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TRACE AND COMMUTATORS OF MEASURABLE OPERATORS AFFILIATED TO A VON NEUMANN ALGEBRA

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Abstract. In this paper, we present new properties of the space $L_1(\mathcal{M},\tau)$ of integrable (with respect to the trace τ) operators affiliated to a semifinite von Neumann algebra $\mathcal{M}$. For self-adjoint τ-measurable operators A and B, we find sufficient conditions of the τ-integrability of the operator $\lambda I - AB$ and the real-valuedness of the trace $\tau(\lambda I - AB)$, where $\lambda \in \mathbb{R}$. Under these conditions, $[A, B] = AB - BA \in L_1(\mathcal{M},\tau)$ and $\tau([A, B]) = 0$. For τ-measurable operators A and $B = B^2$, we find conditions that are sufficient for the validity of the relation $\tau([A, B]) = 0$. For an isometry $U \in \mathcal{M}$ and a nonnegative τ-measurable operator A, we prove that $U - A \in L_1(\mathcal{M},\tau)$ if and only if $I - A, I - U \in L_1(\mathcal{M},\tau)$. For a τ-measurable operator A, we present estimates of the trace of the autocommutator $[A^*, A]$. Let self-adjoint τ-measurable operators $X \geq 0$ and Y be such that $[X^{1/2}, YX^{1/2}] \in L_1(\mathcal{M},\tau)$. Then $\tau([X^{1/2}, YX^{1/2}]) = it$, where $t \in \mathbb{R}$ and $t = 0$ for $XY \in L_1(\mathcal{M},\tau)$.

***Keywords and phrases*:** Hilbert space, linear operator, von Neumann algebra, normal semifinite trace, measurable operator, integrable operator, commutator, autocommutator.

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CONTENTS

1. Introduction

Let a von Neumann algebra $\mathcal{M}$ of operators act in a Hilbert space $\mathcal{H}$ and τ be an exact, normal, semifinite trace on $\mathcal{M}$. We state new properties of the space $L_1(\mathcal{M},\tau)$ of integrable operators affiliated to the algebra $\mathcal{M}$. For an operator $X \in L_1(\mathcal{M},\tau)$, we examine conditions under which $\tau(X) \in \mathbb{R}$ or $\tau(X) = 0$. For self-adjoint τ-measurable operators A and B, we find sufficient conditions of the integrability of the operator $\lambda I - AB$ and the real-valuedness of the trace $\tau(\lambda I - AB)$, where $\lambda \in \mathbb{R}$. Under these conditions, the commutator $[A, B] = AB - BA$ belongs to $L_1(\mathcal{M},\tau)$ and $\tau([A, B]) = 0$ (see Theorems 4.1 and 4.2 and Propositions 4.1–4.4). For τ-measurable operators A and $B = B^2$, we find conditions sufficient for the validity of the relation $\tau([A, B]) = 0$ (Theorem 4.3). Item (ii) of Theorem 4.3 is a generalization of [6, Theorem 2.26].

For an isometry $U \in \mathcal{M}$ and a nonnegative τ-measurable operator A, we prove that $U - A \in L_1(\mathcal{M},\tau)$ if and only if $I - A, I - U \in L_1(\mathcal{M},\tau)$ (Theorem 4.5). For a τ-measurable operator A, we find estimates of the trace of autocommutator $[A^*, A]$ (Corollary 4.4 and Theorem 4.7).

Let self-adjoint, τ-measurable operators $X \geq 0$ and Y be such that $[X^{1/2}, YX^{1/2}] \in L_1(\mathcal{M},\tau)$. Then

$$\tau([X^{1/2}, YX^{1/2}]) = it,$$

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where $t \in \mathbb{R}$ and $t = 0$ for $XY \in L_1(\mathcal{M}, \tau)$ (Theorem 4.8). Our results are new for the $*$-algebra $\mathcal{M} = \mathcal{B}(\mathcal{H})$ of all bounded linear operators in $\mathcal{H}$ equipped with the canonical trace $\tau = \mathrm{tr}$.

2. Notation and Definitions

Let $\mathcal{M}$ be a von Neumann algebra of operators in a Hilbert space $\mathcal{H}$, $\mathcal{M}^{\mathrm{pr}}$ be the lattice of projectors in $\mathcal{M}$, I be the identity operator in $\mathcal{M}$, $P^{\perp} = I - P$ for $P \in \mathcal{M}^{\mathrm{pr}}$, and $\mathcal{M}^{+}$ be the cone of positive elements of $\mathcal{M}$.

A mapping $\varphi : \mathcal{M}^{+} \to [0, +\infty]$ is called a *trace* if

$$\varphi(X + Y) = \varphi(X) + \varphi(Y), \quad \varphi(\lambda X) = \lambda\varphi(X)$$

for any $X, Y \in \mathcal{M}^{+}$ and $\lambda \geq 0$ (moreover, $0 \cdot (+\infty) \equiv 0$) and $\varphi(Z^{*}Z) = \varphi(ZZ^{*})$ for all $Z \in \mathcal{M}$. A trace φ is said to be

(i) *exact* if $\varphi(X) > 0$ for all $X \in \mathcal{M}^{+}$, $X \neq 0$;
(ii) *semifinite* if $\varphi(X) = \sup\{\varphi(Y) : Y \in \mathcal{M}^{+},\ Y \leq X,\ \varphi(Y) < +\infty\}$ for all $X \in \mathcal{M}^{+}$;
(iii) *normal* if for $X_i \nearrow X$ $(X_i, X \in \mathcal{M}^{+})$ we have $\varphi(X) = \sup \varphi(X_i)$.

