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On Extrema of the Mityuk Radius for Doubly Connected Domains

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Abstract—We study extrema of the Mityuk radius depending on the choice of the canonical domain. Turning to the doubly connected case allows us to use the explicit form of mapping functions onto canonical domains. We obtain the results on the localization of critical points of the Mityuk radius for two types of such domains.

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1. INTRODUCTION

Let D be a finitely connected bounded domain in the complex plane and let a holomorphic function

$$F(w, w_0) = (w - w_0)\varphi(w, w_0), \quad \varphi(w_0, w_0) \neq 0, \tag{1}$$

maps D onto the canonical domain $F(D)$, which is obtained from a simply connected domain with a smooth boundary by making an appropriate number of cuts of the prescribed form. In this case, the enclosing contour $L \subset \partial D$ corresponds to a smooth (enclosing) boundary contour of the domain $F(D)$. Let, further, the function $z = f(w)$ be holomorphic in D. I. P. Mityuk's introduction of the concept of a generalized reduced module [1] made it possible to give the following

Definition. *The Mityuk radius of a domain $f(D)$ at a point $f(w)$ with respect to the canonical domain $F(D)$ is defined by*

$$\Omega(w) = |f'(w)|/|\varphi(w, w)|. \tag{2}$$

Function (2) is a multiply connected analog of the conformal radius of a simply connected domain [2]. The definition of the Mityuk radius was first given in [3] for the case when $F(D)$ is a unit disk with circular concentric cuts. It is the case that corresponds to I.P. Mityuk's construction of the generalized reduced modulus of the domain $f(D)$ with respect to the point $f(w)$ and the boundary component $f(L)$; this modulus is equal to the value $M(w) = (2\pi)^{-1} \ln \Omega(w)$ [1].

In the present paper, we consider situations where D is a ring $E_q = \{w : q < |w| < 1\}$, the enclosing contour $F(L)$ of $F(D)$ is a unit circle or a star-shaped curve of a special form, and the only cut of the canonical domain is either concentric circular or radial.

The second section of our paper is devoted to the construction of mapping functions F for the above situations and to the study the behavior of extrema for the corresponding Mityuk radius (2) when $f(w) = w$ is the identity mapping, i.e. functions

$$\Omega(w) = 1/|\varphi(w, w)|. \tag{3}$$

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When constructing maps onto canonical domains, we follow more the geometric approach of Robinson [4] and Alenitsyn [5] than the Komatu method [6] based on the use of elliptic functions. Note that in [4–6] function (1) was constructed for the case of a disk with a circular concentric cut, in [6] — when $F(E_q)$ is a disk with a radial cut. A new trend in the study of the Mityuk radius is set by the recent work [7] devoted to their numerical construction.

In the third section, we present two types of conditions on the function f that provide a simple behavior of the Mityuk radius (2) on segments cut by the ring E_q on rays starting from the origin. One type leads to the uniqueness of an extremum of the restriction of the function (2) on any such segment, the other type — to the absence of such an extremum. Conditions of the first type imply that each critical point of (2) lies on the golden section circle $|w| = \sqrt{q}$ of the ring E_q (see [3]).

In both cases $F(\partial E_q)$ is a unit circle, but in the first case the cut is concentric circular, and in the second case it is radial. Respectively, in the first case, we call Mityuk's radius (2) "circular", in the second we call it "radial".

2. CANONICAL DOMAINS OVER THE RING E_q AND CORRESPONDING MITYUK'S RADII

2.1. Unit Disk with the Concentric Circular Cut

Let us find an expression for the Mityuk radius (3) of the ring E_q with respect to a unit disk with a cut along a circular arc centered at the origin. Let the function (1) maps E_q conformally and univalently onto a domain of this type. The expression for $F(w, w_0)$ is known (see, for example, [5]), its derivation is considered to belong to mathematical folklore, but no simple construction has been found. Here we fill this gap with the help of the symmetry method.

The function $F(w, w_0)$ satisfies the following conditions: *a*) $F(w, w_0)$ is regular inside and on the boundary of E_q; *b*) $|F(w, w_0)|$ is a single-valued function in E_q; *c*) $|F(w, w_0)|$ is constant on the boundary components $|w| = 1$ and $|w| = q$; *d*) $F(w, w_0)$ has a unique (simple) zero at $w_0 \in E_q$.

