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States on Symmetric Logics: Conditional Probability and Independence. II

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Abstract We study the notions of conditional probabilities, independence and ε-independence for states on symmetric logics. We prove that a non-atomic state on the logic with the Lyapunov's property is determined by its specification of independent events. We present the examples of (1) Δ-subadditive but is not subadditive and (2) two-valued non Δ-subadditive states on symmetric logic. We investigate the independence relation transitivity for a Δ-subadditive state.

We also study continuity properties of conditional probabilities and ε-independence relation with respect to natural pseudometric for Δ-subadditive state. Finally, we pose two open problems.

Keywords Quantum logic · State · Conditional probability · Independence · Symmetric difference

1 Introduction and Preliminaries

Measure theory problems for quantum logics (particulary, Boolean algebras and σ-algebras) of sets are an actual field of mathematical activity cf. [6, 7, 15, 16] and references therein. The notion of conditional probability is the principle instrument of the classical probability theory (cf., e.g., [5, Chap. V]).

This paper continues the first author's study begun in [2]; so we retain the notation and terminology used there. Our aim is to study the notions of conditional probabilities, independence and ε-independence for states on symmetric logics. We prove that a non-atomic state on the logic with the Lyapunov's property is determined by its specification of independent events.

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We present the examples of (1) Δ-subadditive but not subadditive and (2) two-valued non Δ-subadditive states on symmetric logics.

We investigate the independence relation transitivity for Δ-subadditive states. We also study continuity properties of conditional probabilities and ε-independence relation with respect to natural pseudometric for Δ-subadditive state. We prove that in this pseudometric space any "triangle" possesses a "perimeter" less than or equal to 2.

Finally, we pose two open problems.

Let us recall [6] that the set $\mathcal{E}$ of subsets of Ω is called a *concrete quantum logic* if the following conditions hold true:

(1) $\Omega \in \mathcal{E}$;
(2) $A \in \mathcal{E} \Rightarrow A^c = \Omega \setminus A \in \mathcal{E}$;
(3) $A, B \in \mathcal{E}, A \cap B = \emptyset \Rightarrow A \cup B \in \mathcal{E}$.

Let $\mathcal{S}(\Omega)$ be the set of all subsets of Ω. Let us consider the following condition for $\mathcal{E} \subset \mathcal{S}(\Omega)$:

(4) $A, B \in \mathcal{E}, A \subset B \Rightarrow B \setminus A \in \mathcal{E}$.

It is seems clear that a family $\mathcal{E} \subset \mathcal{S}(\Omega)$ is a concrete quantum logic if and only if conditions (1) and (4) hold.

We say that the mapping $m : \mathcal{E} \to [0, 1]$ is a *state* (or a *probability measure*) on the concrete logic $\mathcal{E}$, if $m(\Omega) = 1$ and $m(A \cup B) = m(A) + m(B)$ for all $A, B \in \mathcal{E}$, $A \cap B = \emptyset$. Let us denote by $P(\mathcal{E})$ the set of all states on logic $\mathcal{E}$. Recall that the state $m \in P(\mathcal{E})$ is called *subadditive* ([15], p. 829) if for each $A, B \in \mathcal{E}$ there exists a set $C \in \mathcal{E}$ such that $C \supset A \cup B$ and, moreover, $m(C) \leq m(A) + m(B)$.

In what follows the elements of $\mathcal{E}$ will be called events. Every minimal element of $\mathcal{E} \setminus \{\emptyset\}$ with respect to inclusion is called an *atom* in $\mathcal{E}$.

Let $\nu : \mathcal{E} \to \mathbb{R}^n_+$ $(n \geq 1)$ be a (vector) measure ($\nu(A \cup B) = \nu(A) + \nu(B)$ for all $A, B \in \mathcal{E}$, $A \cap B = \emptyset$). An event $A \in \mathcal{E}$ is ν-*atom* if $\nu(A) > 0$ and if for any event $B \subset A$, either $\nu(B) = \nu(A)$ or $\nu(B) = 0$. A (vector) measure ν is *nonatomic* if it has no ν-atoms.

The set $\mathcal{S}(\Omega)$ is a group with respect to the symmetric difference operation: $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Since $(X \Delta Y) \Delta Z = X \Delta (Y \Delta Z)$ for arbitrary $X, Y, Z \in \mathcal{S}(\Omega)$ one can write $X \Delta Y \Delta Z$ without brackets which we need to order operations. Thus

$$A^c \Delta B = (\Omega \Delta A) \Delta B = A \Delta (\Omega \Delta B) = \Omega \Delta (A \Delta B) = (A \Delta B)^c,$$

$$A^c \Delta B^c = (\Omega \Delta A) \Delta (\Omega \Delta B) = (\Omega \Delta A \Delta \Omega) \Delta B = \big((\Omega \Delta \Omega) \Delta A\big) \Delta B = A \Delta B.$$

A concrete logic $\mathcal{E}$ is said to be *symmetric* [12, Definition 3.2], if

(5) $A, B \in \mathcal{E} \Rightarrow A \Delta B \in \mathcal{E}$.