For a trace φ, we set

$$\mathfrak{M}_{\varphi}^{+} = \Big\{X \in \mathcal{M}^{+} : \varphi(X) < +\infty\Big\}, \quad \mathfrak{M}_{\varphi} = \mathrm{lin}_{\mathbb{C}}\, \mathfrak{M}_{\varphi}^{+}.$$

The restriction $\varphi|_{\mathfrak{M}_{\varphi}^{+}}$ can be continuously extended by linearity to a functional on $\mathfrak{M}_{\varphi}$, which will be denoted by the same symbol φ.

An operator in $\mathcal{H}$ (not necessarily bounded or densely definite) is said to be *affiliated to a von Neumann algebra* $\mathcal{M}$ if it commutes with an arbitrary unitary operator from the commutator subalgebra $\mathcal{M}'$ of the algebra $\mathcal{M}$. In the sequel, we denote by τ an exact, normal, semifinite trace on $\mathcal{M}$. A closed operator X affiliated to $\mathcal{M}$ whose domain $\mathcal{D}(X)$ is everywhere dense in $\mathcal{H}$ is said to be *τ-measurable* if for any $\varepsilon > 0$, there exists $P \in \mathcal{M}^{\mathrm{pr}}$ such that $P\mathcal{H} \subset \mathcal{D}(X)$ and $\tau(P^{\perp}) < \varepsilon$. The set $\widetilde{\mathcal{M}}$ of all τ-measurable operators is a $*$-algebra with respect to passing to adjoint operators, multiplication by scalars, and the operations of strong addition and multiplication obtained by the closure of ordinary operations (see [22, 23]). For a family $\mathcal{L} \subset \widetilde{\mathcal{M}}$, we denote by $\mathcal{L}^{+}$ and $\mathcal{L}^{\mathrm{sa}}$ its positive and Hermitian parts, respectively. The partial order in $\widetilde{\mathcal{M}}^{\mathrm{sa}}$ generated by the proper cone $\widetilde{\mathcal{M}}^{+}$ is denoted by $\leq$. Let $i \in \mathbb{C}$, $i^2 = -1$, and $X \in \widetilde{\mathcal{M}}$. For $\mathrm{Re}\, X = (X + X^{*})/2$ and $\mathrm{Im}\, X = (X - X^{*})/(2i)$, we have $X = \mathrm{Re}\, X + i\, \mathrm{Im}\, X$ and $\mathrm{Re}\, X, \mathrm{Im}\, X \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$.

If X is a closed, densely defined linear operator affiliated to $\mathcal{M}$ and $|X| = (X^{*}X)^{1/2}$, then the spectral decomposition $P^{|X|}(\cdot)$ is contained in $\mathcal{M}$ and $X \in \widetilde{\mathcal{M}}$ if and only if there exists $\lambda \in \mathbb{R}$ such that

$$\tau(P^{|X|}((\lambda, +\infty))) < +\infty.$$

If $X \in \widetilde{\mathcal{M}}$ and $X = U|X|$ is the polar decomposition of X, then $U \in \mathcal{M}$ and $|X| \in \widetilde{\mathcal{M}}^{+}$. Moreover, if

$$|X| = \int_{0}^{\infty} \lambda P^{|X|}(d\lambda)$$

is the spectral decomposition, then

$$\tau(P^{|X|}((\lambda, +\infty))) \to 0 \quad \text{as } \lambda \to +\infty.$$

We denote by $\mu_t(X)$ a *permutation* of an operator $X \in \widetilde{\mathcal{M}}$ (see [15, 27]), i.e., a nonincreasing right-continuous function $\mu(X) : (0, \infty) \to [0, \infty)$ defined by the formula

$$\mu_t(X) = \inf\Big\{\|XP\| : P \in \mathcal{M}^{\mathrm{pr}},\ \tau(P^{\perp}) \leq t\Big\}, \quad t > 0.$$

Let m be a linear Lebesgue measure on $\mathbb{R}$. The noncommutative Lebesgue L_p-space ($0 < p < \infty$) associated with $(\mathcal{M}, \tau)$ can be defined as follows:

$$L_p(\mathcal{M}, \tau) = \left\{ X \in \widetilde{\mathcal{M}} : \mu(X) \in L_p(\mathbb{R}^+, m) \right\}$$

with the F-norm (or the norm for $1 \leq p < \infty$) $\|X\|_p = \|\mu(X)\|_p$, $X \in L_p(\mathcal{M}, \tau)$. The restriction $\tau|_{\mathfrak{M}_\tau^+}$ can be extended to a linear bounded functional on $L_1(\mathcal{M}, \tau)$, which will be denoted by the same symbol τ. We have

$$\mathfrak{M}_\tau = \mathcal{M} \cap L_1(\mathcal{M}, \tau), \quad \|X\|_p = \tau(|X|^p)^{1/p}, \quad 0 < p < \infty.$$

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ is the $*$-algebra of all bounded linear operators in $\mathcal{H}$ and $\tau = \mathrm{tr}$ is the canonical trace, then $\widetilde{\mathcal{M}}$ coincides with $\mathcal{B}(\mathcal{H})$. We have

$$\mu_t(X) = \sum_{n=1}^{\infty} s_n(X) \chi_{[n-1,n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^{\infty}$ is sequence of s-numbers of the operator X and χ_A is the indicator of a set $A \subset \mathbb{R}$ (see [17]). Then the space $L_p(\mathcal{M}, \tau)$ is a Schatten–von Neumann ideal $\mathfrak{S}_p$, $0 < p < \infty$.