The multivalued function $\ln F(w, w_0)$ has a constant real part on the boundary components of the ring E_q. In other words, $\ln F(w, w_0)$ is an analogue of the complex Green function for E_q, and its existence follows from the general theorems of the potential theory and the theory of boundary value problems. Let us derive a formula for the function $F(w, w_0)$ based on its geometric properties.

It follows from conditions *a*)–*d*) and the symmetry principle that the function $F(w, w_0)$ can be analytically extended outside of the ring E_q into two rings adjacent to E_q. This extension is given by the relations

$$F(w, w_0)\,\overline{F(1/\bar{w}, w_0)} = \text{const} \quad \text{and} \quad F(w, w_0)\,\overline{F(q^2/\bar{w}, w_0)} = \text{const}. \tag{4}$$

Formulas (4) imply that the points $1/\bar{w}_0$ and $q^2/\bar{w}_0$, which are mirror-symmetric to the point w_0 with respect to the boundary components $|w| = 1$ and $|w| = q$ of the ring, will be simple poles of the extended function $F(w, w_0)$. Applying the symmetry principle again, we continue the function $F(w, w_0)$ into two further rings, and so on, finally to the entire w-plane without the points $w = 0$ and $w = \infty$.

These considerations imply that the function $F(w, w_0)$ must have simple zeros at the points $w = q^{2k} w_0$ and simple poles at the points $w = q^{2k}/\bar{w}_0$, where k is an integer, and that $F(w, w_0)$ has no other zeros and poles. In the context of the above reasoning, consider the function

$$F(w, w_0) = \frac{w - w_0}{1 - \overline{w}_0 w} \prod_{k=1}^{\infty} \left[\frac{(1 - q^{2k} w/w_0)(1 - q^{2k} w_0/w)}{(1 - q^{2k} w \overline{w}_0)(1 - q^{2k}/(w \overline{w}_0))} \right] \tag{5}$$

that has zeros and poles at the required points. Replacing w with $1/\bar{w}$ and with $q^2/\bar{w}$ allows us to refine (4) to the equalities

$$F(w, w_0)\,\overline{F(1/\bar{w}, w_0)} = 1 \quad \text{and} \quad F(w, w_0)\,\overline{F(q^2/\bar{w}, w_0)} = |w_0|^2. \tag{6}$$

It follows from (6) that $|F(w, w_0)| = 1$ when $|w| = 1$ and that $|F(w, w_0)| = |w_0|$ for $|w| = q$, i.e. the function $F(w, w_0)$ satisfies condition *c*), as well as all other requirements of the problem. The following is true.

Theorem 1. *Function $F(w, w_0)$ defined by (5) maps the ring E_q conformally and univalently onto the unit disk with a circular concentric cut at a distance $|w_0|$ from the origin. Mityuk's radius (3) of the ring E_q with respect to the domain $F(E_q)$ is equal to*

$$\Omega_c(w) = H_0(q)^{-2}(1 - |w|^2) \prod_{k=1}^{\infty} (1 - q^{2k}|w|^2)(1 - q^{2k}|w|^{-2}), \tag{7}$$

where $H_0(q) = \prod_{k=1}^{\infty}(1 - q^{2k})$.

Now we can find the critical points of the Mityuk radius (7) of the annular region E_q with respect to the unit disk with a circular concentric cut: the condition $(\partial/\partial w) \ln \Omega_c(w) = 0$ is equivalent to the equation $\sum_{k=1}^{\infty} q^{2k}(r^4 - q^2)(1 - q^{2k}r^2)^{-1}(r^2 - q^{2k+2})^{-1} = 0$, whose solutions, obviously, are all points of the circle $|w| = \sqrt{q}$ and only them. Indeed, for $r > \sqrt{q}$ ($r < \sqrt{q}$) all terms of the series are positive (negative). Thus, we have the following

Theorem 2. *Mityuk's radius of the ring E_q with respect to the unit disk with a circular concentric cut has an infinite set of critical points filling the circle $|w| = \sqrt{q}$.*

2.2. Unit Disk with the Radial Cut

Let us find an expression for the Mityuk radius (3) of the ring E_q with respect to a unit disk with a radial cut. In this case, the function (1) maps E_q conformally and univalently onto such a domain; for convenience, we will assume that the radial cut is located on the real axis.

