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Convergence in Measure and τ-Compactness of τ-Measurable Operators, Affiliated with a Semifinite von Neumann Algebra

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Abstract—Let τ be a faithful normal semifinite trace on a von Neumann algebra. We establish the Leibniz criterion for sign-alternating series of τ-measurable operators and present an analogue of the criterion of series "sandwich" series for τ-measurable operators. We prove a refinement of this criterion for the τ-compact case. In terms of measure convergence topology, the criterion of τ-compactness of an arbitrary τ-measurable operator is established. We also give a sufficient condition of 1) τ-compactness of the commutator of a τ-measurable operator and a projection; 2) convergence of τ-measurable operator and projection commutator sequences to the zero operator in the measure τ.

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Introduction. Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$. Measure topology t_τ on the $*$-algebra $S(\mathcal{M},\tau)$ of all τ-measurable operators plays an important part in the I. Segal's noncommutative integration theory [1], [2], [3]. We give characterizations of various classes of von Neumann algebras in terms of the topologies t_τ and $t_{\tau l}$ [4]; and describe operator "intervals" $I_B = \{A:\ -B \le A \le B\}$, where $A \in S(\mathcal{M},\tau)^{\mathrm{sa}}$ and $B \in S(\mathcal{M},\tau)^+$ ([5]). In [6], [7] in connection with the topology t_τ, convex sets $K_B = \{A \in S(\mathcal{M},\tau):\ A^*A \le B\}$ were investigated.

An operator $A \in S(\mathcal{M},\tau)$ is said to be *τ-essentially left invertible* if there exists an operator $B \in S(\mathcal{M},\tau)$ such that the operator $I - AB$ is τ-compact.

A sufficient condition for an operator $A \in S(\mathcal{M},\tau)$ not to be τ-essentially left invertible is existence of a t_τ-bounded sequence $\{X_n\}_{n=1}^\infty \subset S(\mathcal{M},\tau)$ such that

$$X_n^* \xrightarrow{\tau l} 0, \quad X_n \not\xrightarrow{\tau} 0, \quad AX_n \xrightarrow{\tau} 0 \quad (n \to \infty),$$

([8], Theorem 3.4). On τ-compactness of τ-measurable operators see [9], [10].

1. Notation and definitions. Let $\mathcal{M}$ be a von Neumann algebra of operators in a Hilbert space $\mathcal{H}$, let $\mathcal{M}^{\mathrm{pr}}$ be the lattice of projections in $\mathcal{M}$. Let I be the unit of the algebra $\mathcal{M}$, let $P^\perp = I - P$ for $P \in \mathcal{M}^{\mathrm{pr}}$, and let $\mathcal{M}^+$ be the cone of positive elements from $\mathcal{M}$. A mapping $\varphi:\mathcal{M}^+ \to [0,+\infty]$ is called a *trace* if $\varphi(X+Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$, $\lambda \ge 0$ (here $0 \cdot (+\infty) \equiv 0$) and $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$. A trace φ is said to be *faithful* if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \ne 0$; *normal* if $X_i \nearrow X$ $(X_i, X \in \mathcal{M}^+) \Rightarrow \varphi(X) = \sup\varphi(X_i)$; *semifinite* if $\varphi(X) = \sup\{\varphi(Y):\ Y \in \mathcal{M}^+,\ Y \le X,\ \varphi(Y) < +\infty\}$ for each $X \in \mathcal{M}^+$.

An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be *affiliated to the von Neumann algebra* $\mathcal{M}$ if it commutes with any unitary operator from the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. Let τ be a faithful normal semifinite trace on $\mathcal{M}$. A closed operator X, affiliated to $\mathcal{M}$ and possessing a domain $\mathcal{D}(X)$ everywhere dense in $\mathcal{H}$ is said to be *τ-measurable*, if, for any $\varepsilon > 0$, there exists an operator $P \in \mathcal{M}^{\mathrm{pr}}$ such that $P\mathcal{H} \subset \mathcal{D}(X)$ and $\tau(P^\perp) < \varepsilon$. The set $S(\mathcal{M},\tau)$ of all τ-measurable operators is a $*$-algebra under passage to the adjoint operator, multiplication

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by a scalar, and operations of strong addition and multiplication resulting from the closure of the ordinary operations [1], [3]. For a family $\mathcal{L} \subset S(\mathcal{M}, \tau)$, let $\mathcal{L}^+$ and $\mathcal{L}^{\mathrm{sa}}$ denote its positive and Hermitian parts, respectively. The partial order in $S(\mathcal{M}, \tau)^{\mathrm{sa}}$ generated by the cone $S(\mathcal{M}, \tau)^+$ will be denoted by $\leq$. If $X \in S(\mathcal{M}, \tau)$ then $|X| = \sqrt{X^*X} \in S(\mathcal{M}, \tau)^+$.