3. Lemmas and Examples

Let τ be an exact, normal, semifinite trace on a von Neumann algebra $\mathcal{M}$.

Lemma 3.1 (see [11, Theorem 17])**.** *If $X, Y \in \widetilde{\mathcal{M}}$ and $XY, YX \in L_1(\mathcal{M}, \tau)$, then $\tau(XY) = \tau(YX)$.*

Lemma 3.2 (see [1, Theorem 3] and [2, Theorem 1])**.** *If $X, Y \in \widetilde{\mathcal{M}}^+$ and $XY \in L_1(\mathcal{M}, \tau)$, then $X^{1/2} Y X^{1/2} \in L_1(\mathcal{M}, \tau)$ and $\tau(XY) = \tau(X^{1/2} Y X^{1/2})$.*

Lemma 3.3 (see [3, Theorem 3.1])**.** *If $X, Y \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$ and $XY \in L_1(\mathcal{M}, \tau)$, then $YX \in L_1(\mathcal{M}, \tau)$ and $\tau(XY) = \tau(YX) \in \mathbb{R}$.*

Lemma 3.4 (see [3, Theorem 2.3])**.** *If $X \in L_1(\mathcal{M}, \tau)$, then $\tau(X^*) = \overline{\tau(X)}$.*

Here and below, the bar $\bar{\ }$ means conplex conjugation.

Lemma 3.5 (see [5, Theorem 4.8])**.** *If $\tau(I) = 1$, then for $X \in L_1(\mathcal{M}, \tau)$, the following conditions are equivalent*:

(i) $\tau(X) = 0$;

(ii) $\|I + zX\|_1 \geq 1$ *for all* $z \in \mathbb{C}$.

In particular, if $\tau(I) = 1$ and $A, B \in \mathcal{M}$, then $\|I + z[A, B]\|_1 \geq 1$ for all $z \in \mathbb{C}$. For a type-II_1 factor of the algebra $\mathcal{M}$, commutators of τ-measurable operators were examined in [13]; the problem on the representability of an arbitrary τ-measurable operator X possessing the property $\tau(X) = 0$ as the commutator $X = [A, B]$ was studied in [14].

Lemma 3.6. *Let operators $A, B, D \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$ be such that $T = D - AB \in L_1(\mathcal{M}, \tau)$. Then $[A, B] \in L_1(\mathcal{M}, \tau)$, and if $\tau(T) \in \mathbb{R}$, then $\tau([A, B]) = 0$.*

Proof. Since

$$[A, B] = T^* - T \in L_1(\mathcal{M}, \tau), \tag{1}$$

due to Lemma 3.4 for $\tau(T) \in \mathbb{R}$ we have

$$\tau([A, B]) = \tau(T^* - T) = \tau(T^*) - \tau(T) = \overline{\tau(T)} - \tau(T) = 0. \tag{2}$$

The lemma is proved. □

Lemma 3.7. *For $X \in L_1(\mathcal{M}, \tau)$, the following conditions are equivalent*:

(i) $\tau(X) \in \mathbb{R}$;
(ii) $\tau(\operatorname{Im} X) = 0$.

Lemmas 3.5 and 3.7 imply that if $\tau(I) = 1$ and $X \in L_1(\mathcal{M}, \tau)$, then the condition $\tau(X) \in \mathbb{R}$ is equivalent to the validity of the inequality $\|I + z \operatorname{Im} X\|_1 \geq 1$ for all $z \in \mathbb{C}$.

Example 3.1. Let $\mathcal{M} = \mathbb{M}_n(\mathbb{C})$ and $\tau = \operatorname{tr}$ be a trace on $\mathcal{M}$. The following Jacobi formula is well known:

$$\det e^X = e^{\tau(X)}, \quad X \in \mathcal{M}.$$

In particular, if $\det e^X = 1$, then $\tau(X) = 0$. For $X \in \mathcal{M}$, the following conditions are equivalent:
(i) X is unitary equivalent to a matrix with zero diagonal;
(ii) $\tau(X) = 0$;
(iii) X is a commutator.

A proof of (i)⇔(ii) can be found in [16, Chap. II, problem 209]; the assertion (ii)⇔(iii) is proved in [18, problem 182]. Therefore, each matrix $A \in \mathbb{M}_n(\mathbb{C})$ is unitary equivalent to a matrix with "constant" diagonal and can be represented as the sum $A = \lambda I + X$, where $\tau(X) = 0$ and $\lambda = \operatorname{tr}(A)/n$.

Example 3.2 (see [7, Example 1])**.** Let $0 < p, q < \infty$ and $a_n = 2^{n+1} n^{-q}$, $n \in \mathbb{N}$. We endow the von Neumann algebra $\mathcal{M} = \bigoplus_{n=1}^{\infty} \mathbb{M}_2(\mathbb{C})$ with an exact normal finite trace $\tau = \bigoplus_{n=1}^{\infty} 2^{-n} \operatorname{tr}_2$ and set

$$A = \bigoplus_{n=1}^{\infty} \begin{pmatrix} 1 & a_n \\ 0 & 0 \end{pmatrix}.$$

We have $A = A^2$ and $A \in L_p(\mathcal{M}, \tau)$ for $pq > 1$ and $A \notin L_p(\mathcal{M}, \tau)$ for $pq \leq 1$.

4. Basic Results

Let τ be an exact, normal, semifinite trace on a von Neumann algebra $\mathcal{M}$.

