ISSN 0001-4346, Mathematical Notes, 2022, Vol. 112, No. 3, pp. 24–32. © Pleiades Publishing, Ltd., 2022.
Russian Text © The Author(s), 2022, published in Matematicheskie Zametki, 2022, Vol. 112, No. 3, pp. 350–359.

{FirstPage

Invertibility of the Operators on Hilbert Spaces and Ideals in C^*-Algebras

A. M. Bikchentaev[1*]

[1] *Kazan (Volga Region) Federal University, Kazan, 420008 Russia*

Received April 15, 2022; in final form, May 16, 2022; accepted May 20, 2022

Abstract—Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$, and let $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all linear bounded operators in $\mathcal{H}$. Sufficient conditions for the positivity and invertibility of operators from $\mathcal{B}(\mathcal{H})$ are found. An arbitrary symmetry from a von Neumann algebra $\mathcal{A}$ is written as the product $A^{-1}UA$ with a positive invertible A and a self-adjoint unitary U from $\mathcal{A}$. Let φ be the weight on a von Neumann algebra $\mathcal{A}$, let $A \in \mathcal{A}$, and let $\|A\| \le 1$. If $A^*A - I \in \mathfrak{N}_\varphi$, then $|A| - I \in \mathfrak{N}_\varphi$ and, for any isometry $U \in \mathcal{A}$, the inequality $\|A - U\|_{\varphi,2} \ge \||A| - I\|_{\varphi,2}$ holds. If U is a unitary operator from the polar expansion of the invertible operator A, then this inequality becomes an equality.

DOI: 10.1134/S0001434622090036

Keywords: *Hilbert space, linear operator, invertible operator, von Neumann algebra, C^*-algebra, weight.*

1. INTRODUCTION

{ssec1:x35

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$, and let $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all linear bounded operators in $\mathcal{H}$. Searching for sufficient conditions for the positivity and invertibility of operators from $\mathcal{B}(\mathcal{H})$ is one of the problems of operator theory; see, for example, [1]–[7] and the bibliography therein. Let us describe the results obtained.

Let $A, B \in \mathcal{B}(\mathcal{H})^+$, let B be invertible, let $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ be an operator-monotone function with $f(0) = 0$, and let $\lim_{t\to+\infty} f(t) = +\infty$. Then the operator $f(f^{-1}(A) + B) - A$ belongs to $\mathcal{B}(\mathcal{H})^+$ and is invertible (Theorem 1; the positivity of the operator B is important here). Let $A, B \in \mathcal{B}(\mathcal{H})$, and let A be left-invertible; let $(\lambda - \lambda^2)|A|^2 \ge 2|B|^2$ for some number $0 < \lambda < 1$. Then the operator $|A + B|^2 - |B|^2$ belongs to $\mathcal{B}(\mathcal{H})^+$ and is invertible (Theorem 2). For unital C^*-algebras $\mathcal{A}$ and $S \in \mathcal{A}$, the equivalence of the following conditions was established in [8, Corollary 1]:

(i) $S^2 = I$;

(ii) $S = T^{-1}UT$ for an invertible operator T and a Hermitian unitary U from $\mathcal{A}$. For a von Neumann algebra $\mathcal{A}$, the operator T can be chosen positive (Theorem 4).

The study of traces and weights on operator algebras is an important part of the work on the theory of noncommutative integration (see [9]–[11]) and constantly attracts the attention of researchers; see, for example, [12]–[15] and bibliography therein.

Let φ be the weight on a von Neumann algebra $\mathcal{A}$, $A \in \mathcal{A}$, and let $\|A\| \le 1$. If $A^*A - I \in \mathfrak{N}_\varphi$, then $|A| - I \in \mathfrak{N}_\varphi$ and, for any isometry $U \in \mathcal{A}$, the following inequality holds:

$$\|A - U\|_{\varphi,2} \ge \||A| - I\|_{\varphi,2}.$$

If the operator U is a unitary operator from the polar expansion of the invertible operator A, then this inequality becomes an equality (Theorem 5). Let φ be a finite weight on the unital C^*-algebra $\mathcal{A}$, $A \in \mathcal{A}$, and let $U \in \mathcal{A}$ be an isometry. Then

$$\|A - zU\|_{\varphi,2}^2 \ge \|A\|_{\varphi,2}^2 - \varphi(I)^{-1}|\varphi(U^*A)|^2 \qquad \text{for all} \quad z \in \mathbb{C}$$

(Theorem 6).