The function $F(w, w_0)$ satisfies the conditions a), b) and d) of subsection 2.1, as well as the new condition c), which will state that the equality $|F(w, w_0)| = 1$ is satisfied on the boundary component $|w| = 1$ and the equality $\arg F(w, w_0) = 0$ — on the boundary component $|w| = q$.

Thus, the multivalued function $\ln F(w, w_0)$ has a constant real part on $|w| = 1$ and a constant imaginary part on $|w| = q$. The existence of such a function also follows from the general theorems of potential theory and the theory of boundary value problems. Let us obtain an explicit formula for the function $F(w, w_0)$ from geometric considerations.

According to the symmetry principle, the function $F(w, w_0)$ can be extended analytically beyond E_q to the ring $1 < |w| < 1/q$ adjacent to E_q. This extension is given by the first formula in (6), from which it follows that the point $1/\bar{w}_0$ will be a simple pole of the extended function $F(w, w_0)$.

We conclude from the condition $\arg F(w, w_0) = 0$ on $|w| = q$ that the analytic continuation of the function $F(w, w_0)$ to the ring $q^2 < |w| < q$ is given by the formula

$$F(w, w_0) = \overline{F(q^2/\bar{w}, w_0)}. \tag{8}$$

Therefore, the point $q^2/\bar{w}_0$, symmetrical to the point w_0, will be a simple zero of the analytical continuation of the function $F(w, w_0)$. Applying the symmetry principle again, we extend the function $F(w, w_0)$ to the ring $q^3 < |w| < 1/q^2$, and so on, as a result — to the entire w-plane without the points $w = 0$ and $w = \infty$.

It follows from these considerations that the function $F(w, w_0)$ must have simple zeros at the points $w = q^{4k}w_0$ and $w = q^{4k+2}/\bar{w}_0$, simple poles — at the points $w = q^{4k+2}w_0$ and $w = q^{4k}/\bar{w}_0$, where k is an integer, and that $F(w, w_0)$ has no other zeros and poles.

Based on these considerations, we consider the function

$$F(w, w_0) = \frac{w - w_0}{1 - \overline{w}_0 w} \prod_{k=1}^{\infty} \left[\frac{(1 - q^{2k}w/w_0)(1 - q^{2k}w_0/w)}{(1 - q^{2k}w\overline{w}_0)(1 - q^{2k}/(w\overline{w}_0))} \right]^{(-1)^k} \tag{9}$$

that has zeros and poles at the required points. Replacing w with $1/\bar{w}$ and with $q^2/\bar{w}$, we get $F(1/\bar{w}, w_0) = 1/\overline{F(w, w_0)}$ and $F(q^2/\bar{w}, w_0) = (w_0/\bar{w}_0) \cdot \overline{F(w, w_0)}$. These equalities mean that function (9) satisfies the first relation (6), but does not satisfy (8). The latter, however, can be easily corrected by adding to $F(w, w_0)$ in (9) a "normalization" factor $e^{-i\theta}$, where $\theta = \arg w_0$. Such a "corrected" function

(9) satisfies all the requirements of the problem. In the following assertion we remove the condition that the radial cut is real.

Theorem 3. *Function $F(w, w_0)$ defined by (9) maps the ring E_q conformally and univalently onto the unit disk with a radial cut having an angle of inclination $\theta = \arg w_0$ to the real axis. Mityuk's radius (3) of the ring E_q with respect to the domain $F(E_q)$ is equal to*

$$\Omega_r(w) = H_3(q^2)^2 H_0(q^2)^{-2}(1 - |w|^2) \prod_{k=1}^{\infty} \left[(1 - q^{2k}|w|^2)(1 - q^{2k}|w|^{-2})\right]^{(-1)^k}, \tag{10}$$

where $H_3(h) = \prod_{k=1}^{\infty}(1 - h^{2k-1})$, and the constant H_0 is defined in Theorem 1.

The critical points of function (10) are the roots of the equation $\Phi_q(|w|) = 0$, which obviously has no solutions in E_q, where

$$\Phi_q(r) = \sum_{m=1}^{\infty}(r^{2m} + (q/r)^{2m})/(1 + q^{2m}). \tag{11}$$

So, the following is true.