Theorem 4.1. *Let $A, B \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$, $\lambda \in \mathbb{R}$, $n \in \mathbb{N}$.*
(i) If $T = \lambda A^n - AB \in L_1(\mathcal{M}, \tau)$, then $\tau(T) \in \mathbb{R}$.
(ii) If $T = \lambda I - AB \in L_1(\mathcal{M}, \tau)$ and $A = \sum_{k=1}^{n} a_k P_k$, where $a_k \in \mathbb{R}$ and $P_k \in \mathcal{M}^{\mathrm{pr}}$, $P_k P_j = 0$ for $k \neq j$ for all $k, j = 1, \ldots, n$, then $\tau(T) \in \mathbb{R}$.
In both cases $[A, B] \in L_1(\mathcal{M}, \tau)$ and $\tau([A, B]) = 0$.

Proof. (i) Since

$$T = \begin{cases} A(\lambda I - B) & \text{for } n = 1, \\ A^{n-1}(\lambda A - B) & \text{for } n \geq 2, \end{cases}$$

we have $\tau(T) \in \mathbb{R}$ due to Lemma 3.3.
(ii) For each $k \in \{1, \ldots, n\}$ we have

$$T_k = P_k T = \lambda P_k - a_k P_k B = P_k(\lambda I - a_k B) \in L_1(\mathcal{M}, \tau)$$

and $\tau(T_k) \in \mathbb{R}$ due to Lemma 3.3. For the projector $P = (P_1 + \cdots + P_n)^{\perp}$ we have

$$PT = \lambda P \in L_1(\mathcal{M}, \tau)^{\mathrm{sa}}, \quad \tau(PT) \in \mathbb{R}.$$

Therefore,

$$\tau(T) = \tau(PT) + \sum_{k=1}^{n} \tau(P_k T) \in \mathbb{R}.$$

In both cases, we can apply Lemma 3.6. The theorem is proved. □

Theorem 4.2. *Let operators* $A, B \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$ *and numbers* $\lambda \in \mathbb{R}$ *be such that*

$$T = \lambda I - AB \in L_1(\mathcal{M}, \tau).$$

If A *is invertible in* $\widetilde{\mathcal{M}}$ *or* $I - B \in L_1(\mathcal{M}, \tau)$, *then* $\tau(T) \in \mathbb{R}$. *In both cases,*

$$[A, B] \in L_1(\mathcal{M}, \tau), \quad \tau([A, B]) = 0.$$

Proof. For an invertible operator A, we have

$$T = A(\lambda A^{-1} - B), \quad \lambda A^{-1} - B \in \widetilde{\mathcal{M}}^{\mathrm{sa}};$$

therefore, $\tau(T) \in \mathbb{R}$ due to Lemma 3.3.

Now let $I - B \in L_1(\mathcal{M}, \tau)$. Since

$$T = (\lambda I - A)B + \lambda(I - B),$$

we have

$$(\lambda I - A)B \in L_1(\mathcal{M}, \tau)$$

and due to Lemma 3.3 we obtain

$$\tau((\lambda I - A)B), \ \tau(I - B) \in \mathbb{R}.$$

Therefore, $\tau(T) \in \mathbb{R}$. In both cases we can apply Lemma 3.6. The theorem is proved. □

Proposition 4.1. *Let operators* $A, B \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$ *and numbers* $a_1, a_2, b_1, b_2 \in \mathbb{R}$ *be such that*

$$\lambda = a_1 b_2 + a_2 b_1 \neq 0, \quad T = (a_1 A + b_1 B)(a_2 A - b_2 B) \in L_1(\mathcal{M}, \tau).$$

Then

$$[A, B] \in L_1(\mathcal{M}, \tau), \quad \tau([A, B]) = 0.$$

If $\tau(I) = 1$, *then* $\|I + z[A, B]\|_1 \geq 1$ *for all* $z \in \mathbb{C}$.

Proof. We have $\tau(T) \in \mathbb{R}$ due to Lemma 3.3. Since $T^* \in L_1(\mathcal{M}, \tau)$ and $T^* - T = \lambda[A, B]$, we have $[A, B] \in L_1(\mathcal{M}, \tau)$. Then due to Lemma 3.4 we have

$$\lambda\tau([A, B]) = \tau(T^* - T) = \tau(T^*) - \tau(T) = \overline{\tau(T)} - \tau(T) = 0.$$

For $\tau(I) = 1$ we apply Lemma 3.5. The assertion is proved. □

Proposition 4.2. *Let operators* $X, Y, Z \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$ *and numbers* $n \in \mathbb{N}$, $\lambda \in \mathbb{R}$ *be such that*

$$XY + YZ, XY - \lambda Y^n \in L_1(\mathcal{M}, \tau).$$

Then

$$\tau(XY + YZ) \in \mathbb{R}, \quad \tau([X - Z, Y]) = 0.$$

If $\tau(I) = 1$, *then* $\|I + z[X - Z, Y]\|_1 \geq 1$ *for all* $z \in \mathbb{C}$.