These logics were investigated e.g. in [2, 4, 8, 9, 12, 13]. A family $\mathcal{E} \subset \mathcal{S}(\Omega)$ is a symmetric logic if and only if conditions (1) and (5) hold [2, Proposition 1].

Example 1.1 Let $n \in \mathbb{N}$ and $\Omega = \{1, 2, \ldots, 2n\}$. Then the family

$$\Omega_{\text{even}} = \big\{A \subset \Omega : \text{card}(A) = 2k,\ k = 0, 1, 2, \ldots, n\big\}$$

is a symmetric logic on Ω.

Example 1.2 Let $\mathcal{E} \subset \mathcal{S}(\Omega)$ be a concrete quantum logic and $A \in \Omega$, $A \neq \emptyset$. Then the family $\mathcal{E}_A = \{B \in \mathcal{E} : B \subset A\}$ is a concrete quantum logic with the greatest element A. Moreover, if $\mathcal{E}$ is a symmetric logic, then $\mathcal{E}_A$ is also a symmetric logic.

2 Conditional Probability and Independence

In what follows $\mathcal{E}$ will be a symmetric logic on Ω. Let $m \in P(\mathcal{E})$. For $A, B \in \mathcal{E}$ we define [2]

$$\widetilde{m}(A, B) = \frac{m(A) + m(B) - m(A \Delta B)}{2}.$$

Thus $\widetilde{m}(A, A) = m(A)$, $\widetilde{m}(A, B) = \widetilde{m}(B, A)$, $\widetilde{m}(A, B) + \widetilde{m}(A^c, B) = m(B)$, and $\widetilde{m}(A, B) = 0$ if $A \cap B = \emptyset$.

Definition 2.1 [2] Let $m \in P(\mathcal{E})$ and $A, B \in \mathcal{E}$. Let us say that the conditional probability of an event B under condition of another event A with $m(A) > 0$ is the value

$$\frac{\widetilde{m}(A, B)}{m(A)},$$

which will be denoted by $m(B \mid A)$.

Conditional probabilities may take negative values as well as positive ones. They are indefinite if $m(A) = 0$. If m is a probability on some Boolean algebra $\mathcal{E}$, then

$$\widetilde{m}(A, B) = m(A \cap B), \qquad m(B \mid A) = \frac{m(A \cap B)}{m(A)},$$

thus our definition coincides with the classical one.

Definition 2.2 [2] Let $m \in P(\mathcal{E})$ and $A, B \in \mathcal{E}$. Then two events A and B are independent if

$$\widetilde{m}(A, B) = m(A)m(B).$$

It is proved in Theorem 1 of [2] that the following conditions are equivalent:

(a) events A and B are independent;
(b) events A and B^c are independent;
(c) events A^c and B are independent;
(d) events A^c and B^c are independent.

Since $\widetilde{m}(A, \Omega) = m(A) = m(A)m(\Omega)$, events A and Ω are independent. The events A and B are independent if and only if $m(A \mid B) = m(A \mid B^c)$ [2, Theorem 2]. Two events A and B are independent if $m(A \mid B) = m(A)$ [2, p. 103].

Theorem 2.3 *Let* $m \in P(\mathcal{E})$ *and* $A, B \in \mathcal{E}$, $0 < m(A) < 1$. *The following conditions are equivalent*:

(i) $m(B \mid A) = m(B)$;
(ii) $m(B \mid A^c) = m(B)$;
(iii) $m(B^c \mid A) = m(B^c)$;
(iv) $m(B^c \mid A^c) = m(B^c)$.

Proof All these conditions are equivalent to the independence of A (respectively A^c) and B (respectively B^c). □

Proposition 2.4 *Let $A, B, C \in \mathcal{E}$ and $C \subset B \subset A$, $m(B) > 0$. Then*

$$m(C \mid A) = m(C \mid B)m(B \mid A).$$

Proof As A, B, C are contained in a Boolean subalgebra of $\mathcal{E}$, the classical proof works. □

If $m(A) > 0$ and $m(B) > 0$, then we obtain an analogue of *Bayes formula*:

$$m(A \mid B) = \frac{m(A)m(B \mid A)}{m(B)}.$$

If $0 < m(A) < 1$, then $m(B) = m(B \mid A)m(A) + m(B \mid A^c)m(A^c)$.