Theorem 4. *Mityuk's radius of the ring E_q with respect to the unit disk with a radial cut has no critical points.*

2.3. Star-Shaped Domain with the Radial Cut

Let us find an expression for the Mityuk radius (3) of the ring E_q with respect to a doubly connected domain of a special form, bounded by a star-shaped curve and a cut that lies on the ray starting from the origin. Here the function (1) is univalent and maps E_q conformally onto a domain of such a special form. In this case, conditions a), b) and d) of subsection 2.1 are again preserved, while condition c) states the fulfillment of the equalities $\arg F(w, w_0) = \arg w$ and $\arg F(w, w_0) = \text{const}$, respectively, on the outer $|w| = 1$ and inner $|w| = q$ components of the boundary ∂E_q.

Note that the function $F(w, w_0)$ is determined by these conditions up to a constant positive factor. The function $\ln F(w, w_0)$ is an analogue of the complex Neumann function for the annular domain E_q. Comparison of the formulas for Green's and Neumann's functions for the unit disk allows us to point out an explicit formula for the function $F(w, w_0)$:

$$F(w, w_0) = (w - w_0)(1 - \overline{w}_0 w) \prod_{k=1}^{\infty} \left[\sigma(w/w_0), q^{2k}) \cdot \sigma(w\overline{w}_0, q^{2k})\right], \tag{12}$$

where, for the sake of brevity, we use the notation $\sigma(z, h) = (1 - hz)(1 - h/z)$; moreover, in the formula (12) we have $\arg F(e^{i\theta}, w_0) = \theta$ and $\arg F(qe^{i\theta}, w_0) = -\arg w_0$. The last equalities mean that the function $F(w, w_0)$ satisfies condition c), as well as all other requirements of the problem. We have the following

Theorem 5. *Function $F(w, w_0)$ defined by (12) maps the ring E_q conformally and univalently onto the star-shaped domain with a radial cut having an angle of inclination $\theta = \arg w_0$ to the real axis. Mityuk's radius (3) of the ring E_q with respect to the domain $F(E_q)$ is equal to*

$$\Omega_{s,r}(w) = H_0(q^2)^{-2} \left[(1 - |w|^2) \prod_{k=1}^{\infty}(1 - q^{2k}|w|^2)(1 - q^{2k}|w|^{-2})\right]^{-1},$$

where the constant H_0 is defined in Theorem 1.

In view of the equivalence $(\partial/\partial w) \ln \Omega_c(w) = 0 \Leftrightarrow (\partial/\partial w) \ln \Omega_{s,r}(w) = 0$, the assertion of Theorem 2 holds for critical points of the Mityuk radius $\Omega_{s,r}(w)$.

2.4. Star-Shaped Domain with the Concentric Circular Cut

Let us find an expression for the Mityuk radius (3) of the ring E_q with respect to another canonical domain, namely, doubly connected region of a special form, bounded by a star-shaped curve and now by a circular cut centered at the origin. As usual, we assume that the image of the ring E_q under the mapping by function (1) is a domain of the type just mentioned. Conditions a), b) and d) continue to hold, and condition c) requires the fulfillment of the equalities $\arg F(w, w_0) = \arg w$ and $|F(w, w_0)| = \text{const}$, respectively, on the outer $|w| = 1$ and inner $|w| = q$ components of the boundary ∂E_q.

The function $F(w, w_0)$ is determined by these conditions up to a constant positive factor. It follows from the condition c) that the multivalued function $\ln F(w, w_0)$ has a constant real part on the circle $|w| = q$, and its imaginary part on the circle $|w| = 1$ coincides with $\arg w$. We give an explicit formula for the function $F(w, w_0)$,

$$F(w, w_0) = (w - w_0)(1 - \overline{w}_0 w) \prod_{k=1}^{\infty} \left[\sigma(w/w_0), q^{2k}) \cdot \sigma(w\overline{w}_0, q^{2k})\right]^{(-1)^k}, \qquad (13)$$

where, as above, $\sigma(z, h) = (1 - hz)(1 - h/z)$, $\arg F(e^{i\theta}, w_0) = \theta$, but $|F(qe^{i\theta}, w_0)| = |w_0|$. Thus, the function $F(w, w_0)$ satisfies condition c), as well as all other requirements of the problem. The following is true.

