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Modules Close to SSP- and SIP-Modules

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Abstract—In this paper, we investigate some properties of SIP, SSP and CS-Rickart modules. We give equivalent conditions for SIP and SSP modules; establish connections between the class of semisimple artinian rings and the class of SIP rings. It shows that R is a semisimple artinian ring if and only if R_R is SIP and every right R-module has a SIP-cover. We also prove that R is a semiregular ring and $J(R) = Z(R_R)$ if only if every finitely generated projective module is a CS-Rickart module which is also a $C2$ module.

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1. INTRODUCTION AND NOTATION

Throughout this paper R denotes an associative ring with identity, and modules will be unitary right R-modules. The Jacobson radical ideal in R is denoted by $J(R)$. The notations $N \leq M$, $N \leq_e M$, $N \trianglelefteq M$, or $N \subset_d M$ mean that N is a submodule, an essential submodule, a fully invariant submodule, and a direct summand of M, respectively. We refer to [6, 9, 18], and [23] for all the undefined notions in this paper.

Recall that a module M is called a *SIP module* (respectively, *SSP module*) if the intersection (or the sum) of any two direct summands of M is also a direct summand of M (see [12, 14, 22]). It is known that every Rickart right R-module M (i.e., every endomorphism of M has the kernel a direct summand) has the SIP (see [16, Proposition 2.16]) and every d-Rickart right R-module M (i.e., every endomorphism of M has the image a direct summand) has the SSP ([17, Proposition 2.11]).

A module M is called an *SIP-CS module* if the intersection of any two direct summands of M is essential in a direct summand of M. It is known that every CS-Rickart module has the CS-SIP (see [2, Proposition 1.(4)]).

In this paper, we provide some characterizations of SIP, SSP, SIP-CS and CS-Rickart modules.

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2. SIP MODULES AND SSP MODULES

Let $f : A \to B$ be a homomorphism. We denote by $\langle f \rangle$ the submodule of $A \oplus B$ as follows: $\langle f \rangle = \{a + f(a) \mid a \in A\}$. The following result is obvious and we can omit its proof.

Lemma 2.1. *Let $M = X \oplus Y$ and $f : A \to Y$, a homomorphism with $A \leq X$. Then*

(1) $A \oplus Y = \langle f \rangle \oplus Y$;

(2) $\mathrm{Ker}(f) = X \cap \langle f \rangle$.

We next study some properties of SIP and SSP modules via homomorphisms:

Proposition 2.2. *The following conditions are equivalent for a module M:*

(1) M is SSP;

(2) For any split monomorphism $f : A \to M$ with A a direct summand of M, $A + \mathrm{Im}(f)$ is a direct summand of M;

(3) For any split epimorphism $f : M \to M/A$ with A a direct summand of M, $A + \mathrm{Ker}(f)$ is a direct summand of M.

Proof. (1) $\Rightarrow$ (2), (3) are obvious.

(2) $\Rightarrow$ (1). Assume that $M = A_1 \oplus A_2$ and $f : A_1 \to A_2$ an R-homomorphism. Call $T = \langle f \rangle$ a submodule of M and hence $M = T \oplus A_2$. We consider the homomorphism $\psi : A_1 \to M$ given by $\psi(x) = x + f(x)$. It is easily to see that ψ is a split monomorphism. By (2), $A_1 + \psi(A_1) = A_1 + T$ is a direct summand of M. Furthermore, $A_1 + T = A_1 \oplus \mathrm{Im}(f)$, which implies $\mathrm{Im}(f)$ is a direct summand of A_2.

(3) $\Rightarrow$ (1). Suppose that $M = A_1 \oplus A_2$ and $f : A_1 \to A_2$ an R-homomorphism. Let $T = \langle f \rangle$ be a submodule of M. Then $M = T \oplus A_2$. Call the homomorphism $\psi : M \to M/T$ given by $\psi(a_1 + a_2) = a_2 + T$ for all $a_1 \in A_1, a_2 \in A_2$. Clearly, ψ is a split epimorphism and $\mathrm{Ker}(\psi) = A_1$. By (3), $A_1 + T$ is a direct summand of M. On the other hand, $A_1 + T = A_1 \oplus \mathrm{Im}(f)$, which implies $\mathrm{Im}(f)$ is a direct summand of A_2. □

Corollary 2.3. The following conditions are equivalent for a module M:

(1) M is SSP;

(2) For any two direct summands A_1 and A_2 with $A_1 \simeq A_2$, then $A_1 + A_2$ is a direct summand of M.