Proposition 2.5 *Let $\mathcal{E}$ be a symmetric logic and $m \in P(\mathcal{E})$, $A, B \in \mathcal{E}$ and $m(A), m(B) \in (0, 1)$. The following conditions are equivalent*:

(i) *events A, B, and $A \Delta B$ are pairwise independent*;
(ii) $m(A) = m(B) = m(A \Delta B) = 1/2$.

Proof (i) ⇒ (ii). We have

$$m(A) + m(B) - m(A \Delta B) = 2m(A)m(B), \tag{1}$$

$$m(A) + m(A \Delta B) - m(B) = 2m(A)m(A \Delta B), \tag{2}$$

$$m(B) + m(A \Delta B) - m(A) = 2m(B)m(A \Delta B). \tag{3}$$

The sum of formulas (1) and (2) allows us to have the reduction $m(B) + m(A \Delta B) = 1$; similarly combination of formulas (1) and (3) provides us with the relation $m(A) + m(A \Delta B) = 1$. Thus $m(A) = m(B)$. Again combination of formulas (2) and (3) gives us $m(A) + m(B) = 1$.

The implication (ii) ⇒ (i) can be verified by direct computation. □

Proposition 2.6 *If m is a nonatomic state on the symmetric logic $\mathcal{E}$ and $A \in \mathcal{E}$, $m(A) > 0$, then there exist events $B \subseteq A$, $C \subseteq A$ such that $\widetilde{m}(B, C) = m(B)m(C)$.*

Proof Coincides with the proof of [3, Lemma 1]. □

Proposition 2.7 *Let $\mathcal{E}$ be a symmetric logic and $m \in P(\mathcal{E})$. If $A, B \in \mathcal{E}$, $A \cap B = \emptyset$ and $m(A)m(B) > 0$, then the events A and B are dependent.*

Proof We have

$$0 = \frac{m(A) + m(B) - m(A \Delta B)}{2} \neq m(A)m(B).$$ □

3 Independence and Determination of States

We say that $m, \mu \in P(\mathcal{E})$ *have identical independent events*, if, for any pair of events A and B, $\widetilde{m}(A, B) = m(A)m(B)$ if and only if $\widetilde{\mu}(A, B) = \mu(A)\mu(B)$.

Definition 3.1 A symmetric logic $\mathcal{E}$ has the Lyapunov's property (write $\mathcal{E} \in$ (LP)) if for every pair $m, \mu \in P(\mathcal{E})$ such that the vector measure $\nu = (m, \mu)$ is nonatomic, the range of ν is convex.

Theorem 3.2 *Let $\mathcal{E} \in$ (LP) and $m, \mu \in P(\mathcal{E})$ be so that at least one of them is nonatomic. If they have identical independent event pairs, then they coincide.*

Corollary 3.3 *Let $\mathcal{E} \in$ (LP) and $m, \mu \in P(\mathcal{E})$ be so that at least one of them is nonatomic. If they have identical mutually favorable events in the following sense: for any pair of events A and B, $\widetilde{m}(A, B) \geq m(A)m(B)$ if and only if $\widetilde{\mu}(A, B) \geq \mu(A)\mu(B)$, then m and μ coincide.*

Corollary 3.4 *Let $\mathcal{E} \in$ (LP) and $m, \mu \in P(\mathcal{E})$ be so that at least one of them is nonatomic. If*

$$m(A) = 1/2 \quad \Longleftrightarrow \quad \mu(A) = 1/2,$$

then $m = \mu$.

Proof of Theorem 3.2 coincides with the second proof of Theorem 1 from [3] (we use our Proposition 2.5). It suffices to consider 2 independent events. Proofs of Corollaries 3.3 and 3.4 coincide with the proofs of Corollaries 1 and 2 of [3], respectively. A wide class of quantum structures with the Lyapunov's property was considered in [1].

4 Δ-Subadditive States on Symmetric Logics

Let us say that a state $m \in P(\mathcal{E})$ is Δ-*subadditive* [4] if

$$m(A \Delta B) \leq m(A) + m(B) \quad \text{for any pair } A, B \in \mathcal{E}.$$

In [2] this term was introduced without the prefix Δ. But since the usual subadditivity differs from this notion we feel obliged to correct ourselves.

The set of all Δ-subadditive states is convex. A state $m \in P(\mathcal{E})$ is Δ-subadditive if and only if conditional probabilities are non-negative on it.

