. In the section 1 we shell define the auxiliary functions involving in the construction of the mapping (1) of the domain D onto the unit disk with radial and circular slits. In the Section 2 the existence and the univalence of such a mapping will be proved. In the Section 3 we show that the function (2) has at least one stationary point in D if $n \geq 2$; the doubly-connected example is constructed where the function (2) has no stationary points.

Let us note that a number of the assertions of this note can be transferred to the general case of Jordan domains; we won't stop on details.

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2. FUNCTIONS $S(w,w_0)$, $S^*(w,w_0)$ AND THEIR PROPERTIES

Let $1<m<n$. We let Γ_1 denote the collection of contours $L_0, L_1, \ldots, L_m$, and let Γ_2 be a finite set of curves $L_{m+1}, L_{m+2}, \ldots, L_n$. Let us introduce two fundamental functions $S(w,w_0)$ and $S^*(w,w_0)$ which are the analogies of the Green and Neumann functions for the first and second problems of mathematical physics. The quantity $S(w,w_0)$ is defined by the following properties:

1) the function $s(w,w_0)=S(w,w_0)+\ln|w-w_0|$ is harmonic everywhere in the region D;

2) $S(w,w_0)$ has the following boundary values:

$$S(t,w_0)=0, t\in\Gamma_1, \quad \text{and} \quad (\partial S/\partial n)(t,w_0)=0, t\in\Gamma_2, \tag{3}$$

where $\vec{n}$ is the inner normal to the boundary ∂D, and the notation $t\in\Gamma_k$ means that t varies over all of the contours in the collection Γ_k, $k=1,2$.

The function $S^*(w,w_0)$ is defined analogously with

$$(\partial S^*/\partial n)(t,w_0)=0, \quad t\in\Gamma_1, \quad \text{and} \quad S^*(t,w_0)=0, \quad t\in\Gamma_2, \tag{4}$$

instead of (3). Functions $S(w,w_0)$ and $S^*(w,w_0)$ are uniquely defined by these properties, and their existence follows from the general theorems of the potential theory or from the following reasons. Using the idea of the double of the multiply connected domain we construct the Riemann surface $\mathfrak{R}$, sewing the region D and its copy, $\widetilde{D}$, along boundary contours Γ_2. Due to the second boundary condition in (3) the problem of search of function $S(w,w_0)$ harmonically continues from D to $\widetilde{D}$. At the same time the values $S(w,w_0)$ vanish on each of $2(m+1)$ boundary components of the border of $\mathfrak{R}$. Thus the finding of the function $s(w,w_0)=S(w,w_0)+\ln|w-w_0|$ on the Riemann surface $\mathfrak{R}$ of the genus $n-m-1$ leads to the usual Dirichlet problem which solvability is well-known (see, e.g., [6]).

Integral representation for harmonic functions and boundary behavior of $S(w,w_0)$ and $S^*(w,w_0)$ imply the following

Property 1. *If $u(w)$ is a harmonic function, which is continuously differentiable in the closed domain $\overline{D}$, then we have*

$$u(w)=\frac{1}{2\pi}\int\limits_{\Gamma_1} u(t)\frac{\partial S}{\partial n}(t,w)d\sigma-\frac{1}{2\pi}\int\limits_{\Gamma_2}\frac{\partial u}{\partial n}(t)S(t,w)d\sigma, \tag{5}$$

$$u(w)=-\frac{1}{2\pi}\int\limits_{\Gamma_1}\frac{\partial u}{\partial n}(t)S^*(t,w)d\sigma+\frac{1}{2\pi}\int\limits_{\Gamma_2} u(t)\frac{\partial S^*}{\partial n}(t,w)d\sigma. \tag{6}$$

The following assertion establishes the symmetry of the functions S and S^*.

Property 2. *Functions $S(w,w_0)$ and $S^*(w,w_0)$ are harmonic in D with respect to w_0 (for fixed $w\neq w_0$), and $S(w,w_0)=S(w_0,w)$, $S^*(w,w_0)=S^*(w_0,w)$.*

The proof is completely the same as the one for the Green function to be symmetric with respect to its arguments (see, for instance, [7]).

To formulate the Property 3 we are needed in some constructions which we shell introduce now.

