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RELATIVE ENUMERABILITY
AND THE D-C. E. DEGREES

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Abstract

We study the relationship between the relative enumerability and the d-c. e. degrees. We prove that the degree of the halting problem is splittable into two c. e. degrees such that the upper cone of each of them contains only d-c. e. degrees which are c. e. in another one.

Key words: turing degrees, computably enumerable degrees, relative computable enumerability, splitting, definability.

The Δ_2^0 degrees of unsolvability are basic objects of study in classical computability theory, since they are the degrees of those sets whose characteristic functions are limits of computable functions. A natural tool for understanding the Turing degrees is the introduction of hierarchies to classify various kinds of complexity. The most common such hierarchy, the arithmetical hierarchy, is itself not of much use in the classification of the Δ_2^0 degrees, since it is far too coarse. This fact has led to the introduction of hierarchies based on finer distinctions than quantifier alternation. Two such hierarchies are by now well-established. One, the *CEA hierarchy* independently defined by Arslanov [1] and Jockusch and Shore [2, 3], is, like the arithmetical hierarchy, based on the complexity of the definitions of the involved sets, replacing the alternation of quantifiers with the iteration of computable enumerability in and above a set, a significantly less powerful procedure. The second such hierarchy, the *difference hierarchy* due to Putnam [4] and Ershov [5, 6], is built up by starting with the computable enumerable sets as a base, and then iterating the operation of taking set-theoretic differences, thereby classifying sets on the basis of the difficulty of their construction in comparison with c. e. sets. Analysis of the relationship of the CEA hierarchy to the difference hierarchy is therefore a natural means of comparing the definability of sets to the inherent difficulty of their construction.

We took the first steps toward this analysis in [1]: we proved that there is a Δ_2^0 2-CEA set which is not of n-c. e. degree for any $n < \omega$. Generalizing this result Jockusch and Shore [3] proved that for any computable ordinal $\alpha < \beta$, there is a β-c. e. degree which is not α-CEA, while, on the other hand, that for every uniformly given class of Δ_2^0 degrees, there is a Δ_2^0 2-CEA degree which is not in this class. From the latter result, it follows that for each $\alpha \leq \omega$, there is a 2-CEA set which is not of α-c. e. degree. This result is more interesting when $\alpha > 1$, since d-c. e. sets, and hence 2-CEA sets, not of c. e. degree had already been constructed by Cooper (unpublished). In [7] we took a further step towards analyzing the relationship of this second level of the CEA hierarchy to the difference proving that any ω-c. e. degree which is 2-CEA is also 2-c. e. In this paper we also obtained a second result which imposes a significant limit on possible extensions of this result: there exists a d-c. e. set C such that for every $n \geq 3$, there exists a set A which is simultaneously C-CEA and $(n+1)$-c. e., yet fails to be of n-c. e. degree.

Theorem 1. *There exists a splitting of $\mathbf{0}'$ into c. e. degrees $\mathbf{v}_0$ and $\mathbf{v}_1$ such that for every d-c. e. degree $\mathbf{d}$ and each $i \leq 1$, if $\mathbf{v}_i \leq \mathbf{d}$ then $\mathbf{d}$ is c. e. in $\mathbf{v}_{1-i}$.*

Proof. We will construct c. e. sets V_0 and V_1 so that the degrees $\mathbf{v}_i = \deg V_i$ have the desired properties. We also construct auxiliary c. e. sets U_0, U_1.

This is ensured by the following two types of requirements.

To ensure that $\emptyset' \not\leq_T V_i$, we satisfy requirements

- P_e^i: $U_i \neq \Theta_{i,e}^{V_i}$ (for each partial computable functional $\Theta_{i,e}$).

To ensure that for all d-c. e. sets D, if $V_i \leq_T D$ then D is of degree c. e. in V_{1-i}, we satisfy requirements.

- R_e^i: $D_e = \Delta_{i,e}^{V_i \oplus Q_e^i}$ & $Q_e^i \leq_T V_i \oplus D_e$ (for each d-c. e. set D_e we build an associated d-c. e. set Q_e^i c. e. in V_{1-i} and a partial computable functional $\Delta_{i,e}$).