[*] E-mail: Airat. Bikchentaev@kpfu.ru

2. DEFINITIONS AND NOTATION

{ssec2:x35

The left (respectively, right) ideal of the algebra $\mathcal{A}$ is a vector subspace $\mathcal{J}$ in $\mathcal{A}$ such that

$$A \in \mathcal{A} \quad \text{and} \quad B \in \mathcal{J} \quad \Longrightarrow \quad AB \in \mathcal{J} \quad (\text{respectively, } BA \in \mathcal{J}).$$

By a *C^*-algebra* we mean a complex Banach $*$-algebra $\mathcal{A}$ such that

$$\|A^*A\| = \|A\|^2 \qquad \text{for all} \quad A \in \mathcal{A}.$$

For the C^*-algebra $\mathcal{A}$, we let $\mathcal{A}^{\mathrm{id}}$, $\mathcal{A}^{\mathrm{sa}}$, and $\mathcal{A}^{+}$ denote its subsets of idempotents ($A = A^2$), Hermitian ($A^* = A$) elements, and positive elements, respectively. If $A \in \mathcal{A}$, then $|A| = \sqrt{A^*A} \in \mathcal{A}^+$ and $\mathrm{Re}\{A\} = (A + A^*)/2 \in \mathcal{A}^{\mathrm{sa}}$. If I is the unit in the algebra $\mathcal{A}$, then the formula $S_P = 2P - I$ defines the bijection between $\mathcal{A}^{\mathrm{id}}$ and the set $\mathcal{A}^{\mathrm{sym}}$ of all symmetries ($S^2 = I$) in $\mathcal{A}$. By $\mathcal{A}^{\mathrm{inv}}$ and $\mathcal{A}^{\mathrm{u}}$ we denote the subsets of invertible elements and unitary ($A^*A = AA^* = I$) elements, respectively. An element $A \in \mathcal{A}$ is called an isometr if $A^*A = I$.

By a *weight* on C^*-algebra $\mathcal{A}$ we mean a mapping $\varphi\colon \mathcal{A}^+ \to [0, +\infty]$ such that

$$\varphi(X + Y) = \varphi(X) + \varphi(Y), \quad \varphi(\lambda X) = \lambda\varphi(X) \qquad \text{for all} \quad X, Y \in \mathcal{A}^+, \quad \lambda \geq 0$$

(here $0 \cdot (+\infty) \equiv 0$). A weight φ is sad to be *exact* if $\varphi(X) = 0 \Rightarrow X = 0$, $X \in \mathcal{A}^+$. For the weight φ, we define (see [16, Chap. II, II.6.7.3], [11, Chap. 2, Sec. 11])

- $\mathfrak{M}_\varphi^+ = \{X \in \mathcal{A}^+ : \varphi(X) < +\infty\}$, $\mathfrak{M}_\varphi^{\mathrm{sa}} = \mathrm{lin}_{\mathbb{R}}\, \mathfrak{M}_\varphi^+$;
- $\mathfrak{N}_\varphi = \{A \in \mathcal{A} : A^*A \in \mathfrak{M}_\varphi^+\}$ is a left ideal of $\mathcal{A}$;
- $\|A\|_{\varphi,2} = \sqrt{\varphi(A^*A)}$ ($A \in \mathfrak{N}_\varphi$) be seminorm (norm for exact φ) to $\mathfrak{N}_\varphi$.

The restriction $\varphi|_{\mathfrak{M}_\varphi^+}$ can be extended by linearity to a functional on $\mathfrak{M}_\varphi = \mathrm{lin}_{\mathbb{C}}\, \mathfrak{M}_\varphi^+$, this restriction will be denoted by the same letter φ. Such an extension allows us to identify finite weights (i.e., $\varphi(X) < +\infty$ for all $X \in \mathcal{A}^+$) with positive functionals in $\mathcal{A}$.

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$, let $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all linear bounded operators in $\mathcal{H}$, and let $\sigma(A)$ be the spectrum of the operator $A \in \mathcal{B}(\mathcal{H})$. Any C^*-algebra can be realized as a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$ (Gelfand–Naimark; see [17, Theorem 3.4.1]). A locally convex topology in $\mathcal{B}(\mathcal{H})$, defined by the semi-norms $X \mapsto \|X\xi\|$ ($\xi \in \mathcal{H}$), is called a *strong operator topology* (so-*topology*). By the *commutator* of a set $\mathcal{X} \subset \mathcal{B}(\mathcal{H})$ we mean the set

$$\mathcal{X}' = \{Y \in \mathcal{B}(\mathcal{H}) : XY = YX \text{ for all } X \in \mathcal{X}\}.$$

By a *von Neumann algebra* acting in a Hilbert space $\mathcal{H}$, we mean a $*$-subalgebra $\mathcal{A}$ of the algebra $\mathcal{B}(\mathcal{H})$, for which $\mathcal{A} = \mathcal{A}''$. For a von Neumann algebra $\mathcal{A}$, we let $\mathcal{A}^{\mathrm{pr}}$ denote its lattice of projectors ($A = A^* = A^2$). For an operator $X \in \mathcal{A}$, by $\mathrm{rp}(X)$ we will denote its *rank projector*, i.e., the projector onto the closure of the range of the operator X; we have $\mathrm{rp}(X) \in \mathcal{A}^{\mathrm{pr}}$. For $\dim \mathcal{H} = n < \infty$ the algebra $\mathcal{B}(\mathcal{H})$ is identified with the complete matrix algebra $\mathbb{M}_n(\mathbb{C})$.

Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval. A function $f\colon \mathcal{I} \to \mathbb{R}$ is said to be

- *matrix monotone of order n* or *n-monotone* if, for all $A, B \in \mathbb{M}_n(\mathbb{C})^{\mathrm{sa}}$ with $\sigma(A), \sigma(B) \subseteq \mathcal{I}$, the inequality $A \leq B$ means $f(A) \leq f(B)$;
- *operator-monotone*, If it is n-monotone for all $n \in \mathbb{N}$.

If f is 2-monotone, then $f \in C^1(\mathcal{I})$ and $f' > 0$ for $f \neq \mathrm{const}$. A function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is operator-monotone if and only if it is operator concave, i.e.,

$$f(\lambda A + (1 - \lambda)B) \geq \lambda f(A) + (1 - \lambda) f(B) \qquad \text{for all} \quad A, B \in \mathcal{B}(\mathcal{H})^+$$

and $0 \leq \lambda \leq 1$. Examples:

1) $f(t) = t^p$, $0 \leq p \leq 1$;

2) $f(t) = (t - 1)/\log(t)$, $f(0) := 0$, $f(1) := 1$; see [18, Sec. 2].

3. ON THE INVERTIBILITY OF THE OPERATORS

Lemma 1. *Let* $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ *be an operator-monotone function with* $f(0) = 0$, $\lim_{t\to+\infty} f(t) = +\infty$, *and let* $t_0 > 0$. *Then*

$$\forall\, \varepsilon > 0 \quad \exists\, \delta = \delta(\varepsilon) > 0 \quad \forall\, t \in [0, t_0] \qquad (f(t+\varepsilon) \geq f(t) + \delta).$$

Proof. By [19, Chap. VII, Theorem 4], a 2-monotone function f, $f \neq \mathrm{const}$, can be expressed in the form $f(t) = \int_0^t dx/c(x)^2$, where the function $c(x) > 0$ is also concave for all $x > 0$. Let us rewrite the inequality $f(t+\varepsilon) \geq f(t) + \delta$ in the form $\int_0^{t+\varepsilon} dx/c(x)^2 \geq \int_0^t dx/c(x)^2 + \delta$, i.e.,

$$\int_t^{t+\varepsilon} \frac{dx}{c(x)^2} \geq \delta. \tag{1}$$ {eq1:x350}

Let us prove that the function $1/c(x)^2$ decreases. Suppose that $1/c(x)^2$ is strictly increasing on an interval $(a, b) \subset \mathbb{R}^+$. Then the function

$$f(t) = \int_0^t \frac{dx}{c(x)^2}$$

will be strictly convex on (a, b), which is impossible, because every operator-monotone function on $\mathbb{R}^+$ is concave. It is now clear that the number $\delta = \varepsilon/c(t_0 + \varepsilon)^2$ satisfies inequality (1). □

{th1:x350}

Theorem 1. *Let* $A, B \in \mathcal{B}(\mathcal{H})^+$, *where* B *is invertible, and let* $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ *be an operator-monotone function with* $f(0) = 0$ *and* $\lim_{t\to+\infty} f(t) = +\infty$. *Then*

operator $f(f^{-1}(A) + B) - A \in \mathcal{B}(\mathcal{H})^+$ *and invertible.*

Proof. Since $f^{-1}(A) + B \geq f^{-1}(A)$, holds in view of the fact that the function f is operator-monotone, we have $f(f^{-1}(A) + B) - A \geq 0$. In view of the inequality $B \geq 0$, where B is invertible, there exists a number $\varepsilon > 0$ such that $B \geq \varepsilon I$. Then

$$f(f^{-1}(A) + B) \geq f(f^{-1}(A) + \varepsilon I) \tag{2}$$ {eq2:x350}

because the function f is operator-monotone. Let us prove that there exists a number $\delta = \delta(\varepsilon) > 0$ such that

$$f(f^{-1}(A) + \varepsilon I) \geq A + \delta I. \tag{3}$$ {eq3:x350}

To do this, we use the spectral theorem in the form of multipliers (for the operator A) and Lemma 1 with a sufficiently large parameter t_0, because inequality (3) is in a "commutative" setting. In this case, the operator A is regarded as a nonnegative essentially bounded measurable function on a measure space (Ω, Σ, μ), which is the direct sum of spaces with finite measures. Now, from (2), (3), we obtain $f(f^{-1}(A) + B) - A \geq \delta I$, as required. □

For the function $f(t) = \sqrt{t}$, $t \geq 0$, the choice of the number $\delta = \delta(\varepsilon) > 0$ can be made without using Lemma 1. Let us take a number $\delta > 0$ such that $\varepsilon \geq 2\delta\|A\| + \delta^2$. Then

$$\varepsilon I \geq 2\delta\|A\| I + \delta^2 I \geq 2\delta A + \delta^2 I, \qquad A^2 + \varepsilon I \geq A^2 + 2\delta A + \delta^2 I = (A + \delta I)^2;$$

thus, (3) is established, because the f function is operator-monotone.