Proof. Obviously, $\lambda Y^n + YZ \in L_1(\mathcal{M}, \tau)$. Due to Lemma 3.3 we have

$$\tau(XY + YZ) = \tau((XY - \lambda Y^n) + (\lambda Y^n + YZ)) = \tau((X - \lambda Y^{n-1})Y) + \tau(Y(\lambda Y^{n-1} + Z)) \in \mathbb{R}.$$

Therefore, by Lemma 3.4 we have

$$\tau([X - Z, Y]) = \tau(XY + YZ - (XY + YZ)^*) = \tau(XY + YZ) - \overline{\tau((XY + YZ)} = 0.$$

For $\tau(I) = 1$ we apply Lemma 3.5. The proposition is proved. □

Proposition 4.3. *Let operators $A \in \widetilde{\mathcal{M}}$, $B \in \mathcal{M}$ and a number $n \in \mathbb{N}$ be such that*

$$A - B^n \in L_1(\mathcal{M}, \tau).$$

Then

$$[A, B] \in L_1(\mathcal{M}, \tau), \quad \tau([A, B]) = 0.$$

If $\tau(I) = 1$, then $\|I + z[A, B]\|_1 \geq 1$ for all $z \in \mathbb{C}$.

Proof. We set $X = A - B^n$ and $Y = B$. Then

$$XY, YX \in L_1(\mathcal{M}, \tau), \quad [A, B] = [X, Y].$$

Now due to Lemma 3.1 we have

$$\tau([A, B]) = \tau([X, Y]) = \tau(XY) - \tau(YX) = 0.$$

For $\tau(I) = 1$ we apply Lemma 3.5. The proposition is proved. □

Proposition 4.4. *Let numbers $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$ and operators $A \in \mathcal{M}$, $B \in \widetilde{\mathcal{M}}$ be such that*

$$\lambda_1 I - A, \lambda_2 I - B \in L_1(\mathcal{M}, \tau).$$

Then

$$\lambda_1 \lambda_2 I - AB, [A, B] \in L_1(\mathcal{M}, \tau), \quad \tau([A, B]) = 0.$$

Proof. The operator

$$\lambda_1 \lambda_2 I - AB = \lambda_1 \lambda_2 ((I - \lambda_1^{-1} A) + \lambda_1^{-1} A(I - \lambda_2^{-1} B)) \tag{3}$$

belongs to $L_1(\mathcal{M}, \tau)$. The operators $(\lambda_1 I - A)(\lambda_2 I - B)$ and $(\lambda_2 I - B)(\lambda_1 I - A)$ belong to $L_1(\mathcal{M}, \tau)$; therefore

$$[A, B] = [\lambda_1 I - A, \lambda_2 I - B] \in L_1(\mathcal{M}, \tau)$$

and $\tau([A, B]) = \tau([\lambda_1 I - A, \lambda_2 I - B]) = 0$ due to Lemma 3.1 with $X = \lambda_1 I - A$ and $Y = \lambda_2 I - B$. □

Corollary 4.1. *Let the conditions of Proposition 4.4 be fulfilled and let $\lambda_1, \lambda_2 \in \mathbb{R}$ and $A, B \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$. Then $\tau(\lambda_1 \lambda_2 I - AB) \in \mathbb{R}$.*

This assertion follows from (3) and Lemma 3.3.

Theorem 4.3. *Let $A, B \in \widetilde{\mathcal{M}}$, $B = B^2$, and $[AB, B] \in L_1(\mathcal{M}, \tau)$.*

(i) *The relation $\tau([AB, B]) = 0$ holds.*
(ii) *If $[A, B] \in L_1(\mathcal{M}, \tau)$, then $\tau([A, B]) = 0$.*

Proof. (i) We set

$$X = [AB, B] = AB - BAB, \quad Y = B.$$

Then the operators $XY = X$ and $YX = 0$ belong to $L_1(\mathcal{M}, \tau)$ and due to Lemma 3.1 we have

$$\tau(X) = \tau(XY) = \tau(YX) = \tau(0) = 0.$$

(ii) Since $BA - BAB = AB - BAB - [A, B] \in L_1(\mathcal{M}, \tau)$, the conditions of item (i) are fulfilled for the adjoint operators A^* and B^*:

$$\tau(BA - BAB) = \overline{\tau(A^* B^* - B^* A^* B^*)} = \overline{0} = 0$$

(see Lemma 3.3). Further,

$$\tau([A, B]) = \tau(AB - BAB - (BA - BAB)) = \tau(AB - BAB) - \tau(BA - BAB) = 0 - 0 = 0.$$

The theorem is proved. □

Note that Theorem 4.3(ii) is a generalization of [6, Theorem 2.26]. From Theorem 4.3 and Lemma 3.5 we obtain the following assertion.

Corollary 4.2. *Under the conditions of Theorem 4.3, let $\tau(I) = 1$. Then*

(i) $\|I + z[AB, B]\|_1 \geq 1$ *for all* $z \in \mathbb{C}$;

(ii) *if* $[A, B] \in L_1(\mathcal{M}, \tau)$, *then* $\|I + z[A, B]\|_1 \geq 1$ *for all* $z \in \mathbb{C}$.

Proposition 4.5. *Let $A, B \in \widetilde{\mathcal{M}}$ and $B = B^2$. If $[B, BA] \in L_1(\mathcal{M}, \tau)$, then $\tau([B, BA]) = 0$. Moreover, if $\tau(I) = 1$, then $\|I + z[B, BA]\|_1 \geq 1$ for all $z \in \mathbb{C}$.*

Proof. We set $X = [B, BA]$ and $Y = B$. Then the operators $XY(= 0)$ and $YX(= X)$ belong to $L_1(\mathcal{M}, \tau)$, and due to Lemma 3.1 we have

$$\tau(X) = \tau(YX) = \tau(XY) = \tau(0) = 0.$$

For $\tau(I) = 1$, we apply Lemma 3.5. The proposition is proved. □

Theorem 4.4. *Let $A \in \widetilde{\mathcal{M}}$,*

$$B = \sum_{k=1}^{n} b_k P_k, \quad b_k \in \mathbb{C}, \quad P_k = P_k^2 \in \mathcal{M}, \quad b_k \neq b_j, \quad P_k P_j = 0 \text{ for } k \neq j \text{ and all } k, j = 1, \ldots, n.$$

If $[A, B] \in L_1(\mathcal{M}, \tau)$, then $\tau([A, B]) = 0$.