  The condition $Q_e^i \leq_T V \oplus D_e$ will be met by the usual permitting argument.

To ensure that Q_e^i is c. e. in V_{1-i} we use a common method which works as follows. When an integer x is enumerated into Q_e^i at stage s we appoint a certain marker $\alpha(x)$. Then we allow x to be removed from Q_e^i at a later stage t only if $V_{1-i} \upharpoonright \alpha(x)[t-1] \neq$
$\neq V_{1-i} \upharpoonright \alpha(x)[t]$.

The condition $\emptyset' \leq_T V_0 \oplus V_1$ will follow from the construction directly.

The basic strategy for P-requirements in isolation is the one developed by Friedberg and Muchnik:

(1) Pick an unused witness x from the column associated with this requirement ($<i, \omega>$ for $i \leq 1$), which is larger than all higher-priority restraints, and keep it out of U.

(2) Wait for $\Theta_e^V(x) \downarrow = 0$.

(3) Put x into U and protect $V \upharpoonright (\theta(x) + 1)$.

The basic strategy for R-requirements in isolation is to build $\Delta^{V \oplus Q}$, ensuring that it is total and computes D correctly. Since we build the set V during the construction, we may easily meet this requirement by changing V, if necessary.

While the strategies for the requirements in isolation are thus very simple, there are obviously several conflicts between them. The R-strategy threatens to contribute into V infinitely many numbers while P-restraints of lower priority may obstruct it.

Before giving the explicit construction we first explain the intuition for the P_i- and S_j-requirements below one R_e-strategy.

Basic module for the R_e^i-strategy above P-requirements.
We use an ω-sequence of "cycles", where each cycle k proceeds as follows:

(1) At a stage s set $\Delta_{i,e}^{V_i \oplus Q_e^i}(k) = D_{e,s}(k)$ with a *use* $\delta_{i,e,s}(k) >$ all P^i- and P^{1-i}-restraints, and $\delta_{i,e,s}(k) > \delta_{i,e,s}(k-1)$ and start cycle $k+1$ to run simultaneously with cycle k.

(2) Wait for $D_e(k)$ to change (at a stage t, say).

(3) (i) Enumerate $\delta_{i,e,s}(k)$ into Q_e^i,

   (ii) set $\Delta_{i,e}^{V^i \oplus Q_e^i}(k) = D_{e,t}(k)$ with a new *use* $\delta_{i,e,t}(k) >$ all P^i-restraints, and $\delta_{i,e,t}(k) > \delta_{i,e,s}(k)$, and

   (iii) appoint the marker $\alpha_i(\delta_{i,e,s}(k))$ as the first integer y such that $y \geq \delta_{i,e,t}(k)$ and $y = <2, l>$ for some l.

(4) Wait for $D_e(k)$ to change back (at a stage u, say).

(5) We need
   – to keep Q_e^i below $V^i \oplus D_e$ (at stage t k enters D_e, and we put $\delta_{i,e,s}(k)$ into Q_e^i. Now k leaves D_e).
   – to correct the axiom $\Delta_{i,e}^{(V^i \oplus Q_e^i)}(k) = D_e(k)$

   We have two possibilities to achieve this:

   – either by enumerating $\delta_{i,e,s}(k)$ into V^i
   – or by removing $\delta_{i,e,s}(k)$ from Q_e^i (in this case we need to enumerate $\alpha_i(\delta_{i,e,s}(k))$ into V_{1-i}).

   The crucial point here is that our choice between these two possibilities depends upon the priority ordering of requirements P^i and P^{1-i} that may be injured:

   a) If the highest-priority strategy which would be injured by this correction is a P^i-strategy (or there is no strategy at all that would be injured), then enumerate $\alpha_i(\delta_{i,e,s}(k))$ into V_{1-i} and remove $\delta_{i,e,s}(k)$ from Q_e^i.

   b) Otherwise, put $\delta_{i,e,s}(k)$ into V^i, and set $\Delta_{i,e}^{V^i \oplus Q_e^i}(k) = D_{e,u}(k)$.