By the addition of the conjugate functions $U(w,w_0)$ and $U^*(w,w_0)$ we complete the functions $S(w,w_0)$ and $S^*(w,w_0)$ up to the functions $T(w,w_0)=S(w,w_0)+iU(w,w_0)$ and $T^*(w,w_0)=S^*(w,w_0)+iU^*(w,w_0)$ and which are analytic with respect to w, harmonic with respect to w_0, and multi-valued in D. Let us find their increments when a point w goes around the closed curves homotopic to the boundary contours. It is easy to see that the circuit along the contours $L_k\in\Gamma_1$ doesn't change the function $T^*(w,w_0)$, but gives to $T(w,w_0)$ the increments

$$-2\pi i\xi_k(w)=\int\limits_{L_k} dT(t,w)=i\int\limits_{L_k}\frac{\partial U}{\partial\sigma}(t,w)d\sigma=-i\int\limits_{L_k}\frac{\partial S}{\partial n}(t,w)d\sigma. \tag{7}$$

Similarly, under the circuit of the contour $L_k\in\Gamma_2$ the function $T(w,w_0)$ remains former, but $T^*(w,w_0)$ obtains the increment

$$-2\pi i\xi_k^*(w)=\int\limits_{L_k} dT^*(t,w)=-i\int\limits_{L_k}\frac{\partial S^*}{\partial n}(t,w)d\sigma. \tag{8}$$

Periods $\xi_k(w)$ and $\xi_k^*(w)$ are functions harmonic and single-valued in D. Formulae (5), (6) imply that the boundary values $\xi_k(w)$ vanish on all of the contours Γ_1 except $L_k(\in \Gamma_1)$, where these values are equal to unit; on the rest of the boundary we have $(\partial\xi_k/\partial n)|_{\Gamma_2} = 0$. Analogously, $\xi_k^*|_{L_j} = 0$, $\xi_k^*|_{L_k} = 1$ for $L_k \in \Gamma_2$ and $L_j \in \Gamma_2$, $j \neq k$; $(\partial\xi_k^*/\partial n)|_{\Gamma_1} = 0$.

Functions $\xi_k(w)$ and $\xi_k^*(w)$ are analogs of the harmonic measures in the Dirichlet problem. We apply now to these functions the procedure of completion them by the conjugate functions. As a result we obtain analytic functions $\zeta_k(w)$, $k = \overline{0,m}$, and $\zeta_k^*(w)$, $k = \overline{m+1,n}$, which are determined to within an imaginary constant. They admit integral representations following from (7) and (8):

$$\zeta_k(w) = \frac{1}{2\pi}\int\limits_{L_k} \frac{\partial T}{\partial n}(t,w)d\sigma, \quad \zeta_k^*(w) = \frac{1}{2\pi}\int\limits_{L_k} \frac{\partial T^*}{\partial n}(t,w)d\sigma.$$

When a point w goes around the closed curves homotopic L_j, the functions $\zeta_k(w)$ and $\zeta_k^*(w)$ get the imaginary increments

$$2\pi i A_{jk} = \int\limits_{L_j} d\zeta_k(w) = -i\int\limits_{L_j} \frac{\partial \xi_k}{\partial n}(t)d\sigma, \quad k = \overline{0,m}, \tag{9}$$

and

$$2\pi i A_{jk}^* = \int\limits_{L_j} d\zeta_k^*(w) = -i\int\limits_{L_j} \frac{\partial \xi_k^*}{\partial n}(t)d\sigma, \quad k = \overline{m+1,n},$$

respectively. Owing to the above mentioned boundary behavior of the functions $\xi_k(w)$ and $\xi_k^*(w)$ all of the constants A_{jk} for $j > m$, A_{jk}^* for $j \leq m$ are equal to zero. It follows from the relation

$$A_{jk} = -\frac{1}{4\pi^2}\iint\limits_{L_j L_k} \frac{\partial^2 S(t,\tau)}{\partial n(t)\partial n(\tau)}d\sigma(t)d\sigma(\tau) = A_{kj}$$

that the numbers A_{jk}, $j,k = \overline{0,m}$, form the symmetric matrix. Similar symmetry is obtained for the matrix with entries A_{jk}^*, $j,k = \overline{m+1,n}$. We are able now to formulate and prove the following

Property 3. *Ranks of matrices $\{A_{jk}\}_0^m$ and $\{A_{jk}^*\}_{m+1}^n$ are equal m and $n-m-1$, respectively.*

