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Convergence of Integrable Operators Affiliated to a Finite von Neumann Algebra

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Abstract—In the Banach space $L_1(\mathcal{M},\tau)$ of operators integrable with respect to a tracial state τ on a von Neumann algebra $\mathcal{M}$, convergence is analyzed. A notion of dispersion of operators in $L_2(\mathcal{M},\tau)$ is introduced, and its main properties are established. A convergence criterion in $L_2(\mathcal{M},\tau)$ in terms of the dispersion is proposed. It is shown that the following conditions for $X\in L_1(\mathcal{M},\tau)$ are equivalent: (i) $\tau(X)=0$, and (ii) $\|I+zX\|_1\ge 1$ for all $z\in\mathbb{C}$. A.R. Padmanabhan's result (1979) on a property of the norm of the space $L_1(\mathcal{M},\tau)$ is complemented. The convergence in $L_2(\mathcal{M},\tau)$ of the imaginary components of some bounded sequences of operators from $\mathcal{M}$ is established. Corollaries on the convergence of dispersions are obtained.

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1. INTRODUCTION

Let τ be a faithful normal tracial state on a von Neumann algebra $\mathcal{M}$, $\mathcal{M}^{\mathrm{pr}}$ be the lattice of projectors in $\mathcal{M}$, and I be the identity operator in $\mathcal{M}$. We investigate the convergence in the Banach space $L_1(\mathcal{M},\tau)$ of τ-integrable operators [1, 2]. We introduce the dispersion $\mathbb{D}(X)=\|X-\tau(X)I\|_2^2$ of operators $X\in L_2(\mathcal{M},\tau)$ and establish its main properties (Theorem 4.1 and Corollary 4.2). We show that $\inf_{a\in\mathbb{C}}\|X-aI\|_2^2=\mathbb{D}(X)$ for all $X\in L_2(\mathcal{M},\tau)$ (Theorem 4.4). We propose a convergence criterion for sequences of operators in $L_2(\mathcal{M},\tau)$ in terms of the dispersion (Theorem 4.5). Let $\mathcal{K}_0=\{X\in L_2(\mathcal{M},\tau)\colon \tau(X)=0\}$. For $X_n, X\in\mathcal{K}_0$ $(n\in\mathbb{N})$, we prove the equivalence of the following conditions (Corollary 4.6):

(i) $X_n \xrightarrow{\|\cdot\|_2} X$ as $n\to\infty$, and

(ii) $X_n \xrightarrow{\tau} X$ and $\mathbb{D}(X_n)\to\mathbb{D}(X)$ as $n\to\infty$.

In Theorem 4.8, we show that the following conditions for $X\in L_1(\mathcal{M},\tau)$ are equivalent:

(i) $\tau(X)=0$, and

(ii) $\|I+zX\|_1\ge 1$ for all $z\in\mathbb{C}$.

We complement Padmanabhan's result from [3] on a property of the norm of the space $L_1(\mathcal{M},\tau)$: if an operator $A\in L_1(\mathcal{M},\tau)^+$ is nonsingular, then

$$\forall\varepsilon>0\quad \exists\delta>0\quad \forall P\in\mathcal{M}^{\mathrm{pr}}\qquad \big(\tau(P)\ge\varepsilon\ \Rightarrow\ \|PAP\|_1\ge\delta\big)$$

(Theorem 4.9). We establish the convergence in $L_2(\mathcal{M},\tau)$ of the imaginary components of some bounded sequences of operators in $\mathcal{M}$ (Theorem 4.13) and apply the result to the convergence of dispersions (Corollaries 4.7 and 4.14).

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