   Set $\Delta_{i,e}^{V_i \oplus Q_e^i}(k) = D_{e,u}(k)$ with the same use $\delta_{i,e,u}(k) = \delta_{i,e,t}(k)$.

   In both cases start cycle $k+1$ to run simultaneously.

We now give the construction. We say that the axiom $\Delta_{i,e}^{V^i \oplus Q_e^i}(k) = D_e(k)$ *requires correction* at stage s if at a stage $t < s$ we set $\Delta_{i,e}^{V^i \oplus Q_e^i}(k) = D_{e,t}(k)$ with a *use* $\delta_{i,e,t}(k)$, $D_{e,s}(k) \neq D_{e,t}(k)$, and $(V^i \oplus Q_e^i)_t \upharpoonright \delta_{i,e,t}(k) = (V^i \oplus Q_e^i)_s \upharpoonright \delta_{i,e,t}(k)$.

Stage $s = 0$. Set $U_0 = V_0 = V_1 = \emptyset[0]$. $x_{e,0}^0 = <0, e>$, $x_{e,0}^1 = <1, e>$.

Stage $s > 0$. Fix e such that $s = \langle e, m \rangle$ for some m.

Substage 1 (P_e^0-requirement).

   a) If $\Theta_{0,e}^{V_0}(x_e^0)[s] \downarrow = 0$ and $x_{e,s-1}^0 \notin U_{0,s}$, then enumerate $x_{e,s-1}^0$ into $U_{0,s}$, and protect $V_0 \upharpoonright \theta_{0,e,s}(x_{e,s-1}^0)$ with priority P_e^0.

   b) If $\Theta_{0,e}^{V_0}(x_e^0)[s] \downarrow = U_{0,s}(x_{e,s-1}^0) = 1$, then define

   $$x_{e,s}^0 = (\mu x)[(\exists y)(\forall j)(x = \langle 0, y \rangle \wedge x > \text{all } R_j^i \text{-uses assigned so far}].$$

   Otherwise, set $x_{e,s}^0 = x_{e,s-1}^0$.

Substage 2 (P_e^1-requirement). Similar to the previous case with necessary changes ($\Theta_e^0, V_0, U_0, x_e^0, \theta_{0,e}$ by $\Theta_e^1, V_1, U_1, x_e^1, \theta_{1,e}$ accordingly).

Substage 3. Let z be the greatest integer such that for any $k < z$ there exists a stage $s' < s$ such that at stage s' the axiom $\Delta_{i,e}^{V_i \oplus \tilde{Q}(i,e)}(k) = D_e(k)[s']$ was set. Let $k < z$ be the smallest integer (if any) such that the axiom $\Delta_{i,e}^{(V_i \oplus Q_{i,e})}(k) = D_e(k)$ requires correction at stage s. Let t be a stage at which the axiom $\Delta_{i,e}^{(V_i \oplus Q_{i,e})}(k) = = D_e(k)$ was set.

We consider two cases.

Case 1) $D_{e,s}(k) = 1$. In this case we proceed as in step (3) of the Basic Module:
   (i) enumerate $\delta_{i,e,t}(k)$ into $Q_{i,e}$,

(ii) set $\Delta_{i,e}^{V_i \oplus Q_{i,e}}(k) = D_{e,s}(k)$ with a new *use* $\delta_{i,e,s}(k) >$ all P-restraints, and $\delta_{i,e,s}(k) > \delta_{i,e,t}(k)$, and

(iii) appoint the marker $\alpha_i(\delta_{i,e,s}(k))$ as the first integer y such that $y \geq \delta_{i,e,s}(k)$ and $y = <2, l>$ for some l.

Case 2) $D_{e,s}(k) = 0$. Therefore, there is a stage $u < s$ such that $D_{e,u}(k) = 1$, and at stage u we (re)set the axiom $\Delta_{i,e}^{V_i \oplus Q_{i,e}}(k) = D_e(k)$. It follows also that at stage u we enumerated $\delta_{p,e,t}(k)$ into $Q_{i,e}$. In this case we proceed as in step (5) of the Basic Module:

a) if the highest-priority strategy which would be injured by the $Q_{i,e}(\delta_{i,e,t}(k))-$ or $V_i(\delta_{i,e,t}(k))-$ correction is a P-strategy (or there is no strategy at all that would be injured), then enumerate $\alpha_i(\delta_{i,e,t}(k))$ into V_{1-i} and remove $\delta_{i,e,t}(k)$ from $Q_{i,e}$.

b) Otherwise, put $\delta_{i,e,t}(k)$ into V_i.

