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POISSON STRUCTURES ON WEIL BUNDLES

(submitted by B. Shapukov)

ABSTRACT. In the present paper, we construct complete lifts of covariant and contravariant tensor fields from the smooth manifold M to its Weil bundle $T^{\mathbf{A}}M$ for the case of a Frobenius Weil algebra $\mathbf{A}$. For a Poisson manifold (M, w) we show that the complete lift w^C of a Poisson tensor w is again a Poisson tensor on $T^{\mathbf{A}}M$ and that w^C is a linear combination of some "basic" Poisson structures on $T^{\mathbf{A}}M$ induced by w. Finally, we introduce the notion of a weakly symmetric Frobenius Weil algebra $\mathbf{A}$ and we compute the modular class of (T^AM, w^C) for such algebras.

1. PRELIMINARIES

A *Weil algebra* [5, 16] is an associative commutative algebra $\mathbf{A}$ with unit over the field $\mathbf{R}$ of real numbers, which is of the form $\mathbf{A} = \mathbf{R} \oplus \mathring{\mathbf{A}}$, where $\mathring{\mathbf{A}}$ is a finite-dimensional maximal ideal, consisting of nilpotent elements. In what follows we denote $n = \dim_{\mathbf{R}} \mathring{\mathbf{A}}$.

By $\mathring{\mathbf{A}}^r$ we denote the rth power of $\mathring{\mathbf{A}}$. Let $d_k(\mathbf{A}) = \dim_{\mathbf{R}} \mathring{\mathbf{A}}^k / \mathring{\mathbf{A}}^{k+1}$. The number $d_1(\mathbf{A})$ is usually called the *width* of $\mathbf{A}$. The positive integer q defined by $\mathring{\mathbf{A}}^q \neq 0$, $\mathring{\mathbf{A}}^{q+1} = 0$ is called the *height* of $\mathbf{A}$.

The chain of embedded ideals

$$\mathbf{A} \supset \mathring{\mathbf{A}} \supset \mathring{\mathbf{A}}^2 \supset \cdots \supset \mathring{\mathbf{A}}^q \supset 0$$

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can be extended to the chain of ideals called the Jordan-Hölder composition series [16]

$$\mathbf{A} \supset \overset{\circ}{\mathbf{A}} = \mathbf{I}_1 \supset \mathbf{I}_2 \supset \ldots \supset \mathbf{I}_n \supset 0,$$

where $\mathbf{I}_a/\mathbf{I}_{a+1}$ is a 1-dimensional algebra with the zero multiplication. Here

$$\overset{\circ}{\mathbf{A}}{}^k = \mathbf{I}_{1+d_1(\mathbf{A})+\cdots+d_{k-1}(\mathbf{A})} \quad \text{for} \quad 2 \leq k \leq q.$$

This is the particular case of the general ring construction, see [12]. Using the Jordan-Hölder composition series one can choose the *Jordan-Hölder basis*

$$\{e_a\} = \{e_0, e_{\hat{a}}\}, \quad a = 0, 1, \ldots, n = \dim \overset{\circ}{\mathbf{A}}, \quad \hat{a} = 1, \ldots, n, \tag{1.1}$$

in $\mathbf{A}$ such that $e_0 = 1 \in \mathbf{R}$, $e_{\hat{a}} \in \mathbf{I}_{\hat{a}}$, $e_{\hat{a}} \notin \mathbf{I}_{\hat{a}+1}$. In general, this basis is not unique. For $X = x^a e_a = x^0 + x^{\hat{a}} e_{\hat{a}} \in \mathbf{A}$ we set $\overset{\circ}{X} = x^{\hat{a}} e_{\hat{a}}$, then $X = x^0 + \overset{\circ}{X}$. Let δ^a be the coordinates of unit of $\mathbf{A}$, i.e., $1 = \delta^a e_a$.