Similarly with SIP, we also have some characterizations of SIP-modules:

Proposition 2.4. *The following conditions are equivalent for a module M:*

1. *M is* SIP;
2. *For any split monomorphism $f : A \to M$ with A a direct summand of M, $A \cap f(A)$ is a direct summand of M;*
3. *For any split epimorphism $f : M \to M/A$ with A a direct summand of M, $A \cap \mathrm{Ker}(f)$ is a direct summand of M.*

Corollary 2.5. The following conditions are equivalent for a module M:

1. M is SIP;
2. For any two direct summands A_1 and A_2 with $A_1 \simeq A_2$, then $A_1 \cap A_2$ is a direct summand of M.

Proposition 2.6. *Let R be a ring, M an R-R-bimodule and $T = R \propto M$ the corresponding trivial extension. The following conditions are equivalent:* (1) T has the SSP;

(2) (a) R *has the* SSP;

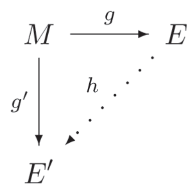
(b) *For every regular x of R with $x = xyx$, we have $xM(1 - xy) = 0$.*

Proof. (1) $\Rightarrow$ (2). By [12, Proposition 4.5].

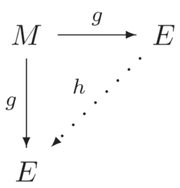
(2) $\Rightarrow$ (1). Assume that $x = xyx$. For any $m \in M$, call $z = xm(1 - xy)$. It follows that $z = (xy)z(1 - xy)$. Note that xy is idempotent of R. By [12, Proposition 4.5], $z = (xy)z(1 - xy) = 0$. □

Let R be a ring and Ω, a class of right R-modules which is closed under isomorphisms and summands. According to Enochs in [10], we study the notion of Ω-envelope and the notion of Ω-cover:

An R-homomorphism $g : M \to E$ is called an *Ω-envelope of a right R-module M*; if $E \in \Omega$ such that any diagram:

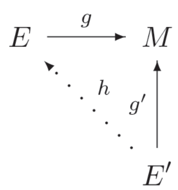


with $E' \in \Omega$, can be completed, and the diagram:

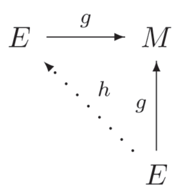


can be completed only by an automorphism h.

An R-homomorphism $g : E \to M$ is called *an Ω-cover of a right R-module M*; if $E \in \Omega$ such that any diagram:



with $E' \in \Omega$, can be completed, and the diagram:



can be completed only by an automorphism h.

A right R-module M is called a *$C3$-module* if whenever A and B are direct summands of M with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M. Dually, M is called a *$D3$-module* if whenever M_1 and M_2 are direct summands of M and $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of M.

Proposition 2.7. *The following conditions are equivalent for a ring R*:

(1) *R is a semisimple artinian ring*;

(2) *Every right R-module has a $D3$-cover*;

(3) *Every* 2*-generated right R-module has a* D3*-cover*;

(4) *Every right R-module has a* D3*-envelope*;

(5) *Every* 2*-generated right R-module has a* D3*-envelope*.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Clear.

(3) $\Rightarrow$ (1). Let S be a simple right R-module. Call $\varphi : R_R \to S$ an epimorphism. By (3), $M = R_R \oplus S$ has a D3-cover, say $\alpha : C \to M$ where C is a D3-module. Let $\iota_1 : S \to M$ and $\iota_2 : R_R \to M$ be the inclusion maps for all $i = 1, 2$. Note that S and R_R are D3-modules, and there are homomorphisms $\beta_1 : S \to C, \beta_2 : R_R \to C$ such that $\alpha\beta_i = \iota_i$. Clearly, $id_M = \iota_1 \oplus \iota_2 = \alpha(\beta_1 \oplus \beta_2)$. This shows that M is isomorphic to a direct summand of C, which implies that M is a D3-module. We deduce that Ker(φ) is a direct summand of R_R by [4, Proposition 4]. It follows that S is a projective module. Thus R is semisimple.

(1) $\Rightarrow$ (4) $\Rightarrow$ (5). Clear.