Set $\Delta_{i,e}^{V_i \oplus Q_{i,e}}(k) = D_{e,s}(k)$.

Substage 4. If none of the axioms $\Delta_{i,e}^{V_i \oplus Q_{i,e}}(k) = D_e(k)$ for $k < z$ requires correction at stage s, then set the new axiom $D_{e,s}(z) = \Delta_{i,e}^{V_i \oplus Q_{i,e}}(z)$ with a *use* $\delta_{i,e,s}(z) >$ all P-, R-restraints.

Substage 5. Go to stage $s + 1$.

End of the construction.

Verification.

Let $\mathbf{v_i} = \deg(V_i)$, $i \leq 1$.

Lemma 1. $Q_{i,e} \leq_T V_i \oplus D_e$.

Proof. To $V_i \oplus D_e$-computably compute whether $x \in Q_{i,e}$, first find a stage u at which a new axiom $D_e(y) = \Delta_{i,e}^{V_i \oplus Q_{i,e}}(y)$ with a *use* $\delta_{i,e,u}(y) \geq x$ is settled. Obviously, such a stage u exists.

Suppose now that $x = \delta_{i,e,s}(k)$ was chosen as a *use* for some $\Delta_{i,e}^{V_i \oplus Q_{i,e}}(k)$ at a stage $s \leq u$ (otherwise, $x \notin Q_{i,e}$). Find a stage $v \geq u$ at which $V_{i,v} \restriction x = V_i \restriction x$ and $D_{e,v}(x) = D_e(x)$. Now $x \in Q_{i,e}$ if and only if $x \in Q_{i,e,v}$. $\square$

Lemma 2. If $D_e = \Phi_e^{U \oplus V}$ then $D_e \leq_T V \oplus Q_e$.

Proof. It follows immediately by construction. $\square$

Lemma 3. $Q_{i,e}$ is c. e. in V_{1-i}.

Proof. It follows immediately from the construction. $\square$

Lemma 4. *For each $i \leq 1$ and $e \in \omega$, requirements P_e^i are eventually satisfied.*

Proof. Fix e and assume by induction that the Lemma holds for all $j < e$. Choose s minimal so that no P_j^i-restraints may be injured by some R-requirement. By construction we may injure P_e^i by finitely many times contributing some integers into V_i to protect the V_{1-i}-restraint of higher priority. But beginning at some stage s we take witnesses for Δ-*uses* greater than the V_{1-i}-restraints, after that we meet the requirement P_e^i. $\square$

Lemma 5. $\mathbf{0}' = \mathbf{v_0} \cup \mathbf{v_1}$.

Proof. Suppose, in contrary, that $\mathbf{v_0} \cup \mathbf{v_1} < \mathbf{0}'$. Then by [8, Theorem 4.1] there exists a d-c. e. degree $\mathbf{d}$ such that $\mathbf{v_0} \cup \mathbf{v_1} < \mathbf{d}$ and $\mathbf{d}$ is not c. e. in $\mathbf{v_0} \cup \mathbf{v_1}$, and therefore it is not c. e. in $\mathbf{v_1}$. We have $\mathbf{v_0} < \mathbf{d}$ and $\mathbf{d}$ is not c. e. in $\mathbf{v_1}$, a contradiction.
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Резюме

М.М. Арсланов. Относительная перечислимость и d-вычислимо перечислимые степени.

В работе изучается связь между относительной перечислимостью и d-вычислимо перечислимыми степенями. Доказано, что тьюринговая степень проблемы остановки машины Тьюринга разложима на такие две вычислимо перечислимые степени, что верхний конус каждой из них состоит только из тех d-вычислимо перечислимых степеней, которые перечислимы относительно второй степени.

Ключевые слова: тьюринговые степени, вычислимо перечислимые степени, относительная перечислимость, разложение, определимость.

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