We denote by (γ^c_{ab}) the structural tensor of $\mathbf{A}$ with respect to the basis (1.1), i.e., $e_a e_b = \gamma^c_{ab} e_c$. It satisfies $\gamma^b_{0a} = \delta^b_a$ (the Kronecker's delta) and $\gamma^c_{a\hat{b}} = 0$ for $a \geq c$. Since $\mathbf{A}$ is commutative and associative, it also satisfies the conditions $\gamma^c_{ab} = \gamma^c_{ba}$ and

$$\gamma^c_{ab}\gamma^b_{ef} = \gamma^b_{ae}\gamma^c_{bf}. \tag{1.2}$$

The conditions of differentiability of a function $f : U \subset \mathbf{A} \to \mathbf{A}$ on a commutative associative algebra $\mathbf{A}$ (or, briefly, $\mathbf{A}$-differentiability), usually called *Scheffers' equations*, are (see [19]):

$$\partial_c f^b \gamma^c_{ag} = \gamma^b_{ac} \partial_g f^c, \tag{1.3}$$

where $\partial_a f^b = \partial f^b / \partial x^a$. Scheffers' equations are equivalent to

$$\partial_a f^b = \gamma^b_{ac} \delta^g \partial_g f^c. \tag{1.4}$$

For $f : U \subset \mathbf{A}^m \to \mathbf{A}$, $f : \{X^i = x^{ia} e_a\} \mapsto f(X^i) = f^b(x^{ia}) e_b$, where $\mathbf{A}^m = \mathbf{A} \times \cdots \times \mathbf{A}$ is the $\mathbf{A}$-module of m-tuples of elements of $\mathbf{A}$, Scheffers' conditions of $\mathbf{A}$-differentiability are of the form [19]:

$$\partial_{ia} f^b = \gamma^b_{ac} \delta^g \partial_{ig} f^c. \tag{1.5}$$

If a function f satisfies (1.5), its differential can be represented in the form $df = f_i dX^i$, where $f_i = \delta^a \partial_{ia} f$ is the partial derivative with respect to $X^i \in \mathbf{A}$. Thus,

$$f_i = \frac{\partial f}{\partial X^i} = \delta^a \frac{\partial f}{\partial x^{ia}}. \tag{1.6}$$

The functions $f_i(X^j)$, $i = 1, \ldots, m$, are also $\mathbf{A}$-differentiable.

The following theorem (see [16]) describes the local structure of an $\mathbf{A}$-differentiable map of the form $F : U \subset \mathbf{A}^m \to \mathbf{A}^k$ for a Weil algebra $\mathbf{A}$. The natural epimorphism $\pi_0^q : \mathbf{A}^m \to \mathbf{R}^m$ determines *the canonical* $\overset{\circ}{\mathbf{A}}{}^m$*-foliation* on $\mathbf{A}^m$. Recall that a smooth map $f : M \to N$ of a foliated manifold $(M, \mathcal{F})$ is called *projectable* or *basic* if f is constant along the leaves of $\mathcal{F}$.

Theorem 1.1 ([16])**.** 1) *Let $U \subset \mathbf{A}^m$ be an open set. Then any $\mathbf{A}$-smooth map $\Phi : U \to \mathbf{A}^k$ is of the form*

$$X^{i'} = \varphi^{i'} + \sum_{|p|=1}^{q} \frac{1}{p!} \frac{D^p \varphi^{i'}}{Dx^p} \overset{\circ}{X}{}^p, \tag{1.7}$$

(where $i = 1, \dots, m$, $i' = 1, \dots, k$, $p = (p_1, \dots, p_m)$ is a multiindex of length m and $\overset{\circ}{X}{}^p = (\overset{\circ}{X}{}^1)^{p_1} \dots (\overset{\circ}{X}{}^m)^{p_m}$) for some basic smooth map $\varphi^{i'} : U \to \mathbf{A}$ which is projectable with respect to the canonical $\overset{\circ}{\mathbf{A}}{}^m$-foliation.

Definition. The map $\Phi : U \to \mathbf{A}^k$ given by the formulas (1.7) is called *the analytic prolongation* of the projectable map $\varphi : U \to \mathbf{A}^k$.

The analytic prolongation of a map φ will be denoted by $\varphi^{\mathbf{A}}$.

Proposition 1.1 ([16])**.** *The analytic prolongation has the following properties:*
1°. $(\varphi + \psi)^{\mathbf{A}} = \varphi^{\mathbf{A}} + \psi^{\mathbf{A}}$.
2°. $(\varphi \cdot \psi)^{\mathbf{A}} = \varphi^{\mathbf{A}} \cdot \psi^{\mathbf{A}}$.
3°. $(\varphi^{\mathbf{A}} \circ \psi)^{\mathbf{A}} = \varphi^{\mathbf{A}} \circ \psi^{\mathbf{A}}$.
4°. $(D^p\varphi / Dx^p)^{\mathbf{A}} = D^p \varphi^{\mathbf{A}} / DX^p$ *for $\varphi : U \subset \mathbf{A}^n \to \mathbf{A}$.*