{ex1:x350}

Example 1. The positivity of the operator B is essential in Theorem 1. In $\mathbb{M}_2(\mathbb{C})$,

$$\text{for} \quad A = \frac{1}{2}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{we have} \quad A^2 = \frac{1}{2}\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$

For an invertible matrix

$$B = \frac{1}{2}\begin{pmatrix} -4 & -1 \\ -1 & 0 \end{pmatrix},$$

the matrix

$$A^2 + B = \sqrt{A^2 + B} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is a projector. Therefore, the matrix

$$\sqrt{A^2 + B} - A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \le 0$$

is invertible.

{th2:x350]

Theorem 2. *Let $A, B \in \mathcal{B}(\mathcal{H})$, let A be left-invertible, and let*

$$(\lambda - \lambda^2)|A|^2 \ge 2|B|^2 \quad \textit{for some number} \quad 0 < \lambda < 1. \tag{4}$$

{eq4:x350]

Then the operator $|A + B|^2 - |B|^2 \in \mathcal{B}(\mathcal{H})^+$ is invertible.

Proof. The operator $A \in \mathcal{B}(\mathcal{H})$ is left-invertible if it is bounded below, i.e.,

$$\exists\, \varepsilon > 0 \quad \forall\, \xi \in \mathcal{H} \qquad (\|A\xi\| \ge \varepsilon\|\xi\|).$$

Then

$$\langle A^*A\xi, \xi\rangle = \langle A\xi, A\xi\rangle = \|A\xi\|^2 \ge \varepsilon^2\|\xi\|^2 = \varepsilon^2\langle I\xi, \xi\rangle \qquad \text{for all} \quad \xi \in \mathcal{H},$$

i.e., $A^*A \ge \varepsilon^2 I$. We have

$$\begin{aligned} |A + B|^2 - |B|^2 &= (1 - \lambda)|A|^2 + \lambda|A|^2 + A^*B + B^*A + \frac{1}{\lambda}|B|^2 - \frac{1}{\lambda}|B|^2 \\ &= (1 - \lambda)|A|^2 + \left|\sqrt{\lambda}\,A + \frac{1}{\sqrt{\lambda}}B\right|^2 - \frac{1}{\lambda}|B|^2 \\ &\ge (1 - \lambda)|A|^2 - \frac{1}{\lambda}|B|^2 \ge \frac{1 - \lambda}{2}|A|^2 \ge \frac{(1 - \lambda)\varepsilon^2}{2}I. \end{aligned}$$

□

{ex2:x350]

Example 2. Condition (4) is essential in Theorem 2. For an arbitrary number $0 < \varepsilon < 1/10$ in $\mathbb{M}_2(\mathbb{C})$, let us put

$$A = \frac{1}{2}\begin{pmatrix} 1 + \varepsilon & 1 - \varepsilon \\ 1 - \varepsilon & 1 + \varepsilon \end{pmatrix},$$

and let $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. We have

$$|A|^2 = \frac{1}{2}\begin{pmatrix} 1 + \varepsilon^2 & 1 - \varepsilon^2 \\ 1 - \varepsilon^2 & 1 + \varepsilon^2 \end{pmatrix} \not\ge \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = 2|B|^2,$$

i.e., condition (4) does not hold for all $0 < \lambda < 1$. The operator

$$|A + B|^2 - |B|^2 = (A + B)^2 - B^2 = \frac{1}{4}\begin{pmatrix} 6 + 4\varepsilon + 2\varepsilon^2 & 4 - 2\varepsilon - 2\varepsilon^2 \\ 4 - 2\varepsilon - 2\varepsilon^2 & 2 + 2\varepsilon^2 \end{pmatrix}$$

is invertible and has a negative eigenvalue.

{lem2:x350

Lemma 2. *Let $\{A_n\}_{n=1}^{\infty} \subset \mathcal{B}(\mathcal{H})^+$, and let the series $\sum_{n=1}^{\infty} A_n$* so-*converge. Let $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{R}^+$, and let*

$$0 < a = \inf_{n \ge 1} \lambda_n \le \sup_{n \ge 1} \lambda_n = b < \infty.$$

Then the series $\sum_{n=1}^{\infty} \lambda_n A_n$ so-*converges and*

$$\sum_{n=1}^{\infty} A_n \quad \textit{invertible} \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} \lambda_n A_n \quad \textit{invertible}.$$

Proof. The series $\sum_{n=1}^{\infty}\lambda_n A_n$ so - converges due to [20, Theorem 2]. Note that

1) $a\sum_{n=1}^{\infty}A_n \le \sum_{n=1}^{\infty}\lambda_n A_n \le b\sum_{n=1}^{\infty}A_n$;

2) $A\in\mathcal{B}(\mathcal{H})^+$ is invertible $\leftrightarrow \exists\,\varepsilon>0$ such that $A\ge\varepsilon I$.