Let $\mathcal{E}$ be a Boolean algebra, $m \in P(\mathcal{E})$ and $A, B_1, B_2 \in \mathcal{E}$. If $B_1 \subset B_2$, then

$$m(B_1 \mid A) \leq m(B_2 \mid A),$$

i.e. the conditional probability is monotonic. A state m on a symmetric logic $\mathcal{E}$ is Δ-subadditive if and only if the conditional probability is monotonic [2, Theorem 3].

It is proved in Lemma 1 of [2] that a state $m \in P(\mathcal{E})$ is Δ-subadditive if and only if

$$m(A \Delta B) \leq m(A \Delta C) + m(C \Delta B) \quad \text{for all } A, B, C \in \mathcal{E}. \tag{4}$$

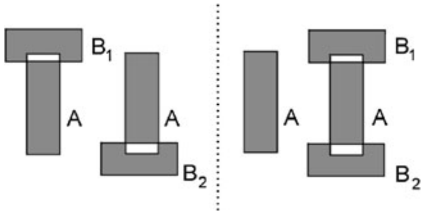
For a Δ-subadditive state $m \in P(\mathcal{E})$ via (4) we get

$$\widetilde{m}(A, B) + \widetilde{m}(A, C) - \widetilde{m}(B, C) \leq m(A) \quad \text{for all } A, B, C \in \mathcal{E}.$$

Example 4.1 Let $\Omega = \{1, 2, 3, 4\}$. Let us define the two-valued state m on symmetric logic Ω_{even} by its values on atoms as follows:

$$m(\{1, 2\}) = m(\{1, 3\}) = m(\{1, 4\}) = 0.$$

Fig. 1 Scheme of the sets $A\Delta B_1$, $A\Delta B_2$ and A, $A\Delta(B_1 \cup B_2)$



Then m is not Δ-subadditive:

$$1 = m\big(\{3,4\}\big) = m\big(\{1,3\}\Delta\{1,4\}\big) > m\big(\{1,3\}\big) + m\big(\{1,4\}\big) = 0.$$

The space of Δ-subadditive states is a tetrahedron (a convex combination of 4 extreme states), the conditional probabilities achieve values between 0 and 1 exactly on this set.

Let $\mathcal{E}$ be a Boolean algebra, $m \in P(\mathcal{E})$ and $A, B_1, B_2 \in \mathcal{E}$. If the event A does not depend on the events B_1, B_2 and $B_1 \cap B_2 = \emptyset$, then the events A and $B_1 \cup B_2$ are independent.

Theorem 4.2 *Let $\mathcal{E}$ be a symmetric logic of subsets of a set Ω, $\mathcal{A}$ be a Boolean algebra of subsets of Ω, $\mathcal{E} \subset \mathcal{A}$ and $A, B_1, B_2 \in \mathcal{E}$. Let a state m on $\mathcal{E}$ allow for an extension,* $\mathbf{m}$, *over $\mathcal{A}$ as a signed measure. If the event A does not depend on the events B_1, B_2 and $B_1 \cap B_2 = \emptyset$, then the events A and $B_1 \cup B_2$ are independent.*

Proof Let $B = B_1 \cup B_2$. Since

$$m(A) + m(B_1) - m(A\Delta B_1) = 2m(A)m(B_1),$$
$$m(A) + m(B_2) - m(A\Delta B_2) = 2m(A)m(B_2),$$

we have

$$2m(A) + m(B) - m(A\Delta B_1) - m(A\Delta B_2) = 2m(A)m(B). \tag{5}$$

Since (see Fig. 1)

$$\begin{aligned} &m(A\Delta B_1) + m(A\Delta B_2) \\ &\quad = m(A) + m(B_1) - 2\mathbf{m}(A \cap B_1) + m(A) + m(B_2) - 2\mathbf{m}(A \cap B_2) \\ &\quad = m(A) + m(A\Delta B), \end{aligned}$$

we have via (5) the relation $m(A) + m(B) - m(A\Delta B) = 2m(A)m(B)$. This completes the proof. □

Let $n \in \mathbb{N}$ and $\Omega = \{1, 2, \ldots, 2n\}$. By Theorem 2.1 [4] every $m \in P(\Omega_{\mathrm{even}})$ can be extended to a signed measure $\mathbf{m}$ over $\mathcal{S}(\Omega)$, the (Boolean) power algebra of Ω. Thus Theorem 4.2 holds for any state m on Ω_{even}.