Let $\mu(X)$ stand for the *rearrangement* of the operator $X \in S(\mathcal{M}, \tau)$, i.e., nonincreasing right-continuous function $\mu(X)\colon (0, \infty) \to [0, \infty)$, given by the formula

$$\mu_t(X) = \inf\{\|XP\| : \quad P \in \mathcal{M}^{\mathrm{pr}}, \quad \tau(P^\perp) \leq t\}, \quad t > 0.$$

The set of τ-compact operators $S_0(\mathcal{M}, \tau) = \{X \in S(\mathcal{M}, \tau) : \lim\limits_{t \to \infty} \mu_t(X) = 0\}$ is an ideal in $S(\mathcal{M}, \tau)$.

The sets $U(\varepsilon, \delta) = \{X \in S(\mathcal{M}, \tau) : \quad (\|XP\| \leq \varepsilon \text{ and } \tau(P^\perp) \leq \delta \text{ for some } P \in \mathcal{M}^{\mathrm{pr}})\}$, where $\varepsilon > 0$, $\delta > 0$, form a base at 0 for metrizable vector topology t_τ on $S(\mathcal{M}, \tau)$, called *the measure topology* [3].

It is well known that $\langle S(\mathcal{M}, \tau), t_\tau \rangle$ is a complete topological $*$-algebra in which $\mathcal{M}$ is dense. For $X_n, X \in S(\mathcal{M}, \tau)$ we write $X_n \xrightarrow{\tau} X$ if a sequence $\{X_n\}_{n=1}^\infty$ is t_τ-converges to X. By definition, a sequence $\{X_n\}_{n=1}^\infty$ *converges τ-locally to* $X \in S(\mathcal{M}, \tau)$ (notation: $X_n \xrightarrow{\tau l} X$) if $X_n P \xrightarrow{\tau} XP$ for all $P \in \mathcal{M}^{\mathrm{pr}}$ with $\tau(P) < +\infty$ ([7], p. 114). For properties of such convergence, see [4], [11]–[13].

Let m be the linear Lebesgue measure on $\mathbb{R}$. The noncommutative Lebesgue L_1-space associated with $(\mathcal{M}, \tau)$ can be defined as $L_1(\mathcal{M}, \tau) = \{X \in S(\mathcal{M}, \tau) : \mu(X) \in L_1(\mathbb{R}^+, m)\}$ with norm $\|X\|_1 = \|\mu(X)\|_1$, $X \in L_1(\mathcal{M}, \tau)$.

Lemma 1 ([14])**.** *Let $X, Y \in S(\mathcal{M}, \tau)$. Then*

1) $\mu_t(X) = \mu_t(|X|) = \mu_t(X^*)$ *for all* $t > 0$;
2) *if* $|X| \leq |Y|$ *then* $\mu_t(X) \leq \mu_t(Y)$ *for all* $t > 0$;
3) $\mu_{s+t}(X + Y) \leq \mu_s(X) + \mu_t(Y)$ *for all* $s, t > 0$;
4) $\mu_{s+t}(XY) \leq \mu_s(X)\mu_t(Y)$ *for all* $s, t > 0$;
5) $\mu_t(AXB) \leq \|A\|\|B\|\mu_t(X)$ *for all* $A, B \in \mathcal{M}$ *and* $t > 0$;
6) $\mu_t(f(|X|)) = f(\mu_t(X))$ *for all continuous increasing functions* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *and* $t > 0$.

Lemma 2. *Let $A_j, A \in S(\mathcal{M}, \tau)$, $j \in J$. Then*

1) $A_j \xrightarrow{\tau} A \Leftrightarrow \mu_t(A_j - A) \xrightarrow{j} 0$ *for every* $t > 0$;
2) *a net* $\{A_j\}_{j \in J}$ *is t_τ-bounded* $\Leftrightarrow \sup_{j \in J} \mu_t(A_j) < +\infty$ *for every* $t > 0$.

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ is the $*$-algebra of all bounded linear operators on $\mathcal{H}$ and $\tau = \mathrm{tr}$ is the canonical trace, then $S(\mathcal{M}, \tau)$ coincides with $\mathcal{B}(\mathcal{H})$, and $S_0(\mathcal{M}, \tau)$ coincides with the ideal $\mathfrak{S}(\mathcal{H})$ of compact operators on $\mathcal{H}$, τ-local convergence coincides with strong-operator (*so*-topology) convergence. We have

$$\mu_t(X) = \sum_{n=1}^{+\infty} s_n(X) \chi_{[n-1, n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^\infty$ is the sequence of s-numbers of the operator X ([15], p. 46), χ_A is the indicator of the set $A \subset \mathbb{R}$. Then the space $L_1(\mathcal{M}, \tau)$ is a trace-class ideal $\mathfrak{S}_1(\mathcal{H})$.