$\square$

{th3:x350}

Theorem 3. *Let $A, B\in\mathcal{B}(\mathcal{H})$, and let $\lambda,\mu>0$. Then*

(i) *if the operator A^*B+B^*A is invertible then so is $\lambda A^*A+\mu AA^*$:*

(ii) *if $A\ge 0$ and AB^*+BA is invertible then so is $\lambda A+\mu BAB^*$.*

Proof. By virtue of Lemma 2, we can put $\lambda=\mu=1$. Let $X\in\mathcal{B}(\mathcal{H})^+$, $Y\in\mathcal{B}(\mathcal{H})^{\mathrm{sa}}$, and let $-X\le Y\le X$. If Y is invertible, then so is X [21, Corollary 2].

(i) Since $(A\pm B)^*(A\pm B)\ge 0$, we have

$$-A^*A-B^*B\le A^*B+B^*A\le A^*A+B^*B.$$

(ii) Since $(\sqrt{A}\pm B\sqrt{A})(\sqrt{A}\pm B\sqrt{A})^*\ge 0$, we have

$$-A-BAB^*\le AB^*+BA\le A+BAB^*.$$

$\square$

4. SYMMETRIES AND IDEALS IN VON NEUMANN ALGEBRAS

{ssec4:x35

For the unital C^*-algebras $\mathcal{A}$ and $S\in\mathcal{A}$, the equivalence of the following conditions was established in [8, Corollary 1]:

(i) $S^2=I$, i.e., $S\in\mathcal{A}^{\mathrm{sym}}$;

(ii) $S=T^{-1}UT$ for some $T\in\mathcal{A}^{\mathrm{inv}}$ and $U\in\mathcal{A}^{\mathrm{u}}\cap\mathcal{A}^{\mathrm{sa}}$.

Let us prove that, for a von Neumann algebra $\mathcal{A}$, the operator T can be chosen to be positive.

{th4:x350}

Theorem 4. *Let $\mathcal{A}$ be a von Neumann algebra, and let $S\in\mathcal{A}$. Then the following conditions are equivalent*:

(i) *$S^2=I$;*

(ii) *$S=A^{-1}UA$ for some $A\in\mathcal{A}^+\cap\mathcal{A}^{\mathrm{inv}}$ and $U\in\mathcal{A}^{\mathrm{u}}\cap\mathcal{A}^{\mathrm{sa}}$.*

Proof. (i) $\Rightarrow$ (ii) Formula $S_P=2P-I$ defines the bijection between the sets $\mathcal{A}^{\mathrm{id}}$ and $\mathcal{A}^{\mathrm{sym}}$. If $P\in\mathcal{A}^{\mathrm{id}}$, then there exists a $T\in\mathcal{A}^{\mathrm{inv}}$ such that

$$Q\equiv T^{-1}PT\in\mathcal{A}^{\mathrm{pr}}$$

[22, Lemma 16]. If $T^{-1}=V|T^{-1}|$ be polar expansion of the operator T^{-1}, then $V=T^{-1}|T^{-1}|^{-1}\in\mathcal{A}^{\mathrm{u}}$ and $T=|T^{-1}|^{-1}V^*$ by virtue of the theorem about the converse of the product of operators. Now

$$P=TQT^{-1}=|T^{-1}|^{-1}V^*QV|T^{-1}|=A^{-1}RA \qquad \text{c} \quad A=|T^{-1}|\in\mathcal{A}^+\cap\mathcal{A}^{\mathrm{inv}}$$

and $R=V^*QV\in\mathcal{A}^{\mathrm{pr}}$. Therefore,

$$S=2P-I=2A^{-1}RA-I=A^{-1}(2R-I)A$$

and we can put $U=2R-I$. $\square$

{lem3:x350

Lemma 3. *Let $\mathcal{J}$ be a left (or right) ideal in a unital C^*-algebra $\mathcal{A}$, and let $A \in \mathcal{A}^+$, $I - A \in \mathcal{J}$. Then*

(i) *$I - A^{1/2^n} \in \mathcal{J}$ for all $n \in \mathbb{N}$;*

(ii) *if $\mathcal{A}$ is a von Neumann algebra and $A \leq I$, then $I - \mathrm{rp}(A) \in \mathcal{J}$; for* so*-closed $\mathcal{J}$, the condition $A \leq I$ can be omitted.*

Proof. (i) Let $\mathcal{J}$ be a left ideal in a unital C^*-algebra $\mathcal{A}$. Since

$$(I + \sqrt{A})(I - \sqrt{A}) = I - A \in \mathcal{J}$$

and the element $I + \sqrt{A}$ is invertible, we have $I - \sqrt{A} \in \mathcal{J}$. Since

$$(I + A^{1/4})(I - A^{1/4}) = I - \sqrt{A} \in \mathcal{J}$$

and the element $I + A^{1/4}$ is invertible, we have $I - A^{1/4} \in \mathcal{J}$. Continuing this process, we obtain the required result.