Proof. Since

$$[A, B] = \sum_{k=1}^{n} b_k (AP_k - P_k A) \in L_1(\mathcal{M}, \tau), \tag{4}$$

for all $k, j = 1, \ldots, n$, $k \neq j$, we have

$$P_j[A, B] = P_j AB - b_j P_j A \in L_1(\mathcal{M}, \tau), \tag{5}$$

and also $P_j[A, B]P_k = (b_k - b_j)P_j A P_k \in L_1(\mathcal{M}, \tau)$; therefore, $P_j A P_k \in L_1(\mathcal{M}, \tau)$. Now from (5) we obtain

$$P_j A P_j - P_j A \in L_1(\mathcal{M}, \tau) \quad \text{for all} \quad j = 1, \ldots, n. \tag{6}$$

Considering the operators $[A, B]P_j$ instead of (5), we similarly obtain

$$P_j A P_j - A P_j \in L_1(\mathcal{M}, \tau) \quad \text{for all} \quad j = 1, \ldots, n.$$

This and (6) imply that $[A, P_j] \in L_1(\mathcal{M}, \tau)$ for all $j = 1, \ldots, n$. Due to [6, Theorem 2.26] we obtain $\tau([A, P_j]) = 0$ for all $j = 1, \ldots, n$ and from (4) we obtain $\tau([A, B]) = 0$. □

Theorem 4.5. *For an isometry $U \in \mathcal{M}$ and an operator $A \in \widetilde{\mathcal{M}}^+$, the following conditions are equivalent*:

(i) $U - A \in L_1(\mathcal{M}, \tau)$;

(ii) $I - A,\ I - U \in L_1(\mathcal{M}, \tau)$.

Proof. (i)⇒(ii) Let

$$A = \int_0^\infty \lambda P^A(d\lambda)$$

be the spectral decomposition of the operator $A \in \widetilde{\mathcal{M}}^+$. We represent A as the sum

$$A = AP^A([0; 1]) + AP^A((1; \infty)) \equiv A_1 + A_2.$$

Then

$$A_1 \in \mathcal{M}, \quad A_2 = (U - A_1) - (U - A) \in L_1(\mathcal{M}, \tau) + \mathcal{M}.$$

Therefore, there exists a number $k \in \mathbb{N}$ such that $\tau P^{A_2}((k; \infty)) < \infty$. Note that

$$P^{A_2}((n; \infty)) = P^A((n; \infty)) \quad \forall n \in \mathbb{N}.$$

Thus, the operator $B_2 = P^{A_2}((k;\infty))$ belongs to the class $L_1(\mathcal{M},\tau)^+$. For $B_1 = A - B_2 \in \mathcal{M}^+$, we have $U - B_1 \in \mathfrak{M}_\tau$ and the operator $I + B_1$ are invertible in $\mathcal{M}$. Due to [10, Theorem 2], the operators $I - B_1$ and $I - U$ lie in $\mathfrak{M}_\tau$. Therefore,

$$I - A = I - B_1 - B_2 \in L_1(\mathcal{M},\tau).$$

(ii)⇒(i) We have $U - A = I - A - (I - U) \in L_1(\mathcal{M},\tau)$. □

Corollary 4.3. *Under the conditions of Theorem 4.5, we have*

(i) $[U, A] \in L_1(\mathcal{M},\tau)$;
(ii) $\tau(U - A) \in \mathbb{R}$ *if and only if* $\tau(I - U) \in \mathbb{R}$;
(iii) *if, in addition,* $U = U^*$, *then* $\tau([U, A]) = 0$.

Proof. (i) We have

$$[U, A] = (I - A)U - U(I - A) \in L_1(\mathcal{M},\tau).$$

(iii) Due to Lemma 3.3, we obtain $\tau((I - A)U) \in \mathbb{R}$ and hence

$$\tau([U, A]) = \tau((I - A)U) - \tau(U(I - A)) = \tau((I - A)U) - \tau(((I - A)U)^*)$$
$$= \tau((I - A)U) - \overline{\tau((I - A)U)} = 0.$$

For $\tau(I) = 1$, due to Lemma 3.5, we have $\|I + z[U, A]\|_1 \geq 1$ for all $z \in \mathbb{C}$. □

Proposition 4.6. *If* $U \in \mathcal{M}$ *is a unitary operator and* $A \in \widetilde{\mathcal{M}}$, *then* $|[U, A]| = |A - U^*AU|$.