2. Main results. We present operator analogs of the classical statements for functional series.

Theorem 1 (the Leibniz criterion for $S(\mathcal{M}, \tau)$). *Let $A_n \in S(\mathcal{M}, \tau)^+$ and $A_1 \geq A_2 \geq \ldots \geq \geq A_n \geq \ldots$, $A_n \xrightarrow{\tau l} 0$ as $n \to \infty$. Then the sign-alternating series $\sum_{n=1}^\infty (-1)^{n+1} A_n$ $t_{\tau l}$-converges (t_τ-converges for $A_1 \in S_0(\mathcal{M}, \tau)$) and its sum does not exceed A_1.*

Theorem 2. *Let $A_n \in S(\mathcal{M}, \tau)^+$ be such that the series $\sum_{n=1}^\infty A_n$ $t_{\tau l}$-converges. Let $B_n \in S(\mathcal{M}, \tau)^{\mathrm{sa}}$ be such that $-A_n \leq B_n \leq A_n$ for all $n \in \mathbb{N}$. Then the series $\sum_{n=1}^\infty B_n$ also $t_{\tau l}$-converges.*

Theorem 3. *Let $A_n \in S_0(\mathcal{M}, \tau)^+$ be such that the series $\sum_{n=1}^\infty A_n$ t_τ-converges. Let $B_n \in S(\mathcal{M}, \tau)^{\mathrm{sa}}$ be such that $-A_n \leq B_n \leq A_n$ for all $n \in \mathbb{N}$. Then the series $\sum_{n=1}^\infty B_n$ also t_τ-converges in $S_0(\mathcal{M}, \tau)^+$.*

Lemma 3. *Consider $A_j, B_j \in S(\mathcal{M}, \tau)$, $j \in J$, and let a net $\{A_j\}_{j\in J}$ be t_τ-bounded, $B_j \xrightarrow{\tau l} 0$. Then $A_j B_j \xrightarrow{\tau l} 0$.*

It is shown in Lemma 3.3 of [12] that if $\{A_j\}_{j\in J} \subset S(\mathcal{M}, \tau)^+$ and $A_j \xrightarrow{\tau l} 0$ then $A_j^q \xrightarrow{\tau l} 0$ for every $0 < q < 1$.

Proposition 1. *For a t_τ-bounded net $\{A_j\}_{j\in J} \subset S(\mathcal{M}, \tau)^+$ the following conditions are equivalent:*

(i) $A_j \xrightarrow{\tau l} 0$,

(ii) $A_j^q \xrightarrow{\tau l} 0$ *for every* $q > 0$.

Theorem 4. *Let $A_n \in \mathcal{M}^+$ and $A = \sum_{n=1}^{\infty} A_n \in \mathcal{M}^+$, the series so-converges. Then for every $q > 0$ we have $A_n^q \xrightarrow{\tau l} 0$ as $n \to \infty$.*

Sketch of proof. By positive homogeneity and normality of the trace τ for $P \in \mathcal{M}^{\rm pr}$ with $\tau(P) < +\infty$ we have

$$+\infty > \|A\|\tau(P) \geq \tau(PAP) = \sum_{n=1}^{\infty} \tau(PA_nP).$$

Since a number series converges, its common term tends to zero:

$$\tau(PA_nP) = \|PA_nP\|_1 \to 0 \ \text{ as } \ n \to \infty.$$

Since the space $\langle L_1(\mathcal{M}, \tau), \|\cdot\|_1\rangle$ is continuously embedded to the topological algebra $\langle S(\mathcal{M}, \tau), t_\tau\rangle$, we have $PA_nP \xrightarrow{\tau} 0$ as $n \to \infty$. For every $t > 0$ by item 1) of Lemma 2 and items 1), 6) of Lemma 1 we obtain

$$\mu_t(A_n^{1/2}P)^2 = \mu_t(PA_n^{1/2} \cdot A_n^{1/2}P) = \mu_t(PA_nP) \to 0 \ \text{ as } \ n \to \infty.$$

Therefore, $A_n^{1/2}P \xrightarrow{\tau} 0$ as $n \to \infty$ via item 1) of Lemma 2. Thus, $A_n^{1/2} \xrightarrow{\tau l} 0$ as $n \to \infty$ and we apply Lemma 3.

Corollary 1. Let $\{P_n\}_{n=1}^{\infty} \subset \mathcal{M}^{\rm pr}$ and $P_nP_m = 0$ for $n \neq m$, $n, m \in \mathbb{N}$. Then $P_n \xrightarrow{\tau l} 0$ as $n \to \infty$.