Proof is carried out for the matrix $\{A_{jk}\}_0^m$ (the matrix $\{A_{jk}^*\}_{m+1}^n$ is treated similarly). Following to [8] we form a harmonic function in D, $\xi(w) = \sum_{k=0}^m \alpha_k \xi_k(w)$, and calculate an integral

$$\frac{1}{2\pi}\iint\limits_D [\mathrm{grad}\xi(w)]^2 du dv = -\frac{1}{2\pi}\int\limits_{\partial D} \xi(t)\frac{\partial \xi}{\partial n}(t)d\sigma = -\frac{1}{2\pi}\int\limits_{\Gamma_1} \xi(t)\frac{\partial \xi}{\partial n}(t)d\sigma.$$

Using the inequalities (9) and taking into account the boundary properties of the function $\xi(w)$, we get

$$\frac{1}{2\pi}\iint\limits_D [\mathrm{grad}\xi(w)]^2 du dv = \sum_{j=0}^m \sum_{k=0}^m A_{jk}\alpha_j\alpha_k. \tag{10}$$

The quantity in the left-hand side of the equality (10) is always non-negative: it vanishes only in the case $\mathrm{grad}\xi \equiv 0$ in D, i.e. when $\xi(w) \equiv \mathrm{const}$. But then all of the boundary values, α_k, of the function $\xi(w)$ on the contours Γ_1 are equal to each other.

Conversely, if all of the boundary values, α_k, of the function $\xi(w)$ are equal to each other, then we obtain the boundary value problem for $\xi(w)$, $\xi|_{\Gamma_1} = \alpha$, $(\partial\xi/\partial n)|_{\Gamma_2} = 0$, which, by the uniqueness, has only one solution, $\xi(w) \equiv \alpha$ (it is received by the transfer to the double of the domain D). Thus we have proved the validity of the non-negativity relation

$$\sum_{j=0}^m \sum_{k=0}^m A_{jk}\alpha_j\alpha_k \geq 0,$$

where the equality is attained only when all of α_k's are equal to each other.

We set now $\alpha_0 = 0$. The quadratic form

$$\sum_{j=1}^{m}\sum_{k=1}^{m} A_{jk}\alpha_j\alpha_k = \sum_{j=0}^{m}\sum_{k=0}^{m} A_{jk}\alpha_j\alpha_k$$

vanishes only when all of α_k's are equal to α_0, i.e. equal to zero. Therefore, the matrix $\{A_{jk}\}_1^m$ is positive definite, and hence it has non-zero determinant. Now we will show that the latter is the rank minor.

In fact, it follows from the properties of periods that $\sum_{j=0}^{m} A_{kj} = 0$, $k = \overline{0, m}$, and, moreover, as it is noted above, all of A_{jk}'s vanish for $j > m$. Consequently,

$$A_{k0} + \sum_{j=1}^{m} A_{kj} = 0, \tag{11}$$

whence $\det\{A_{kj}\}_0^m = 0$. Together with the just proved inequality, $\det\{A_{kj}\}_1^m \neq 0$, it yields to us the desirable result, $rank\{A_{kj}\}_0^m = m$.

3. MAPPING ONTO THE UNIT DISK WITH CIRCULAR AND RADIAL SLITS

Let $\{B_{jk}\}_1^m$ and $\{B_{jk}^*\}_{m+2}^n$ are the inverse matrices to $\{A_{jk}\}_1^m$ and $\{A_{jk}^*\}_{m+2}^n$. We have the following

Lemma 1. *Let D be $(n+1)$-ly connected Jordan domain with analytic boundary. Function*

$$F(w, w_0) = \exp\{-T(w, w_0) - \sum_{j=1}^{m}\sum_{k=1}^{m} B_{jk}\zeta_k(w)\xi_j(w_0)\} \tag{12}$$

maps D conformally and univalently onto the unit disk with m cuts along the concentric circular arcs centered at the origin and with $n-m$ radial slits lying on the lines meeting at the origin.