Proof. We have

$$|[U, A]|^2 = A^*A - A^*U^*AU - U^*A^*UA + U^*A^*AU = |A - U^*AU|^2,$$

and the assertion follows from the uniqueness of the square root of a nonnegative τ-measurable operator. □

Theorem 4.6. *Let operators* $A, B \in \mathcal{M}$ *be such that* $I - A, I - B \in \mathfrak{M}_\tau$. *Then* $[A, B] \in \mathfrak{M}_\tau$ *and*

$$|\tau([A, B])| \leq (1 + \|B\|)\|I - A\|_1 + (1 + \|A\|)\|I - B\|_1.$$

Proof. Recall that

$$|\tau(XY)| \leq \|X\|\tau(|Y|) \quad \text{for all} \quad X \in \mathcal{M}, \quad Y \in \mathfrak{M}_\tau \tag{7}$$

(see [26, Chap. V, Sec. 2, formula (2)]). We have

$$I - AB = A(I - B) + I - A \in \mathfrak{M}_\tau$$

and due to the triangle inequality for $\mathbb{C}$ and (7), we obtain

$$\begin{aligned}|\tau([A, B])| &= |\tau(I - BA - (I - AB))| \leq |\tau(I - BA)| + |\tau(I - AB)| \\ &= |\tau(B(I - A) + I - B)| + |\tau(A(I - B) + I - A)| \\ &\leq |\tau(B(I - A))| + |\tau(I - B)| + |\tau(A(I - B))| + |\tau(I - A)| \\ &\leq (1 + \|B\|)\|I - A\|_1 + (1 + \|A\|)\|I - B\|_1.\end{aligned}$$

The theorem is proved. □

Corollary 4.4. *Let an operator* $A \in \mathcal{M}$ *be such that* $I - A \in \mathfrak{M}_\tau$. *Then*

$$[A^*, A] \in \mathfrak{M}_\tau, \quad |\tau([A^*, A])| \leq 2(1 + \|A\|)\|I - A\|_1.$$

Theorem 4.7. *Let* $A \in \widetilde{\mathcal{M}}$, $0 < p, q, r \leq \infty$, *and* $1/p + 1/q = 1/r$. *If*

$$\operatorname{Re} A \in L_p(\mathcal{M},\tau), \quad \operatorname{Im} A \in L_q(\mathcal{M},\tau),$$

then

$$[A^*, A] \in L_r(\mathcal{M},\tau), \quad \|[A^*, A]\|_r \leq 2^{\max\{1+1/r,2\}}\|\operatorname{Re} A\|_p \|\operatorname{Im} A\|_q.$$

Proof. We set $\|\cdot\|_\infty = \|\cdot\|$ and $L_\infty(\mathcal{M},\tau) = \mathcal{M}$. Note that

$$2[A^*, A] = [A + A^*, A - A^*] = 4i[\operatorname{Re} A, \operatorname{Im} A]. \quad (8)$$

Due to [20, Proposition 6], we obtain for $0 < r \le 1$

$$\|X + Y\|_r \le 2^{1/r-1}(\|X\|_r + \|Y\|_r) \quad \text{for all} \quad X, Y \in L_r(\mathcal{M},\tau). \quad (9)$$

If $X \in L_p(\mathcal{M},\tau)$ and $Y \in L_q(\mathcal{M},\tau)$, then $XY \in L_r(\mathcal{M},\tau)$ and, due to [20, Lemma 1], we have

$$\|XY\|_r \le \|X\|_p \|Y\|_q. \quad (10)$$

Using the triangle inequality (for $r \ge 1$) or (9) (for $0 < r \le 1$) and then applying the inequality (10), we obtain the required estimate from (8). The theorem is proved. □

Remark 4.1. If operators $A \in \widetilde{\mathcal{M}}^+$ and $P \in \mathcal{M}^{\rm pr}$ are such that $AP + PA \ge 0$, then $[A, P] = 0$ due to [8, Lemma 2]. In [4], sufficient conditions of the validity of the inclusions $XY, YX \in L_1(\mathcal{M},\tau)$ for operators $X, Y \in \widetilde{\mathcal{M}}$ were obtained. For such operators, we have $\tau([X, Y]) = 0$ owing to Lemma 3.1. In [9], sufficient conditions of the τ-compactness of the product of τ-measurable operators were established. Sometimes, these conditions provide the τ-compactness of commutators of these operators.

Theorem 4.8. *Let operators $X \in \widetilde{\mathcal{M}}^+$ and $Y \in \widetilde{\mathcal{M}}^{\rm sa}$ be such that $[X^{1/2}, YX^{1/2}] \in L_1(\mathcal{M},\tau)$. Then $\tau([X^{1/2}, YX^{1/2}]) = it$, where $t \in \mathbb{R}$ and $t = 0$ for $XY \in L_1(\mathcal{M},\tau)$.*

Proof. We have $X^{1/2}YX^{1/2} - XY = ([X^{1/2}, YX^{1/2}])^* \in L_1(\mathcal{M},\tau)$. We set

$$A = X^{1/2}, \quad B = [X^{1/2}, Y].$$

Then the operators $XY - X^{1/2}YX^{1/2} = AB$ and $X^{1/2}YX^{1/2} - YX = BA = [X^{1/2}, YX^{1/2}]$ lie in $L_1(\mathcal{M},\tau)$ and $\tau(AB) = \tau(BA)$ due to Lemma 3.1. Since $AB = -(BA)^*$, by Lemma 3.4 we have

$$\tau(AB) = \tau(-(BA)^*) = -\tau((BA)^*) = -\overline{\tau(BA)} = -\overline{\tau(AB)}.$$