(5) $\Rightarrow$ (1). Let S be a simple right R-module. Call $\varphi : R_R \to S$ an epimorphism. By (5), $M = R_R \oplus S$ has a D3-envelope, named $\iota : M \to E$ where E is a D3-module. Since S and R are D3-modules, there exist $f_1 : E \to S,\ f_2 : E \to R$ such that $f_i\iota = \pi_i$, where $\pi_1 : M \to S$ and $\pi_2 : M \to R$ are the projections. There exists $\phi : E \to M$ such that $\pi_i\phi = f_i$ for all $i = 1, 2$. It follows that $\phi\iota = id_M$, and hence ι is a split monomorphism. Thus $N \oplus E(N)$ is isomorphic to a direct summand of E. This gives that $S \oplus R$ is also a D3-module. We deduce that Ker(φ) is a direct summand of R_R. So S is a projective module. Thus R is semisimple. □

Corollary 2.8. The following conditions are equivalent for a ring R:

(1) R is a semisimple artinian ring;

(2) R_R is SIP and every right R-module has a SIP-cover;

(3) R_R is SIP and every 2-generated right R-module has a SIP-cover;

(4) R_R is SIP and every right R-module has a SIP-envelope;

(5) R_R is SIP and every 2-generated right R-module has a SIP-envelope.

A ring R is called a right *V-ring* if every simple right R-module is injective.

Proposition 2.9. *The following conditions are equivalent for a ring R:*

(1) *R is a right* V-*ring*;

(2) *Every finitely cogenerated right R-module has a* C3-*envelope*;

(3) *Every finitely cogenerated right R-module has a* C3-*cover*.

Proof. (1) $\Rightarrow$ (2), (3) are obvious.

(2) $\Rightarrow$ (1) Let N be an arbitrary simple module. Assume that $\iota : M = N \oplus E(N) \to E$ is the C3-envelope, where E is a C3-module. Since N and $E(N)$ are C3-modules, there exist $f_1 : E \to N,\ f_2 : E \to E(N)$ such that $f_i\iota = \pi_i$, where $\pi_1 : M \to N_i$ and $\pi_2 : M \to E(N)$ are the projections. There exists $\phi : E \to M$ such that $\pi_i\phi = f_i$ for all $i = 1, 2$. It follows that $\phi\iota = id_M$, and so the monomorphism ι splits. Thus $N \oplus E(N)$ is isomorphic to a direct summand of E. It follows that $N \oplus E(N)$ is also a C3-module. Therefore N is a direct summand of $E(N)$. This gives N is injective. Thus R is a right V-ring.

(3) $\Rightarrow$ (1) The proof is similar to the proof (3) $\Rightarrow$ (1) of Proposition 2.7. □

Similarly, we also get the following result for injectivity of semisimple modules:

Proposition 2.10. *The following conditions are equivalent for a ring R:*

(1) *R is a right Noetherian right* V-*ring*;

(2) *Every right R-module with essential socle has a* C3-*envelope*;

(3) *Every right R-module with essential socle has a* C3-*cover*.

3. SIP-CS MODULES

A module M is called *relatively CS-Rickart to* N (or N*-CS-Rickart*) if for every $\varphi \in \mathrm{End}_R(M,N)$, $\mathrm{Ker}\varphi$ is an essential submodule of a direct summand of M. A module M is called *relatively d-CS-Rickart to* N (or N*-d-CS-Rickart*) if for every $\varphi \in \mathrm{End}_R(N,M)$, $\mathrm{Im}\varphi$ lies above a direct summand of M. A module M is called *CS-Rickart (d-CS-Rickart)* if M is M-CS-Rickart (resp., M-d-CS-Rickart). M is called a *SIP-CS module* if A_i is essential in a direct summand of M for all $i \in \mathcal{I}, \mathcal{I}$ is a finite index set, then $\bigcap_{i\in\mathcal{I}} A_i$ is essential in a direct summand of M. M is called a *lifting SSP module* if A_i lies above a direct summand of M for all $i \in \mathcal{I}, \mathcal{I}$ is a finite index set, then $\sum_{i\in\mathcal{I}} A_i$ lies above a direct summand of M. The class of CS-Rickart (d-CS-Rickart, SIP-CS, lifting SSP) modules is studied by the authors in [1, 2].

Lemma 3.1. *The following implications hold for a module* $M = M_1 \oplus \ldots \oplus M_n$:

(1) *if* M *is relatively* CS-Rickart *to* N *then* M_i *relatively* CS-Rickart *to* N;

(2) *if* M *is relatively* d-CS-Rickart *to* N *then* M_i *relatively* d-CS-Rickart *to* N.