Proof. To find out the geometrical properties of the function (12) we shell determine the multivalence character of $\arg F(w, w_0)$. The function

$$\sum_{j=1}^{m}\sum_{k=1}^{m} B_{jk}\zeta_k(w)\xi_j(w_0) \tag{13}$$

is analytic in w; its periods under the circuit along the curves homotopic to the contours from the collection Γ_2 are equal to zero. Increments, which (13) receives under the circuit of $L_l \in \Gamma_1$, are equal to

$$2\pi i\sum_{j=1}^{m}\sum_{k=1}^{m} B_{jk}A_{kl}\xi_j(w_0) = 2\pi i\xi_l(w_0)$$

by virtue of (9). The function $-T(w, w_0)$ has the same periods, therefore the expression

$$\ln F(w, w_0) = -T(w, w_0) - \sum_{j=1}^{m}\sum_{k=1}^{m} B_{jk}\zeta_k(w)\xi_j(w_0) \tag{14}$$

has no periods with respect to the curves homotopic to inner boundary contours L_k ($k = \overline{1, n}$). Let us find an increment of the function (14) relative to L_0. For this purpose we take into account that the periods $T(w, w_0)$ and $\zeta_k(w)$ under the circuit of L_0 are equal to $2\pi i\xi_0(w_0)$ and, in view of (11), to $2\pi i A_{k0} = -2\pi i\sum_{j=1}^{m} A_{kj}$, respectively. Hence the increment, which $\ln F(w, w_0)$ receives under the circuit of the contour L_0, is equal to $+2\pi i$. Thus the function $F(w, w_0)$ is single-valued in D and has the simple zero at the point $w = w_0$.

For the values of (12) we shell determine the modules on the contours of Γ_1 and the arguments on the contours of Γ_2. For this purpose we shell calculate the real part of the function $\ln F(w, w_0)$ on Γ_1, and its imaginary part on Γ_2. We consider the known boundary values of $T(w, w_0)$ and $\zeta_k(w)$ to find that

$$\ln|F(t, w_0)| = \begin{cases} 0, & t \in L_0, \\ -\sum_{j=1}^{m} B_{jk}\xi_j(w_0), & t \in L_k, k = \overline{1, m}, \end{cases}$$

$$\frac{\partial}{\partial\sigma}\arg F(t,w_0)=0,\quad t\in L_k, k=\overline{m+1,n}.$$

It follows that the value $|F(t,w_0)|$ is constant on every contour $L_k\in\Gamma_1$, and $\arg F(t,w_0)$ is constant on every contour $L_k\in\Gamma_2$. We conclude from these facts that the function $F(w,w_0)$ maps D on the domain $D_F=F(D)$ such that the images of the boundary curves in Γ_1 lie on the circles, and the images of the boundary curves in Γ_2 lie on the straight lines $\arg F(t,w_0)=\mathrm{const}$. An argument of the function $F(w,w_0)$ don't change when w circumscribes any contour L_k, $k\neq 0$, therefore the image of every such contour will be the cut along an arc of the circle ($k=\overline{1,m}$) or radial slit ($k=\overline{m+1,n}$) on some Riemann surface over the plane of the variable F. Since $\arg F(t,w_0)$ increases by 2π when w circumscribes L_0, then the image of the boundary contour L_0 will be the unit circle $|F|=1$.

Having used the argument principle, we will prove the univalence of the mapping $F=F(w,w_0)$. If a is a point of the domain D_F out of concentric circles and straight lines along which the images of boundary contours L_k, $k=\overline{1,n}$, are located, then

$$\mathrm{var}_{\partial D}\arg[F(w,w_0)-a]=2\pi. \tag{15}$$

Indeed, when $w\in L_k$, $k=\overline{m+1,n}$, the difference $F-a$ lies in the half plane bounded by a straight line $\arg F(t,w_0)=\mathrm{const}$, and hence the variation of $\arg[F(w,w_0)-a]$ is equal to zero. When $w\in L_k$, $k=\overline{1,m}$, we represent $F-a$ as $a\,[F(w,w_0)/a-1]$ if $|a|>|F(t,w_0)|$, $t\in L_k$, or by the expression $F(w,w_0)\,[1-a/F(w,w_0)]$ if $|a|<|F(t,w_0)|$, $t\in L_k$; then it is easy to see that the relation

$$\mathrm{var}_{L_k}\arg[F(w,w_0)-a]=0,\quad k=\overline{1,m},$$

holds. By virtue of simplicity of the closed curve $F(L_0)$ representing the unit circle an increment of the argument of $F(w,w_0)-a$ along L_0 is equal to $+2\pi$, and (15) is proved.

If a point a is located at the infinite part of the complement to the unit circle $|F|=1$, then one can show by the same reason that the variation of $\arg[F(w,w_0)-a]$ along the boundary ∂D is equal to zero. In view of the argument principle the function $F(w,w_0)-a$, which doesn't have poles in the domain D, is nonzero if $|a|>1$, and is equal to zero only once if $|a|<1$. For the correctness of the latter conclusion it remains to check that the analogous reasons are valid for the points lying on the curves excluded above.