Therefore, $\tau(AB) = \tau([X^{1/2}, YX^{1/2}]) = it$ with some $t \in \mathbb{R}$. Therefore,

$$\tau(XY + YX - 2X^{1/2}YX^{1/2}) = 0. \quad (11)$$

Now let $XY \in L_1(\mathcal{M},\tau)$ and $Y = Y_+ - Y_-$ be the Jordan decomposition, where $Y_+, Y_- \in \widetilde{\mathcal{M}}^+$ and $Y_+Y_- = 0$, and let $P_+, P_- \in \mathcal{M}^{\rm pr}$ be the supports of the operators Y_+ and Y_-, respectively. If $A \in \mathcal{M}$ and $B \in \widetilde{\mathcal{M}}$, then

$$\mu_t(AB) \le \|A\|\mu_t(B)$$

for all $t > 0$ (see [15, 27]). Therefore, the operators

$$XY_+ = XYP_+, \quad XY_- = XYP_-$$

lie in $L_1(\mathcal{M},\tau)$. Owing to Lemma 3.2, we have

$$X^{1/2}Y_+X^{1/2},\ X^{1/2}Y_-X^{1/2} \in L_1(\mathcal{M},\tau);$$

therefore, $X^{1/2}YX^{1/2} \in L_1(\mathcal{M},\tau))$ and

$$\tau(XY) = \tau(XY_+) - \tau(XY_-) = \tau(X^{1/2}Y_+X^{1/2}) - \tau(X^{1/2}Y_-X^{1/2}) = \tau(X^{1/2}YX^{1/2}) \ge 0.$$

Hence

$$\tau(YX) = \tau((XY)^*) = \overline{\tau(XY)} = \overline{\tau(X^{1/2}YX^{1/2})} = \tau(X^{1/2}YX^{1/2})$$

due to Lemma 3.4. The theorem is proved. □

Corollary 4.5. *Let $\tau(I) = 1$ and operator $X \in \widetilde{\mathcal{M}}^+$ and $Y \in \widetilde{\mathcal{M}}^{\rm sa}$ be such that $[X^{1/2}, YX^{1/2}] \in L_1(\mathcal{M},\tau)$. Then $\|I + z(XY + YX - 2X^{1/2}YX^{1/2})\|_1 \ge 1$ for all $z \in \mathbb{C}$.*

Proof. This assertion follows from (11) and Lemma 3.5. □

A vector subspace $\mathcal{E}$ in $\widetilde{\mathcal{M}}$ is called an *ideal space* on $(\mathcal{M}, \tau)$ if

(1) $X \in \mathcal{E}$ implies $X^* \in \mathcal{E}$;

(2) the conditions $X \in \mathcal{E}$, $Y \in \widetilde{\mathcal{M}}$, and $|Y| \leq |X|$ imply that $Y \in \mathcal{E}$.

As examples, we mention $\mathcal{M}$ and the set of elementary operators $\mathcal{F}(\mathcal{M})$, $\widetilde{\mathcal{M}}_0$, $(L_1 + L_\infty)(\mathcal{M}, \tau)$ and $L_p(\mathcal{M}, \tau)$ for $0 < p < +\infty$. If $\mathcal{E}$ is an ideal space on $(\mathcal{M}, \tau)$, $X \in \mathcal{E}$, and $Y, Z \in \mathcal{M}$, then $YXZ \in \mathcal{E}$.

The following hypothesis strengthens Theorem 3 from [1] and Theorem 1 from [2] (see Lemma 3.2).

Hypothesis. Let τ be an exact, normal, semifinite trace on the von Neumann algebra $\mathcal{M}$ and $\mathcal{E}$ be an ideal space on $(\mathcal{M}, \tau)$. If $X, Y \in \widetilde{\mathcal{M}}^+$ and $XY + YX \in \mathcal{E}$, then $X^{1/2}YX^{1/2}$, $Y^{1/2}XY^{1/2} \in \mathcal{E}$.

We show that in the particular case where

$$Y = \sum_{k=1}^{n} \lambda_k P_k, \quad \lambda_k > 0, \qquad P_k \in \mathcal{M}^{\text{pr}}, \qquad P_k P_j = 0 \quad \text{for } k \neq j,\ k, j = 1, \ldots, n,$$

the hypothesis is valid. We have

$$P = \sum_{k=1}^{n} P_k \in \mathcal{M}^{\text{pr}}.$$

The operator

$$Z = P(XY + YX)P = 2\sum_{k=1}^{n} \lambda_k P_k X P_k + \sum_{\substack{k=1,\\ j<k}}^{n} (\lambda_k + \lambda_j)(P_k X P_j + P_j X P_k)$$

lies in $\mathcal{E}$. Then $P_k X P_j = (\lambda_k + \lambda_j)^{-1} P_k Z P_j \in \mathcal{E}$, $k, j = 1, \ldots, n$. We have

$$Y^{1/2} X Y^{1/2} = \sum_{k=1}^{n} \lambda_k^{1/2} P_k \cdot X \cdot \sum_{k=1}^{n} \lambda_k^{1/2} P_k = \sum_{k=1}^{n} \lambda_k P_k X P_k + \sum_{\substack{k=1,\\ j<k}}^{n} (\lambda_k \lambda_j)^{1/2} (P_k X P_j + P_j X P_k) \in \mathcal{E}.$$

Let $X^{1/2}Y^{1/2} = U|X^{1/2}Y^{1/2}|$ be the polar decomposition of the operator $X^{1/2}Y^{1/2}$. Then

$$X^{1/2}YX^{1/2} = (Y^{1/2}X^{1/2})^*(Y^{1/2}X^{1/2}) = U\,Y^{1/2}X^{1/2}(Y^{1/2}X^{1/2})^*\,U^* = U\,Y^{1/2}XY^{1/2}\,U^* \in \mathcal{E}.$$

Remark 4.2. The hypotheses is valid for $X \in \mathcal{E} \cap \mathcal{M}^+$ and $Y \in \mathcal{M}^+$ (see [21, Proposition 14]; for $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \text{tr}$ see [19]). In [12], commutator inequalities related to the polar decompositions of τ-measurable operators are stated. In [24, 25], [1, Theorem 3] and [2, Theorem 1] were strengthened.

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