Proof. We only need to prove for the case $n = 2$, $i = 1$.

(1) Assume that $M = M_1 \oplus M_2$ is relatively CS-Rickart to N. There exists $\varphi : M \to N$ such that $\varphi = \psi \oplus 0|_{M_2}$ for each $\psi : M_1 \to N$. By assumption, there exists a direct summand D of M such that $\mathrm{Ker}(\varphi) \leq_e D$. Since $\mathrm{Ker}(\varphi) = \mathrm{Ker}(\psi) \oplus M_2$ and $D = (D \cap M_1) \oplus M_2$, it follows that $\mathrm{Ker}(\psi) \oplus M_2 \leq_e (D \cap M_1) \oplus M_2$. Therefore $\mathrm{Ker}(\psi) \leq_e D \cap M_1$. Since D is a direct summand of M, $D \cap M_1$ is a direct summand of M_1. Hence M_1 is relatively CS-Rickart to N.

(2) Assume that $M = M_1 \oplus M_2$ is relatively d-CS-Rickart to N. There exists $\varphi : M \to N$ such that $\varphi = \psi \oplus 0|_{M_2}$ for each $\psi : M_1 \to N$. By assumption, $\mathrm{Im}\varphi = \mathrm{Im}\psi$ lies above a direct summand of N. Thus, M_1 is relatively d-CS-Rickart to N. □

Proposition 3.2. *The following implications hold for a module* M:

(1) *if* M *is a* SIP-CS *module with* $C2$ *condition and* $M = M_1 \oplus M_2$ *then* M_1 *relatively* CS-Rickart *to* M_2;

(2) *if* M *is a lifting* SSP *module with* $D2$ *condition and* $M = M_1 \oplus M_2$ *then* M_1 *relatively* d -CS-Rickart to M_2.

Proof. Let $f : M_1 \to M_2$ be an R-homomorphism. Then $M = \langle f \rangle \oplus M_2$.

(1) We have that $\mathrm{Ker}(f) = \langle f \rangle \cap M_1 \leq_e eM$ for some $e^2 = e \in S$, by M is SIP-CS. Let $\pi_1 : M_1 \oplus M_2 \to M_1$ be the canonical projection and hence $eM \cap M_2 = 0$, implies that $\pi_1(eM) \cong eM$. Since M is a $C2$ module, $\pi_1(eM)$ is a direct summand of M. Then, since $\mathrm{Ker}(f) \leq_e eM$, $\mathrm{Ker}(f) = \pi_1(\mathrm{Ker}(f)) \leq_e \pi_1(eM)$. Hence, M_1 is relatively CS-Rickart to M_2.

(2) We have that $\mathrm{Im}(f) \oplus M_1 = \langle f \rangle + M_1$ lies above eM for some $e^2 = e \in S$, by M is lifting SSP. Since $\langle f \rangle + M_1 + M_2 = M$, $eM + M_2 = M$ by [8, 3.2.(1)]. Let $\pi_1 : M_1 \oplus M_2 \to M_1$ be the canonical projection. Then $\pi_{1|eM}$ splits by $D2$ condition. It follows that $eM = (eM \cap M_2) \oplus N$ by $\mathrm{Ker}(\pi_{1|eM}) = eM \cap M_2$. We have $N \oplus M_2 = N + (eM \cap M_2 + M_2) = eM + M_2 = M$ and obtain that $N \oplus \mathrm{Im}(f) = M_1 \oplus \mathrm{Im}(f) \supset eM$.

By modular law, $eM = N \oplus eM \cap \mathrm{Im}(f)$. As $\dfrac{\mathrm{Im}(f) \oplus M_1}{eM} \ll \dfrac{M}{eM}$, we have that $\dfrac{N \oplus \mathrm{Im}(f)}{N \oplus eM \cap \mathrm{Im}(f)} \ll \dfrac{N \oplus M_2}{N \oplus eM \cap \mathrm{Im}(f)}$. This is equivalent to $\dfrac{\mathrm{Im}(f)}{eM \cap \mathrm{Im}(f)} \ll \dfrac{M_2}{eM \cap \mathrm{Im}(f)}$, which implies that $\mathrm{Im}(f)$ lies above the direct summand $eM \cap \mathrm{Im}(f)$ of M. □

Corollary 3.3. The following implications hold for a module $M = M_1 \oplus \ldots \oplus M_n$:

(1) if M is a SIP-CS module with $C2$ condition then M_i is relatively CS-Rickart to M_j for every $i \neq j$;

(2) if M is a lifting SSP with $D2$ condition then M_i is relatively d-CS-Rickart to M_j for every $i \neq j$.