Really, let, for example, the module of the point a coincides with $F(t,w_0)$, $t\in L_k$, $k=\overline{1,m}$. If the point b ($|b|\neq|a|$) belongs to sufficiently small neighborhood of a, then, by Rouche's theorem, the functions $F(w,w_0)-b$ and $F(w,w_0)-a=[F(w,w_0)-b]+[b-a]$ have the same number of zeros. Therefore the function $F(w,w_0)$ takes the value a only once.

Thus we have established that the function $F(w,w_0)$ produces the univalent mapping from the domain D onto D_F, the unit disk with circular and radial slits. Lemma 1 is proved.

Remark 1. The function $F(w,w_0)$ is determined to within a factor depending, in general, on w_0 and which is in modulus equal to unit. This factor will be uniquely determined if we require that one of the radial slits, which is the image of some contour in Γ_2 (say, the contour L_{m+1}), has the zero inclination.

4. MITYUK'S FUNCTION

We will present the function $F(w,w_0)$ in the form (1); it is clear that $f(w_0,w_0)=F'_w(w_0,w_0)\neq 0$. Function (2) acts as a generalized reduced module of the domain D at a point w with respect to the canonical domain D_F ([1]). Let us call the quantity (2) Mityuk's function, and a quantity

$$\Omega(w)=\exp[2\pi M(w)]=1/|f(w,w)| \tag{16}$$

Mityuk's radius of the domain D at a point w (see [9, 10]). For finding the critical points of the function $M(w)$ we get the equation

$$f_1'(w,w)=0, \tag{17}$$

where $f_1'(w,w_0)$ means an application of the operator

$$\frac{\partial}{\partial w}=\frac{1}{2}\left(\frac{\partial}{\partial u}-i\frac{\partial}{\partial v}\right),\quad w=u+iv,$$

to the function $f(w, w_0)$ with respect to the first argument. There holds the following

Theorem 1. *Equation (17) is solvable in $(n+1)$-ly connected domain D with analytic boundary for any $n \neq 1$.*

Proof. As in the case of Gakhov's equation (see, for example, [11]), we will reduce the solvability of (17) to the existence problem of stationary points for some real surface connected with that equation. Having used the representation (12) of the function $F(w, w_0)$, on the base of the symmetry of the function $S(w, w_0) = \operatorname{Re} T(w, w_0)$ with respect to its arguments and of the symmetry of the matrix $\{B_{jk}\}_{j,k=1}^{m}$ we derive the equality $\ln|F(w, w_0)| = \ln|F(w_0, w)|$, whence

$$\ln|f(w, w_0)| = \ln|f(w_0, w)|. \tag{18}$$

The relation (18) is principal in our reasonings. Further transforms of the equation (17) coincide with those which were used in [11] for Gakhov's equation. By means of (18) the left-hand side of (17) may be presented as

$$f_1'(w, w) = f(w, w)\frac{\partial}{\partial w}\ln|f(w, w)|, \tag{19}$$

Thus it is possible to write the equation (17) in the form $(\partial/\partial w)M(w) = 0$ where $M(w)$ is Mityuk's function (2).

Equality (19) means that the root of the equation (17) is a stationary, or critical point of the surface with the equation $\Omega = \Omega(w)$, where $\Omega(w)$ is the Mityuk's radius (16). We will explain the boundary properties of function (16).

Lemma 2. *Function (16) is infinitely differentiable in the domain D and has the following limit values on the boundary ∂D:*

$$\lim_{w\to t}\Omega(w) = 0, \quad t \in \Gamma_1, \tag{20}$$

$$\lim_{w\to t}\Omega(w) = +\infty, \quad t \in \Gamma_2. \tag{21}$$

Proof of the lemma 2. Function $\Omega(w)$ inherits the property of infinite differentiability from the functions entering its representation. Having used (12), we will rewrite the expression for $\Omega(w)$ in the form

$$\ln\Omega(w) = s(w, w) + \sum_{j=1}^{m}\sum_{k=1}^{m} B_{jk}\xi_j(w)\xi_k(w),$$

where $s(w, w)$ is a regular part of the function $S(w, w_0)$ calculated at a point $w = w_0$, that is $s(w, w) = \lim_{w_0\to w}[S(w, w_0) + \ln|w - w_0|]$.