Example 4.3 Let $\Omega = \{0, 1, 2, 3, 4, 5\}$ and the symmetric logic $\mathcal{E}$ contain the sets

$$A = \{1,2,3\}, \qquad B_1 = \{0,1\}, \qquad B_2 = \{3,4\}, \qquad C = \{2,5\},$$
$$D = \{0,2,3\}, \qquad E = \{1,2,4\}, \qquad F = \{0,2,4\}, \qquad \Omega$$

and their complements. The logic $\mathcal{E}$ has 16 elements. Let m be a Δ-subadditive state on $\mathcal{E}$ such that

$$m(A) = m(D) = m(E) = 1/2, \qquad m(B_1) = m(B_2) = 1/3, \qquad m(F) = 3/8.$$

Then the event A does not depend on the events B_1, B_2, but the events A and $B_1 \cup B_2$ are not independent. Since $\{0\} = E^c \cap F = A^c \cap D$, $\{1\} = A \cap B_1$, $\{2\} = A \cap C$, $\{3\} = B_2 \cap F^c$, $\{4\} = A^c \cap B_2$, and $\{5\} = C \cap D^c$, the Boolean algebra generated by $\mathcal{E}$ is $\mathcal{S}(\Omega)$. Note that m cannot be extended as a signed measure on $\mathcal{S}(\Omega)$.

Example 4.4 Let $\Omega = \{1, 2, 3, 4, 5, 6\}$. Again we define the state m on the symmetric logic Ω_{even} by its values on atoms as follows:

$$m\big(\{1,2\}\big) = \frac{2}{3}, \qquad m\big(\{1,3\}\big) = m\big(\{1,4\}\big) = \frac{2}{9}, \qquad m\big(\{1,5\}\big) = \frac{1}{3},$$
$$m\big(\{1,6\}\big) = m\big(\{2,3\}\big) = m\big(\{2,4\}\big) = \frac{4}{9}, \qquad m\big(\{2,5\}\big) = \frac{5}{9},$$
$$m\big(\{2,6\}\big) = \frac{2}{3}, \qquad m\big(\{3,4\}\big) = 0, \quad m\big(\{3,5\}\big) = \frac{1}{9}, \qquad m\big(\{3,6\}\big) = \frac{2}{9},$$
$$m\big(\{4,5\}\big) = \frac{1}{9}, \qquad m\big(\{4,6\}\big) = \frac{2}{9}, \qquad m\big(\{5,6\}\big) = \frac{1}{3}.$$

Then m is Δ-subadditive.

If $\mathcal{E}$ is a symmetric logic, then every subadditive state $m \in P(\mathcal{E})$ is Δ-subadditive (hint: $C \supset A \cup B \supset A \Delta B$), but the reverse implication is not true in general. The state m from Example 4.4 is Δ-subadditive and is not subadditive: for $A = \{3, 4\}$, $B = \{1, 2, 4, 5\}$ we have the unique set $C = \{1, 2, 3, 4, 5, 6\}$ such that $C \supset A \cup B$ and $1 = m(C) > m(A) + m(B) = 0 + 7/9 = 7/9$.

Theorem 4.5 *Let $m \in P(\mathcal{E})$. The following conditions are equivalent*:

(i) *m is Δ-subadditive*;
(ii) *$\widetilde{m}(A, B) \le \min\{m(A), m(B)\}$ for all $A, B \in \mathcal{E}$.*

Proof (i) $\Rightarrow$ (ii). We have $B = (B \Delta A) \Delta A$, therefore $m(B) \le m(A \Delta B) + m(A)$, i.e. $m(B) - m(A \Delta B) \le m(A)$. The latter inequality is equivalent to the inequality $\widetilde{m}(A, B) \le m(A)$.

(ii) $\Rightarrow$ (i). We have $m(B) - m(A) - m(A \Delta B) \le 0$ for all $A, B \in \mathcal{E}$. We replace B by B^c; then $m(B^c) - m(A) - m(A \Delta B^c) \le 0$, i.e. $1 - m(B) - m(A) - 1 + m(A \Delta B) \le 0$. This completes the proof. □

Theorem 4.6 *Let $m \in P(\mathcal{E})$ be Δ-subadditive and $A, B, C \in \mathcal{E}$. Then $|\widetilde{m}(A, B) - \widetilde{m}(A, C)| \le m(B \Delta C)$.*

Proof We have $B = (B \Delta C) \Delta C$ and $C = (B \Delta C) \Delta B$, therefore $m(B) \le m(B \Delta C) + m(C)$ and $m(C) \le m(B \Delta C) + m(B)$, i.e.

$$\big|m(B) - m(C)\big| \le m(B \Delta C). \tag{6}$$

By (4) we get $m(A\Delta B) \le m(A\Delta C) + m(C\Delta B)$ and $m(A\Delta C) \le m(A\Delta B) + m(B\Delta C)$, therefore

$$\left|m(A\Delta C) - m(A\Delta B)\right| \le m(B\Delta C). \tag{7}$$

Now we have via triangle inequality and formulas (6), (7) the relations

$$\begin{aligned}\left|\widetilde{m}(A,B) - \widetilde{m}(A,C)\right| &= \frac{|m(B) - m(C) - m(A\Delta B) + m(A\Delta C)|}{2} \\ &\le \frac{|m(B) - m(C)|}{2} + \frac{|m(A\Delta C) - m(A\Delta B)|}{2} \\ &\le m(B\Delta C).\end{aligned}$$