Proof. If M is a SIP-CS module with $C2$ condition (respectively, lifting SSP with $D2$ condition), then by Proposition 3.2, $\bigoplus_{i\neq j} M_i$ is relatively CS-Rickart to M_j (respectively, relatively d-CS-Rickart to M_j). By Lemma 3.1, M_i is relatively CS-Rickart to M_j (respectively, relatively d-CS-Rickart to M_j) for every $i \neq j$. □

Corollary 3.4. The following implications hold for a module M:

(1) if $M \oplus M$ is a SIP-CS module with $C2$ condition then M is a CS-Rickart module;

(2) if $M \oplus M$ is a lifting SSP with $D2$ condition then M is a d-CS-Rickart module.

Proof. Follow from Corollary 3.3. □

The singular submodule $Z(M)$ of a right R-module M is defined as $Z(M) = \{m \in M : ann_R^r(m)$ is an essential right ideal of $R\}$ where $ann_R^r(m)$ denotes the right annihilator of m in R. The singular submodule of R_R is called the (right) singular ideal of the ring R and is denoted by $Z(R_R)$. It is well known that $Z(R_R)$ is indeed an ideal of R.

Next we give a necessary and sufficient condition for a ring over which every finitely generated projective module to be a SIP-CS-module which is also a $C2$ module.

Theorem 3.5. *The following conditions are equivalent for a ring R:*

(1) *R is a semiregular ring and $J(R) = Z(R_R)$;*

(2) *Every finitely generated projective module is a* CS-Rickart *module which is also a $C2$ module;*

(3) *Every finitely generated projective module is a* SIP-CS *module which is also a $C2$ module;*

(4) *Every finitely generated projective module is a* SIP-CS *module which is also a $C3$ module.*

Proof. (1) ⇒ (2). Follows from [2, Theorem 2].

(2) ⇒ (3). Follows from [2, Proposition 1].

(3) ⇒ (2). Let P be a finitely generated projective module. By the hypothesis, P is a SIP-CS module which is also a $C2$ module. Then $P \oplus P$ is a SIP-CS module which is also a $C2$ module. Since Proposition 3.2, P is relatively CS-Rickart to P, it means that P is a CS-Rickart module.

(3) ⇔ (2). Follows from [1, Corollary 3.5]. □

Lemma 3.6. *The following conditions are equivalent for a module M:*

(1) *M is a* SIP-CS *module;*

(2) *Intersection of every pair of direct summands of M is essential in a direct summand of* M.

Proof. It is obvious. □

Proposition 3.7. *Assume that M is a* SIP-CS *module. Then for any decomposition $M = M_1 \oplus M_2$ and $f : M_1 \to M_2$ is a homomorphism, then* Ker(f) *is essential in a direct summand of M.*

Proof. Assume that $M = M_1 \oplus M_2$ and $f : M_1 \to M_2$ an R-homomorphism. Call $T = \langle f \rangle$ a submodule of M. So $M = T \oplus M_2$ and $\mathrm{Ker}(f) = T \cap M_1$. On the other hand, by the hypothesis, M is a SIP-CS and hence Ker(f) is essential in a direct summand of M by Lemma 3.6. □

Corollary 3.8. Let M be a module and N, a nonsingular module. If $M \oplus N$ is a SIP-CS module, then every homomorphism from M to N has the kernel a direct summand of M.

Proof. Let $f : M \to N$ be a non-zero homomorphism. By Proposition 3.7, Ker(f) is essential in a direct summand of $M \oplus N$. Assume that A is a direct summand of $M \oplus N$ such that $\mathrm{Ker}(f) \leq_e A$. Call $\pi_M : M \oplus N \to M$ the canonical projection and $h = (f \circ \pi_M)|_A : A \to N$. Therefore $\mathrm{Ker}(h) = \mathrm{Ker}(f) \oplus (N \cap A)$. We have that $\mathrm{Ker}(f) \leq_e A$ and obtain that $\mathrm{Ker}(f) \oplus (N \cap A) \leq_e A$. It follows that $\mathrm{Ker}(f) \oplus (N \cap A) = A$. Thus Ker$(f)$ is a direct summand of M. □

Corollary 3.9. Let M be an indecomposable module and N be a nonsingular module. If $M \oplus N$ is a SIP-CS module, then every nonzero homomorphism from M to N is a monomorphism.