(ii) Suppose that $\mathcal{A}$ is a von Neumann algebra, and $A \leq I$. Since $\mathrm{rp}(A) \geq A$, it follows that $I - A \geq I - \mathrm{rp}(A) \geq 0$ and $\mathrm{rp}(A)A = A\,\mathrm{rp}(A) = A$. If $X, Y \in \mathcal{A}^+$ and $X \leq Y$, then there exists a $Z \in \mathcal{A}$ with $\|Z\| \leq 1$ such that $\sqrt{X} = Z\sqrt{Y}$ [23, Chap. 1, Sec. 1, Lemma 2]. Therefore,

$$\sqrt{I - \mathrm{rp}(A)} = I - \mathrm{rp}(A) = Z\sqrt{I - A}$$

for some $Z \in \mathcal{A}$ with $\|Z\| \leq 1$. By virtue of the spectral theorem in the form of multipliers, we have

$$\begin{aligned} I - \mathrm{rp}(A) &= \sqrt{I - \mathrm{rp}(A)} = \sqrt{(I - \mathrm{rp}(A))(I - A)} = (I - \mathrm{rp}(A))\sqrt{I - A} \\ &= Z\sqrt{I - A} \cdot \sqrt{I - A} = Z(I - A) \in \mathcal{J}. \end{aligned}$$

Now let the left ideal $\mathcal{J}$ be so-closed. Since $\mathcal{J}$ is a convex subset in $\mathcal{A}$, by [9, Chap. II, Sec. 2, item (iv) of Theorem 2.6], it follows that $\mathcal{J}$ is σ-weakly closed. Therefore, by virtue of [9, Chap. II, Sec. 3, Proposition 3.12] $\mathcal{J}$ contains a single projector E such that $\mathcal{J} = \mathcal{A}E$. Let $X \in \mathcal{A}$ for which $I - A = XE$. Then

$$I - \mathrm{rp}(A) = I - A - \mathrm{rp}(A)(I - A) = (I - \mathrm{rp}(A))(I - A) = (I - \mathrm{rp}(A))XE \in \mathcal{J}$$

and the lemma is proved. □

{th5:x350}

Theorem 5. *Let φ be the weight on a von Neumann algebra $\mathcal{A}$, $A \in \mathcal{A}$, and let $\|A\| \leq 1$. If $A^*A - I \in \mathfrak{N}_\varphi$, then $|A| - I \in \mathfrak{N}_\varphi$ and, for any isometry $U \in \mathcal{A}$, the following inequality holds*:

$$\|A - U\|_{\varphi,2} \geq \||A| - I\|_{\varphi,2}. \tag{5}$$ {eq5:x350}

If the operator U is a unitary operator from the polar expansion of the invertible operator A, then the equality is achieved in (5).

Proof. Since $|A| = \sqrt{A^*A}$ and $\mathfrak{N}_\varphi$ is a left ideal of $\mathcal{A}$, we have $|A| - I \in \mathfrak{N}_\varphi$ by virtue of item (i) of Lemma 3. To prove inequality (5) without loss of generality, it suffices to consider the case of an arbitrary isometry $U \in \mathcal{A}$, for which $A - U \in \mathfrak{N}_\varphi$. Let $B = |A| - I$; then the polar representation of the operator A can be expressed as $A = W(I + B)$, where W is a unitary operator. Let $V = W^*U$; then V is an isometry and

$$\|A - U\|_{\varphi,2}^2 = \|I + B - V\|_{\varphi,2}^2 = \varphi((B + I - V)^*(B + I - V)).$$

Therefore,

$$\|A - U\|_{\varphi,2}^2 = \varphi(B^2) + \varphi(D), \tag{6}$$ {eq6:x350}

where

$$D = 2I + 2B - V - V^* - BV - V^*B \in \mathfrak{M}_\varphi^{\mathrm{sa}}.$$

Let us prove that $D \geq 0$. Since $\|A\| \leq 1$, we have $|A|^2 \leq |A|$. Then

$$\begin{aligned} D &= 2I + |A| - I - V - V^* - |A|V^* + V^* - V|A| + V \\ &= I + |A| - |A|V - V^*|A| = (|A| - V)^*(|A| - V) + (|A| - |A|^2) \geq 0, \end{aligned}$$

as the sum of two nonnegative operators, and (5) holds due to (6) and the fact that the function $f(t) = \sqrt{t}\,$, $t \geq 0$, is monotone.