□

Theorem 4.7 *Let $m \in P(\mathcal{E})$ be Δ-subadditive and $A_1, A_2, \dots$ and $B_1, B_2, \dots$ be two sequences of events, where $m(B_n) \to 1$ as $n \to \infty$. Then*

(I) $\lim_{n\to\infty} m(A_n) = \lim_{n\to\infty} \widetilde{m}(A_n, B_n)$ *under the following condition: there exists at least one of indicated limits*;

(II) *if*

$$\liminf_{n\to\infty} m(A_n) \ge a > 0, \tag{8}$$

then

$$\lim_{n\to\infty} \frac{m(A_n)}{\widetilde{m}(A_n, B_n)} = 1.$$

Proof (I) We have

$$m(A_n) = \widetilde{m}(A_n, B_n) + \widetilde{m}\big(A_n, B_n^c\big) \quad \text{for all } n \in \mathbb{N}. \tag{9}$$

By Theorem 4.5, $0 \le \widetilde{m}(A_n, B_n^c) \le m(B_n^c) \to 0$ as $n \to \infty$. Hence $\widetilde{m}(A_n, B_n^c) \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} m(A_n) = \lim_{n\to\infty} \widetilde{m}(A_n, B_n)$.

(II) Let $i \in \mathbb{N}$ be so that $m(A_n) \ge a/2$ for all $n \ge i$, and let $j \in \mathbb{N}$ be so that $m(B_n) \ge 1 - a/4$ for all $n \ge j$. We have for $n \ge \max\{i, j\}$

$$\widetilde{m}(A_n, B_n) = \frac{m(A_n) + m(B_n) - m(A_n \Delta B_n)}{2} \ge \frac{a/2 + 1 - a/4 - 1}{2} = \frac{a}{8}.$$

Then for $n \ge \max\{i, j\}$ via (9) and Theorem 4.5 the relations

$$1 = \frac{m(A_n)}{m(A_n)} \le \frac{m(A_n)}{\widetilde{m}(A_n, B_n)} = \frac{\widetilde{m}(A_n, B_n) + \widetilde{m}(A_n, B_n^c)}{\widetilde{m}(A_n, B_n)} = 1 + \frac{\widetilde{m}(A_n, B_n^c)}{\widetilde{m}(A_n, B_n)} \le 1 + \frac{8}{a} m\big(B_n^c\big)$$

hold. Since $m(B_n^c) \to 0$ as $n \to \infty$, this completes the proof. □

Example 4.8 Condition (8), in general, cannot be omitted. Consider $\Omega = [0, 1]$, Borel σ-algebra $\mathcal{E} \subset \mathcal{S}(\Omega)$ with Lebesgue measure m and the events $A_n = [0, 1/n]$, $B_n = [1/(2n), 1]$.

5 On Independent Events

The notion of independence is fundamental in probability theory.

Theorem 5.1 *Let $m \in P(\mathcal{E})$ be Δ-subadditive. The following conditions are equivalent*:

(i) *all the events from $\mathcal{E}$ are mutually independent*;
(ii) $m(A) \in \{0, 1\}$ *for all* $A \in \mathcal{E}$;
(iii) *the independence relation on $\mathcal{E}$ is transitive*.

Proof (i) $\Rightarrow$ (ii). Assume that there exists $A \in \mathcal{E}$ with $0 < m(A) < 1$. Then we have $0 < m(A^c) < 1$ and

$$\widetilde{m}\big(A, A^c\big) = \frac{m(A) + m(A^c) - m(A \Delta A^c)}{2} = 0 \neq m(A)m\big(A^c\big). \tag{10}$$

(ii) $\Rightarrow$ (i). Step 1. Let $m(A) = m(B) = 0$. Then

$$0 \leq m(A \Delta B) \leq m(A) + m(B) = 0 \tag{11}$$

and $\widetilde{m}(A, B) = 0 = m(A)m(B)$.

Step 2. Let $m(A) = 0$, $m(B) = 1$. Then $m(B^c) = 0$ and by Step 1 the events A and B^c are independent. Therefore, the events A and B are independent via Theorem 1 of [2].

Step 3. Let $m(A) = m(B) = 1$. Then $m(A^c) = m(B^c) = 0$ and by Step 1 the events A^c and B^c are independent. Therefore, the events A and B are independent via Theorem 1 of [2].