Proposition 3.10. *Let M be a nonsingular right R-module. If $(R \oplus M)_R$ is a* SIP-CS *module, then every cyclic submodule of M is projective.*

Proof. Let m be a non-zero arbitrary element of M. Call the homomorphism $\varphi : R_R \to M$ given by $\varphi(x) = mx$. As $(R \oplus M)_R$ is a SIP-CS module, Ker(φ) is a direct summand of R_R by Corollary 3.8. It follows that Im(φ) is isomorphic to a direct summand of R_R. Thus mR is a projective module. □

A ring R is called right *(semi)hereditary* if every (finitely generated) right ideal of R is projective.

Theorem 3.11. *The following statements are equivalent for a right nonsingular ring R:*

(1) *R is right hereditary;*

(2) *Every projective right R-module is a* SIP-CS *module.*

Proof. (1) ⇒ (2) is obvious.

(2) $\Rightarrow$ (1) Let I be a right ideal of R. We will show that I is a projective module. Call an epimorphism $\varphi : F \to N$ for some free right R-module F. Let ι be the inclusion map from I to R_R. Consider the homomorphism $\iota \circ \varphi : F \to R_R$. By (2), $F \oplus R_R$ is a SIP-CS module. We have from Corollary 3.8, $\mathrm{Ker}(\varphi) = \mathrm{Ker}(\iota \circ \varphi)$ is a direct summand F. This gives that $F = \mathrm{Ker}(\varphi) \oplus B$ for some submodule B of F. Thus, I is projective. □

The author Warfied proved that if R is right serial, then R is right nonsingular if and only if R is right semihereditary.

The same argument of the proof of Theorem 3.11, we also have the following result of semihereditary rings:

Theorem 3.12. *The following statements are equivalent for a right nonsingular ring R:*

(1) *R is right semihereditary;*

(2) *Every finitely generated projective right R-module is a* SIP-CS *module;*

(3) *Every finitely generated free right R-module is a* SIP-CS *module.*

Let M be a right R-module and $S = \mathrm{End}(M)$. We denote

$$\Delta(S) = \{f \in S \mid \mathrm{Ker}(f) \leq_e M\}.$$

An R-module is called a *self-generator* if it generates all its submodules.

Theorem 3.13. *The following conditions are equivalent for a self-generator module M with $S = \mathrm{End}(M)$:*

(1) *S is a semiregular ring with $J(S) = \Delta(S)$;*

(2) *M is a* CS-Rickart *and* C2 *module.*

Proof. (1) $\Rightarrow$ (2) Assume that S is a semiregular ring with $J(S) = \Delta(S)$. As M is a self-generator, $J(S) = \Delta(S) \leq Z(S_S)$. We deduce that S is right C2. This gives that M is a C2-module by [20, Theorem 7.14(1)]. Let $\alpha : M \to M$ be an endomorphism of M. As S is a semiregular ring, there exists $\beta \in S$ such that $\beta = \beta\alpha\beta$ and $\alpha - \alpha\beta\alpha \in J(S)$ by [19, Theorem 2.9]Ni. Call $e = 1 - \beta\alpha$. Then $e^2 = e \in S$. As $\alpha - \alpha\beta\alpha \in \Delta(S)$, $\mathrm{Ker}(\alpha - \alpha\beta\alpha) \leq_e M$ and hence $\mathrm{Ker}(\alpha - \alpha\beta\alpha) \cap e(P) \leq_e e(M)$. It is easily to check that $\mathrm{Ker}(\alpha - \alpha\beta\alpha) \cap e(M) = \mathrm{Ker}(\alpha)$. We deduce that $\mathrm{Ker}(\alpha) \leq_e e(M)$.

(2) $\Rightarrow$ (1) By [21, Theorem 3.2]. □

Corollary 3.14. The following conditions are equivalent for a self-generator module M with $S = \mathrm{End}(M^{(\mathbb{N})})$:

(1) S is a semiregular ring with $J(S) = \Delta(S)$;

(2) $M^{(\mathbb{N})}$ is a CS-Rickart and C2 module;

(3) $M^{(\mathbb{N})}$ is a SIP-CS and C2 module.

*Proof.*By Proposition 3.2 and Theorem 3.13. □

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