Theorem 6. *Function $F(w, w_0)$ defined by (13) maps the ring E_q conformally and univalently onto the star-shaped domain with a circular concentric cut centered at the origin. Mityuk's radius (3) of the ring E_q with respect to the domain $F(E_q)$ is equal to*

$$\Omega_{s,c}(w) = H_3(q^2)^2 H_0(q^2)^{-2} \left[(1 - |w|^2) \prod_{k=1}^{\infty} \left[(1 - q^{2k}|w|^2)(1 - q^{2k}|w|^{-2})\right]^{(-1)^k}\right]^{-1},$$

where the constants H_0 and H_3 are defined in Theorems 1 *and* 3.

It is easy to see that Theorem 4 holds for the critical points in the case under consideration (the designations H_0 and H_3 are inherited from [8], p. 274).

3. TWO TYPES OF MITYUK'S RADIUS FOR HOLOMORPHIC RING IMAGES

3.1. "Circular" Mityuk's Radius

In the case when the image $F(D)$ of a multiply connected domain D under the mapping by the function (1) is a unit disk with circular concentric cuts, the value (2) will be called the "circular" Mityuk radius. For $D = E_q$, function (2) has the form

$$\Omega(w) = |f'(w)|\,\Omega_c(w), \qquad (14)$$

where the radius $\Omega_c(w)$ is defined in (7). In the general case, a number of properties of the "circular" radius of Mityuk were studied in [9], in a doubly connected one — in the article [3], where, in particular, it was shown that the function (14) can be represented as $\Omega(w) = |f'(w)|/f_q'(|w|)$ with

$$f_q'(w) = H_0(q)^2 \int_{\sqrt{q}}^{w} \left\{\prod_{k=1}^{\infty} \left[(1 - q^{2k-2}\zeta^2)(1 - q^{2k}\zeta^{-2})\right]\right\}^{-1} d\zeta,$$

and the constant $H_0(q)$ is defined in Theorem 1.

Let H_q be a class of functions f holomorphic and locally univalent in E_q, for which the equality $\mathrm{Re}\{e^{i\theta} f''(w)/f'(w)\} = 0$ is true for $|w| = \sqrt{q}$ ($\theta = \arg w$); Λ_q is a family of functions of the form $af_q(\varepsilon w) + b$, $a, b \in \mathbb{C}$, $|\varepsilon| = 1$. The following result was established in the paper [3].

Theorem 7. *The critical points of Mityuk's "circular" radius (14) for the function $f \in H_q \setminus \Lambda_q$ are concentrated on the circle $|w| = \sqrt{q}$ if one of the following two inequalities is fulfilled for $w \in E_q$ and $\theta = \arg w$:*

$$1)\ \mathrm{Re}\,\{e^{i2\theta} S_f(w)\} \le S_{f_q}(|w|); \quad 2)\ \mathrm{Re}\,\{e^{i2\theta}(f''/f')'(w)\} \le (f_q''/f_q')'(|w|),$$

where $S_f(w) = (f''/f')'(w) - (f''/f')^2(w)/2$ is the Schwarzian derivative of the function f.

3.2. "Radial" Mityuk's Radius

In the case when the image $F(D)$ of a multiply connected domain D under the mapping by the function (1) is a unit disk with radial cuts, we will call the value (2) "radial" Mityuk's radius. For $D = E_q$ the function (2) has the form

$$\Omega(w) = |f'(w)|\,\Omega_r(w), \tag{15}$$

where the radius $\Omega_r(w)$ is defined in (10). In the following statement, the situation is considered for functions defined in the disk $\mathbb{D} = \{w : |w| < 1\}$ containing the ring E_q.

Theorem 8. *Let the function f be holomorphic and locally univalent in $\mathbb{D}$ and let the conformal radius $R(w) = |f'(w)|(1 - |w|^2)$ of the simply connected domain $f(\mathbb{D})$ does not increase on the intervals $\{w = r\varepsilon : 0 < r < 1\}$, $|\varepsilon| = 1$. Then the "radial" Mityuk radius (15) of the doubly connected domain $f(E_q)$ has no critical points in the ring E_q.*

Proof is based on a chain of inequalities

$$(\partial/\partial r)\ln\Omega(w) = 2\{\varepsilon f''(w)/f'(w)\} - 2\Phi_q(r)/r < (\partial/\partial r)\ln R(w) \leq 0$$

for $w = r\varepsilon \in E_q$ $(0 < r < 1, |\varepsilon| = 1)$, where the function $\Phi_q(r)$ is defined in (11). □

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