The implication (i) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (ii). Assume that the independence relation on $\mathcal{E}$ is transitive and there exists $A \in \mathcal{E}$ with $0 < m(A) < 1$. Then (a) A is independent of $\emptyset$; (b) $\emptyset$ is independent from A^c, but A dependent from A^c by (10). We obtain a contradiction. This completes the proof. □

Proposition 5.2 *Let $\mathcal{E}$ be a symmetric logic and $m \in P(\mathcal{E})$ be Δ-subadditive. Then the family $\mathcal{E}_m = \{A \in \mathcal{E} : m(A) \in \{0, 1\}\}$ also form a symmetric logic*.

Proof Since $\Omega \in \mathcal{E}_m$, it suffices to show that $A \Delta B \in \mathcal{E}_m$ for all $A, B \in \mathcal{E}_m$.

If $m(A) = m(B) = 0$, then $m(A \Delta B) = 0$ by (11) and $A \Delta B \in \mathcal{E}_m$.

If $m(A) = m(B) = 1$, then $m(A^c) = m(B^c) = 0$ and $m(A^c \Delta B^c) = 0$ by (10). Since $A^c \Delta B^c = A \Delta B$, we have $A \Delta B \in \mathcal{E}_m$.

If $m(A) = 1$ and $m(B) = 0$, then $m(A^c) = 0$ and $m(A \Delta B) = 1 - m(A^c \Delta B) = 1$. □

From Theorem 5.1 and Proposition 5.2 we have

Corollary 5.3 *Let $\mathcal{E}$ be a symmetric logic and $m \in P(\mathcal{E})$ be Δ-subadditive. Then all events from $\mathcal{E}_m$ are mutually independent and the independence relation on $\mathcal{E}_m$ is transitive*.

6 On ε-Independent Events

Definition 6.1 Let $m \in P(\mathcal{E})$ and $A, B \in \mathcal{E}$. For $\varepsilon > 0$ two events A and B are ε-independent, if

$$\big|\widetilde{m}(A, B) - m(A)m(B)\big| \leq \varepsilon.$$

Proposition 6.2 *If an event A is ε-independent with itself for $0<\varepsilon\le 1/4$, then either $m(A)\le 2\varepsilon$ or $m(A)\ge 1-2\varepsilon$.*

Proof We have $\widetilde{m}(A,A)=m(A)$ and

$$\left|\widetilde{m}(A,A)-m(A)m(A)\right|=\left|m(A)-m(A)m(A)\right|=\left|m(A)\big(1-m(A)\big)\right|\le\varepsilon.$$

By solving this quadratic inequality with respect to $m(A)$, we get

$$m(A)\le\frac{1-\sqrt{1-4\varepsilon}}{2}\quad\text{or}\quad m(A)\ge\frac{1+\sqrt{1-4\varepsilon}}{2}.$$

Finally, we apply the following inequalities for $0<\varepsilon\le 1/4$:

$$\frac{1-\sqrt{1-4\varepsilon}}{2}\le 2\varepsilon,\qquad \frac{1+\sqrt{1-4\varepsilon}}{2}\ge 1-2\varepsilon.$$ □

Theorem 6.3 *Let $m\in P(\mathcal{E})$ and $A,B\in\mathcal{E}$. The following conditions are equivalent*:

(i) *events A and B are ε-independent*;
(ii) *events A and B^c are ε-independent*;
(iii) *events A^c and B are ε-independent*;
(iv) *events A^c and B^c are ε-independent.*

Proof Since $X^{cc}=X$ ($X\in\mathcal{S}(\Omega)$), it suffices to show that (i) $\Rightarrow$ (ii). The following relations prove it:

$$\begin{aligned}&\left|\widetilde{m}\big(A^c,B\big)-m\big(A^c\big)m(B)\right|\\&=\left|\frac{1-m(A)+m(B)-1+m(A\Delta B)}{2}-\big(1-m(A)\big)m(B)\right|\\&=\left|-\widetilde{m}(A,B)+m(A)m(B)\right|\le\varepsilon.\end{aligned}$$ □

It seems useful to compare Theorems 2.3 and 6.3 with Theorem 1 of [2]. Also the following assertion can be compared with Proposition 2.7.