Let U be a unitary operator from the polar expansion of the operator A, i.e., $U = A|A|^{-1}$. Then

$$\begin{aligned} \|A - U\|^2_{\varphi,2} &= \varphi((A - A|A|^{-1})^*(A - A|A|^{-1})) \\ &= \varphi(|A|^2 - |A|^{-1}|A|^2 - |A|^2|A|^{-1} + |A|^{-1}|A|^2|A|^{-1}) \\ &= \varphi(|A|^2 - 2|A| + I) = \varphi((|A| - I)^2) = \||A| - I\|^2_{\varphi,2}, \end{aligned}$$

i.e., in (5), equality is achieved. □

For $\mathcal{A} = \mathcal{B}(\mathcal{H})$, $\varphi = \mathrm{tr}$, and the unitary operator U, the assertion of Theorem 5 was obtained in [24, Chap. VI, Lemma 3.1] and it was shown that, in this particular case, the equal sign in (5) is realized if and only if U is a unitary operator from the polar expansion of an invertible operator A.

{lem4:x350

Lemma 4. *For the numbers $a, c > 0$ and $b \in \mathbb{C}$, let us define a function $f\colon \mathbb{C} \to \mathbb{R}$, assuming that $f(z) = c|z|^2 - 2\Re\{z\overline{b}\} + a$ for all $z \in \mathbb{C}$. Then*

$$\min_{z\in\mathbb{C}} f(z) = f\left(\frac{b}{c}\right) = a - \frac{|b|^2}{c}\,.$$

Proof. For all $z \in \mathbb{C}$, we have

$$f(z) = \left(\sqrt{c}\,z - \frac{b}{\sqrt{c}}\right)\left(\sqrt{c}\,\overline{z} - \frac{\overline{b}}{\sqrt{c}}\right) - \frac{|b|^2}{c} + a = \left|\sqrt{c}\,z - \frac{b}{\sqrt{c}}\right|^2 - \frac{|b|^2}{c} + a.$$

□

{th6:x350}

Theorem 6. *Let φ be the weight on a unital C^*-algebra $\mathcal{A}$, $A \in \mathcal{A}$, and let $U \in \mathcal{A}$ be an isometry. Then*

(i) *$A \in \mathfrak{N}_\varphi \Leftrightarrow UA \in \mathfrak{N}_\varphi$ and $\|UA\|_{\varphi,2} = \|A\|_{\varphi,2}$;*

(ii) *if φ is finite, then $\|A - zU\|^2_{\varphi,2} \geq \|A\|^2_{\varphi,2} - \varphi(I)^{-1}|\varphi(U^*A)|^2$ for all $z \in \mathbb{C}$.*

Proof. (i) We have

$$\|UA\|^2_{\varphi,2} = \varphi(A^*U^*UA) = \varphi(A^*A) = \|A\|^2_{\varphi,2}.$$

(ii) The extension of a finite weight φ to the whole algebra $\mathcal{A}$ will be denoted by the same letter φ. Since $\varphi(X^*) = \overline{\varphi(X)}$ and $\varphi(\mathrm{Re}\{X\}) = \Re\{\varphi(X)\}$ for all $X \in \mathcal{A}$, it follows that, for all $z \in \mathbb{C}$,

$$\begin{aligned} \|A - zU\|^2_{\varphi,2} &= \varphi((A^* - \overline{z}U^*)(A - zU)) = \varphi(A^*A) - \varphi(2\,\mathrm{Re}\{\overline{z}U^*A\}) + |z|^2\varphi(I) \\ &= \|A\|^2_{\varphi,2} - 2\Re\{\overline{z}\varphi(U^*A)\} + |z|^2\varphi(I). \end{aligned}$$

The minimum of this expression (for fixed A and U) is attained at the point $z = \varphi(I)^{-1}\overline{\varphi(U^*A)}$, and it is equal to $\|A\|^2_{\varphi,2} - \varphi(I)^{-1}|\varphi(U^*A)|^2$; see Lemma 4. □

From Theorem 5 and Theorem 6 with $z = 1$, we obtain the following statement.

{cor1:x350

Corollary 1. *Let φ be finite weight on a von Neumann algebra $\mathcal{A}$, $A \in \mathcal{A}$ with $\|A\| \leq 1$, and let $U \in \mathcal{A}$ be an isometry. Then*

$$\|A - U\|^2_{\varphi,2} \geq \max\{\||A| - I\|^2_{\varphi,2},\ \|A\|^2_{\varphi,2} - \varphi(I)^{-1}|\varphi(U^*A)|^2\}.$$

{ex3:x350}

Example 3. Suppose that the positive functional $\varphi\colon \mathbb{M}_2(\mathbb{C}) \to \mathbb{C}$ is given by the density matrix $S_\varphi = \operatorname{diag}(t+s, t-s)$ with fixed $t > 0$ and $0 \le s \le t$. Then

$$\varphi(X) = \operatorname{tr}(S_\varphi X) = (t+s)x_{11} + (t-s)x_{22} \qquad \text{for all} \quad X = [x_{ij}]_{i,j=1}^2 \in \mathbb{M}_2(\mathbb{C}).$$

Let us put $A := \operatorname{diag}(1,0)$, $U := \operatorname{diag}(1,-1)$. Then

$$\|A - U\|_{\varphi,2}^2 = \||A| - I\|_{\varphi,2}^2 = t - s$$

and, in the inequality of Theorem 5, the equality is achieved for all $0 \le s \le t$. We have $\varphi(I) = 2t$ and

$$\|A\|_{\varphi,2}^2 = \varphi(U^*A) = \varphi(A) = t + s,$$

because A is a projector. Inequality in item (ii) of Theorem 6 becomes

$$t - s \ge t + s - \frac{(t+s)^2}{2t} = \frac{t}{2} - \frac{s^2}{2t}$$

and, for $s = t$, this inequality becomes an equality. Thus, for $s = t$, the inequality of Corollary 1 also becomes an equality.