We will prove a limit relation (21) (proof of the equality (20) is completely analogous). By virtue of the boundedness of the functions $\xi_k(w)$ in the closed domain $\overline{D_w}$ it is sufficient to establish that

$$\lim_{w\to t} s(w, w) = +\infty, \quad t \in \Gamma_2. \tag{22}$$

We will prove (22) for one of the contours $L_k \in \Gamma_2$.

It follows from the equalities (3) that the function $S(w, w_0)$ is the conformal invariant. In order to write this fact it will be convenient to correct notations for a short time. Domains connected by conformal mappings are denoted by the same letter D, but they differ in lower indices corresponding to variables which run over these D's; so the new notation for our D is D_w. Furthermore, we mark the dependence on the domains in the notations of fundamental and related functions defined in these domains: $S(w, w_0) = S(D_w; w, w_0)$, etc. As a result we receive the formula expressing change of $s(D_w; w, w)$ under the conformal transformations of the domain D_w :

$$s(D_w; w, w) = s(D_\zeta; \zeta, \zeta) + \ln|\phi'(\zeta)|, \tag{23}$$

where $D_w = \phi(D_\zeta)$, $w = \phi(\zeta)$. Considering the law of change (23), we pass to auxiliary circular domain D_ζ (unit disk minus n non-intersecting closed disks lying within it), and we map it onto the domain D_w by a function $w = \phi(\zeta)$. We take the unit circle centered at the origin as an outer circle $L_{\zeta 0} \in \partial D_\zeta$ corresponding to the contour $L_k = L_{wk}$.

It is well-known that for the domain D_w with smooth boundary ∂D_w whose tangent slope angle as a function of the arc parameter σ satisfies the Hölder condition the derivative of the conformal mapping $w = \phi(\zeta)$ from D_ζ onto D_w has the bounded module $|\phi'(\zeta)|$. Therefore on the basis of (23) we will reduce the equality (22) to the form

$$\lim_{\zeta \to e^{i\theta}} s(\zeta, \zeta) = +\infty. \tag{24}$$

Let us consider the function $S(\zeta, \zeta_0)$ harmonic in $D_\zeta \backslash \{\zeta_0\}$ whose normal derivative vanishes on the outer circle $L_{\zeta 0}$. Due to this fact the function $S(\zeta, \zeta_0)$ may be harmonically extend beyond the unit circle by the symmetry principle (see, e.g., [12], p. 471). This extension is given by the formula

$$\widetilde{S}(\zeta, \zeta_0) = \begin{cases} S(\zeta, \zeta_0), & |\zeta| < 1, \\ S(1/\overline{\zeta}, \zeta_0), & |\zeta| > 1. \end{cases}$$

Let's note that $S(1/\overline{\zeta}, \zeta_0) = -\ln|(1 - \zeta\overline{\zeta}_0)/\zeta| + s(1/\overline{\zeta}, \zeta_0)$, $|\zeta| > 1$. The first summand here is the function harmonic in the whole plane except the point $1/\overline{\zeta}_0$. Hence the function $S(\zeta, \zeta_0)$ is represented in the form $S(\zeta, \zeta_0) = -\ln|(\zeta - \zeta_0)(1 - \zeta\overline{\zeta}_0)| + h(\zeta, \zeta_0)$, at the same time

$$s(\zeta, \zeta_0) = -\ln|1 - \zeta\overline{\zeta}_0| + h(\zeta, \zeta_0), \tag{25}$$

where the function $h(\zeta, \zeta_0)$ harmonic in ζ has no singularities at the point $\zeta = \zeta_0$. Besides this, the normal derivative of the function $h(\zeta, \zeta_0)$ vanishes on the unit circle. Using an integral representation (6) for $h(\zeta, \zeta_0)$ and setting $\zeta_0 = \zeta$ we will come to the formula

$$h(\zeta, \zeta) = -\frac{1}{2\pi} \sum_{k=1}^{m} \int\limits_{L_{\zeta k}} \frac{\partial h(t, \zeta)}{\partial n} S^*(t, \zeta) d\theta + \frac{1}{2\pi} \sum_{k=m+1}^{n} \int\limits_{L_{\zeta k}} h(t, \zeta) \frac{\partial S^*(t, \zeta)}{\partial n} d\theta. \tag{26}$$

If now $\zeta \to e^{i\theta} \in L_{\zeta 0}$, then, as it is seen from (26), the quantity $h(e^{i\theta}, e^{i\theta})$ is bounded. This and (25) imply (24), and, consequently, (22). Lemma 2 is proved.