Theorem 5. *For an operator $A \in S(\mathcal{M}, \tau)$ the following conditions are equivalent:*

(i) $A \in S_0(\mathcal{M}, \tau)$,

(ii) $X_nA \xrightarrow{\tau} 0$ $(n \to \infty)$ *for all t_τ-bounded sequences $\{X_n\}_{n=1}^{\infty} \subset S(\mathcal{M}, \tau)$ such that* $X_n \xrightarrow{\tau l} 0$ $(n \to \infty)$.

Sketch of proof. (ii)⇒(i) If an operator A is not τ-compact then $a := \lim_{t\to\infty} \mu_t(A) > 0$. Since the trace τ is semifinite, there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of pairwise orthogonal projections in $\mathcal{M}$ and a number $b > 0$ such that $0 < b \leq \tau(P_n) < +\infty$ and $aP_n \leq A$ for all $n \in \mathbb{N}$ (see [5], Proposition 4.1). We multiply both sides of the relation $aP_n \leq A$ by the projection P_n from the left and the right and obtain $aP_n \leq P_nAP_n$ for all $n \in \mathbb{N}$. Then

$$\mu_t(P_nA) \geq \mu_t(P_nAP_n) \geq \mu_t(aP_n) = a\mu_t(P_n) \geq a\chi_{(0,b)}(t) \ \text{ for all } \ t > 0$$

by items 2), 5) of Lemma 1. Thus $P_n \xrightarrow{\tau l} 0$ $(n \to \infty)$ by Corollary 1, but the sequence $\{P_nA\}_{n=1}^{\infty}$ is not t_τ-convergent by item 1) of Lemma 2.

Let $T, P \in S(\mathcal{M}, \tau)$ and $P = P^2$. If the commutator $[T, P]$ belongs to $S_0(\mathcal{M}, \tau)$ then $TP - PTP = [T, P]P \in S_0(\mathcal{M}, \tau)$.

Theorem 6. *Let $T \in S(\mathcal{M}, \tau)$ and $P \in \mathcal{M}^{\rm pr}$ be such that $PTT^*P \leq TPT^*$.*

(i) *If $TP - PTP \in S_0(\mathcal{M}, \tau)$ then $[T, P] \in S_0(\mathcal{M}, \tau)$.*

(ii) *If $TP = PTP$ then $[T, P] = 0$.*

Sketch of proof. (i) Since $X = TP - PTP \in S_0(\mathcal{M}, \tau)$, we have $X^* = PT^* - PT^*P \in S_0(\mathcal{M}, \tau)$ and

$$|[T, P]^*|^2 = (TP - PT)(PT^* - T^*P) = TPT^* - T \cdot PT^*P - PTP \cdot T^* + PTT^*P \leq$$

$$\leq TPT^* - T(PT^* - X^*) - (TP - X)T^* + TPT^* = TX^* + XT^* \in S_0(\mathcal{M}, \tau)^+.$$

Then by items 1), 2), 6) of Lemma 1 we obtain

$$0 \leq \mu_t([T, P]) = \mu_t(|[T, P]^*|^2)^{1/2} \leq \mu_t(TX^* + XT^*)^{1/2} \to 0 \text{ as } t \to \infty.$$

Thus, $[T, P] \in S_0(\mathcal{M}, \tau)$.

Note that if a hyponormal operator $T \in S(\mathcal{M}, \tau)$ and $P \in \mathcal{M}^{\mathrm{pr}}$ are such that the operator TP is cohyponormal then $PTT^*P \leq TPT^*$.

Proposition 2. *Let $P, T_n \in S(\mathcal{M}, \tau)$ and $P^2 = P$.*

(i) *If $[T_n, P] \xrightarrow{\tau} 0$ as $n \to \infty$ then $T_nP - PT_nP \xrightarrow{\tau} 0$ as $n \to \infty$.*

(ii) *If $[T_n, P] \xrightarrow{\tau l} 0$ as $n \to \infty$ then $T_nP - PT_nP \xrightarrow{\tau l} 0$ as $n \to \infty$.*

Theorem 7. *Let a t_τ-bounded sequence $\{T_n\}_{n=1}^\infty \subset S(\mathcal{M}, \tau)$ and $P \in \mathcal{M}^{\mathrm{pr}}$ be such that $PT_nT_n^*P \leq T_nPT_n^*$ for all $n \in \mathbb{N}$. If $T_nP - PT_nP \xrightarrow{\tau} 0$ as $n \to \infty$ then $[T_n, P] \xrightarrow{\tau} 0$ as $n \to \infty$.*

Theorem 8. *If an operator $A \in L_1(\mathcal{M}, \tau)$ anticommutes with some invertible operator $T \in \mathcal{M}$ then $\tau(A) = 0$.*

Corollary 2. If $\tau(I) = 1$ then in conditions of Theorem 8 we have $\|I + zA\|_1 \geq 1$ for all $z \in \mathbb{C}$.

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