FUNDING

This work was carried out within the framework of the Development Program of the Scientific and Educational Mathematical Center of the Volga Federal Region (agreement no. 075-02-2022-882).

REFERENCES

1. O. G. Avsyankin, "Invertibility of multidimensional integral operators with bihomogeneous kernels," Math. Notes **108** (2), 277–281 (2020).
2. A. M. Bikchentaev, "Существенно обратимые измеримые операторы, присоединенные к полуконечной алгебре фон Неймана, и коммутаторы," Sib. Mat. Zh. **63** (2), 272–282 (2022).
3. A. M. Bikchentaev, "On invertibility of some operator sums," Lobachevskii J. Math. **33** (3), 216–222 (2012).
4. A. M. Bikchentaev, "On τ-essentially invertibility of τ-measurable operators," Int. J. Theor. Phys. **60** (2), 567–575 (2021).
5. M. Kostadinov, "Injectivity of linear combinations in $\mathcal{B}(\mathcal{H})$," Electron. J. Linear Algebra **37**, 359–369 (2021).
6. M. H. Mortad, "On the invertibility of the sum of operators," Anal. Math. **46** (1), 133–145 (2020).
7. X. Ding, Sh. Wu, and X. Zhao, "Invertibility and spectral properties of dual Toeplitz operators," J. Math. Anal. Appl. **484** (2), Article ID 123762 (2020).
8. A. M. Bikchentaev and Kh. Fawwaz, "Differences and commutators of idempotents in C^*-algebras," Izv. Vyssh. Uchebn. Zaved. Mat., No. 8, 16–26 (2021).
9. M. Takesaki, *Theory of the Operator Algebras. I*, in *The Operator Algebras and Non-Commutative Geometry, 5* (Springer-Verlag, Berlin, 2002).
10. M. Takesaki, *Theory of the Operator Algebras. II*, in *The Operator Algebras and Non-Commutative Geometry, 6* (Springer-Verlag, Berlin, 2003).
11. A. N. Sherstnev, *Methods of Bilinear Forms in the Noncommutative Theory of the Measure and Integral* (Fizmatlit, Moscow, 2008) [in Russian].
12. L. Zsidó, "On the equality of operator valued weights," J. Funct. Anal. **282** (9), Article ID 109416 (2022).
13. A. Bikchentaev, "Trace inequalities for Rickart C^*-algebras," The positivity **25** (5), 1943–1957 (2021).
14. A. M. Bikchentaev, "Seminorms associated with subadditive weights on C^*-algebras," Math. Notes **107** (3), 383–391 (2020).
15. K. Thomsen, "The factor type of dissipative KMS weights on graph C^*-algebras," J. Noncommut. Geom. **14** (3), 1107–1128 (2020).
16. B. Blackadar, *The Operator Algebras. Theory of C^*-Algebras and von Neumann Algebras*, in *The Operator Algebras and Non-Commutative Geometry, III* (Springer-Verlag, Berlin, 2006).
17. Gerard J. Murphy, *C^*-Algebras and Operator Theory* (Academic Press, Boston, MA, 1990).
18. F. Hiai, "Matrix analysis: matrix monotone functions, matrix means, and majorization," Interdiscip. Inform. Sci. **16** (2), 139–248 (2010).
19. W. F. Donoghue (Jr.), *Monotone Matrix Functions and Analytic Continuation*, in *Die Grundlehren Math. Wiss.* (Springer, New York, 1974), Vol. 207.

20. A. M. Bikchentaev, “Convergence in measure and τ-compactness of τ-measurable operators, affiliated with a semifinite von Neumann algebra,” Russian Math. (Iz. VUZ) **64** (5), 79–82 (2020).
21. A. M. Bikchentaev, “On a lemma of Berezin,” Math. Notes **87** (5), 768–773 (2010).
22. I. Kaplansky, “Modules over operator algebras,” Amer. J. Math. **75** (4), 839–858 (1953).
23. J. Dixmier, *Les algebres d'opérateurs dans l'espace Hilbertien (algebres de von Neumann)* (Gauthier-Villars, Paris, 1969).
24. I. Ts. Gokhberg and M. G. Krein, *Introduction to the Theory of Linear Non-Self-Adjoint Operators in Hilbert Space* (Nauka, Moscow, 1965) [in Russian].