Proposition 6.4 *Let $m\in P(\mathcal{E})$ be Δ-subadditive and $A\in\mathcal{E}$ be so that $m(A)\le\varepsilon$ or $m(A)\ge 1-\varepsilon$. Then A and an arbitrary event B are ε-independent.*

Proof If $m(A)\le\varepsilon$, then $m(A)m(B)\le\varepsilon$ and by Theorem 4.5 we have also $\widetilde{m}(A,B)\le m(A)\le\varepsilon$. Therefore,

$$\left|\widetilde{m}(A,B)-m(A)m(B)\right|\le\max\left\{\widetilde{m}(A,B),m(A)m(B)\right\}\le\varepsilon.$$

If $m(A)\ge 1-\varepsilon$, then $m(A^c)\le\varepsilon$ and, consequently, A^c and an arbitrary event B are ε-independent. By Theorem 6.3, A and an arbitrary event B are also ε-independent. This completes the proof. □

If $m\in P(\mathcal{E})$ is Δ-subadditive, then the mapping $d_m:\mathcal{E}\times\mathcal{E}\to[0,\infty)$, defined by the formula $d_m(A,B)=m(A\Delta B)$ ($A,B\in\mathcal{E}$) is a pseudometric on $\mathcal{E}$ [2, Theorem 4]. Via (2) a state m is uniform continuous on $\langle\mathcal{E},d_m\rangle$. Theorem 4.6 shows us that for every fixed $A\in\mathcal{E}$ the mapping $X\mapsto\widetilde{m}(A,X)$ ($X\in\mathcal{E}$) is uniform continuous on $\langle\mathcal{E},d_m\rangle$.

Theorem 6.5 *Let $m \in P(\mathcal{E})$ be Δ-subadditive, $A, A_n, B, B_n \in \mathcal{E}$ $(n \in \mathbb{N})$. If $A_n \to A$, $B_n \to B$ $(n \to \infty)$ on $\langle \mathcal{E}, d_m \rangle$ and the events A_n and B_n are ε-independent for any $n \in \mathbb{N}$, then the events A and B are ε-independent.*

Proof It follows from Theorem 4 of [2] that $m(A_n) \to m(A)$ and $m(B_n) \to m(B)$ as $n \to \infty$. Via Theorem 5 of [2] we have $m(A_n \Delta B_n) \to m(A \Delta B)$ as $n \to \infty$. Therefore $\widetilde{m}(A_n, B_n) \to \widetilde{m}(A, B)$ as $n \to \infty$ and the inequality

$$\left|\widetilde{m}(A, B) - m(A)m(B)\right| \leq \varepsilon$$

holds. This completes the proof. □

Proposition 6.6 *Let $m \in P(\mathcal{E})$ be Δ-subadditive. Then*

$$d_m(A, B) + d_m(B, C) + d_m(C, A) \leq 2 \quad \textit{for all } A, B, C \in \mathcal{E}. \tag{12}$$

Proof We have

$$m(A \Delta B) = 1 - m\left(A \Delta B^c\right),$$
$$m(B \Delta C) = 1 - m\left(B^c \Delta C\right),$$
$$m(C \Delta A) = 1 - m\left(A^c \Delta C\right).$$

Thus $d_m(A, B) + d_m(B, C) + d_m(C, A) = 3 - m(A \Delta B^c) - m(B^c \Delta C) - m(A^c \Delta C)$. Since

$$1 = m\left(A^c \Delta A\right) \leq m\left(A^c \Delta C\right) + m(C \Delta A) \leq m\left(A^c \Delta C\right) + m\left(C \Delta B^c\right) + m\left(B^c \Delta A\right),$$

we have (12). □

Corollary 6.7 *Let $\mathcal{E}$ be a symmetric logic and $m \in P(\mathcal{E})$ be Δ-subadditive. Then*

$$\widetilde{m}(A, B) + \widetilde{m}(B, C) + \widetilde{m}(C, A) \geq m(A) + m(B) + m(C) - 1 \quad \textit{for all } A, B, C \in \mathcal{E}.$$

7 Open Problems

If $\mathcal{E}$ is a Boolean algebra then any state $m \in P(\mathcal{E})$ is subadditive. There exists a concrete quantum logic which is not a Boolean algebra and all of its states are subadditive. The result was established in [14] with substantial help from the techniques developed in [10] and [11] (see also [15], p. 831).

Problem 7.1 Let $\mathcal{E}$ be a symmetric logic such that any state $m \in P(\mathcal{E})$ is Δ-subadditive. Is it true that $\mathcal{E}$ is a Boolean algebra?

If $\mathcal{E}$ is a σ-algebra then the pseudometric space $\langle \mathcal{E}, d_m \rangle$ is complete for any state $m \in P(\mathcal{E})$.

Problem 7.2 Let $\mathcal{E}$ be both a symmetric logic and a σ-class, $m \in P(\mathcal{E})$ be Δ-subadditive. Is it true that the pseudometric space $\langle \mathcal{E}, d_m \rangle$ is complete?

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