Let's continue the proof of Theorem 1 returning to old notations and introducing the new one: we will denote by Ω the surface $\Omega = \Omega(w)$ of Mityuk's radius (16).

According to Lemma 2 the smooth surface Ω is attached to the boundary contours from the collection Γ_1 and has the form of cylinder over each of the components from Γ_2. It is clear that such a surface, in general, doesn't possess a maximum (top) over the domain D. We will prove the existence of a stationary point of the surface D using the properties of the planar vector fields. Let's consider the gradient vector field in D,

$$\operatorname{grad} \Omega(w) = \left(\frac{\partial \Omega}{\partial u}, \frac{\partial \Omega}{\partial v} \right), \quad w = u + iv, \tag{27}$$

whose singular points are exactly the roots of the equation (17). Suppose the domain D doesn't contain the singular points of (27). Then the components of the level lines $\Omega(w) = \varepsilon$ and $\Omega(w) = N$, where ε and N are sufficiently small and large positive numbers, respectively, are the simple closed curves that approximate the boundary of the domain D. We consider the ($(n+1)$-ly connected) domain D^* which is bounded by these curves, and we define the winding number of the vector field (27) along the boundary of the domain D^* by the equality

$$\gamma = -\frac{1}{2\pi i} \int\limits_{\partial D^*} d \ln \frac{\partial \Omega}{\partial w}. \tag{28}$$

It is known from the theory of the planar vector fields [13] that if the winding number (28) is nonzero, then there exists at least one singular point of the vector field (27). Hence in order to prove the solvability of the equation (17) it is enough to show that $\gamma \neq 0$.

We will consider the behavior of $\operatorname{grad} \Omega(w)$ on the boundary of the domain D^*. Since D^* consists of the level lines of the function (16) where the equality

$$\frac{\partial \Omega}{\partial u} du + \frac{\partial \Omega}{\partial v} dv = 0$$

is fulfilled, then the vector grad $\Omega(w)$ is orthogonal to tangent vector (du, dv), i.e. the vector field (27) coincides with the field of normals on the boundary D^*. Therefore $\gamma = 1 - n$. Thus in the case of doubly connected domain ($n = 1$) the winding number of the field (27) is equal to zero. If the connectivity order of the domain D is greater than two, then $\gamma \neq 0$, as was required.

Theorem 1 is proved.

Let's consider the doubly connected case excluded in Theorem 1. We show that the equation (17) can really be unsolvable (cf. [14]).

Example. The function

$$F(w, w_0) = \frac{w - w_0}{1 - \overline{w}_0 w} \prod_{k=1}^{\infty} \left[\frac{(1 - q^{2k} w/w_0)(1 - q^{2k} w_0/w)}{(1 - q^{2k} w\overline{w}_0)(1 - q^{2k}/(w\overline{w}_0))} \right]^{(-1)^k}$$

maps the ring $E_q = \{w : q < |w| < 1\}$ conformally onto the unit disk with one radial slit. For doubly connected domain E_q the equation (17) takes the form

$$0 = \frac{r^2}{1 - r^2} + \sum_{k=1}^{\infty} (-1)^k \left[\frac{q^{2k} r^2}{1 - q^{2k} r^2} - \frac{q^{2k}/r^2}{1 - q^{2k}/r^2} \right] \tag{29}$$

($r = |w|$). We will expand every term of the series at the left-hand side of (29) in powers of r and we will group together the terms with the identical powers of r. This sequence of operations is reflected in the following chain of equalities:

$$0 = \sum_{m=1}^{\infty} r^{2m} + \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} (-q^{2m})^k \left[r^{2m} - r^{-2m} \right] = \sum_{m=1}^{\infty} \left[r^{2m} - \frac{(r^{2m} - r^{-2m}) q^{2m}}{1 + q^{2m}} \right]$$

$$= \sum_{m=1}^{\infty} \frac{(r^{2m} + r^{-2m}) q^{2m}}{1 + q^{2m}}.$$

The result demonstrates that the equation (17) has no solutions in the doubly connected domain E_q.

In conclusion we will note that the given example of unsolvability of the equation (17) in doubly connected domain is not the unique.

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