We denote by $\mathcal{M}f$ the category of smooth manifolds and by $\mathcal{F}M$ that of fibered manifolds. To each Weil algebra $\mathbf{A}$ there corresponds a functor $T^{\mathbf{A}} : \mathcal{M}f \to \mathcal{F}M$ called the *Weil functor* which maps a smooth manifold M to the fibered manifold $\pi_{\mathbf{A}} : T^{\mathbf{A}}M \to M$ called the *Weil bundle* (see [5, 16, 20]). A.P. Shirokov proved [15] that $T^{\mathbf{A}}M$ carries the structure of a smooth manifold over $\mathbf{A}$. Weil functors preserve products, i.e., $T^{\mathbf{A}}(M \times N) \cong T^{\mathbf{A}}M \times T^{\mathbf{A}}N$. Moreover, under some additional conditions (locality and regularity) each product preserving bundle functor $F : \mathcal{M}f \to \mathcal{F}M$ is equivalent to a Weil functor $T^{\mathbf{A}}$ for a Weil algebra $\mathbf{A}$ [5].

A Weil algebra $\mathbf{A}$ is said to be *Frobenius* (cf. [19, 2]) if there exists a nondegenerate bilinear form $q : \mathbf{A} \times \mathbf{A} \to \mathbf{R}$, satisfying the following condition of associativity:

$$q(XY, Z) = q(X, YZ) \quad \text{for any} \quad X, Y, Z \in \mathbf{A}. \tag{1.8}$$

Frobenius algebras play an important role in the theory of smooth manifolds over algebras in constructing realizations of tensor operations [8].

With respect to the basis (1.1) the condition (1.8) is written as

$$q_{bc}\gamma^c_{ef} = \gamma^c_{be}q_{cf}. \tag{1.9}$$

We will call q a *Frobenius form.* A Frobenius form is not unique (if exists). For a Frobenius algebra $\mathbf{A}$ we define the *Frobenius covector* $p : \mathbf{A} \to \mathbf{R}$ by $p(X) := q(X, 1)$. Its coordinates with respect to the basis (1.1) satisfy

$$p_a\gamma^a_{bc} = q_{bc}. \tag{1.10}$$

Contracting (1.10) with δ^c and taking into consideration that $\delta^c = \delta^c_0$ (the Kronecker's delta) with respect to the basis (1.1), we obtain

$$p_b = q_{bc}\delta^c. \tag{1.11}$$

From (1.8) and (1.10) it easily follows that

$$q(X, Y) = p(XY) \quad \text{for each} \quad X, Y \in \mathbf{A}. \tag{1.12}$$

We denote the set of all Frobenius covectors on $\mathbf{A}$ by $\mathbf{A}^*_{Fr}$.

Example 1.1. The important example of a Frobenius Weil algebra is the algebra of *dual numbers* $\mathbf{D} = \mathbf{R}(\varepsilon) = \{x_0 + x_1\varepsilon \mid x_0, x_1 \in \mathbf{R}, \varepsilon^2 = 0\}$. To this algebra there corresponds the tangent bundle functor: $T^{\mathbf{R}(\varepsilon)}M = TM$.

Example 1.2. Another example is the algebra $\mathbf{D}^n = \mathbf{R}(\varepsilon^n) = \{x_0 + x_1\varepsilon + \cdots + x_{n-1}\varepsilon^{n-1} \mid x_i \in \mathbf{R}, \varepsilon^n = 0\}$. of *plural numbers* which is a generalization of the previous one. To this algebra there corresponds the functor of jet bundle of higher order.

If $\mathbf{A}$ and $\mathbf{B}$ are Frobenius algebras, then $\mathbf{A} \otimes \mathbf{B}$ is also a Frobenius algebra (see, e.g. [19]).

In what follows we assume all Weil algebras under consideration to be Frobenius algebras.

2. The structure of a Frobenius Weil algebra

Let $\mathbf{A}$ be a Frobenius Weil algebra of height q and let $n = \dim \mathring{\mathbf{A}}$. Let us choose a Jordan-Hölder basis (1.1) in $\mathbf{A}$.

Lemma 2.1. $\dim \mathring{\mathbf{A}}^q = 1$*, i.e.,* $\mathring{\mathbf{A}}^q = \mathbf{I}_n$.

Proof. On the contrary, suppose that $\dim_{\mathbf{R}} \mathring{\mathbf{A}}^q \geq 2$. Then at least two last elements e_{n-1} and e_n of a basis (1.1) belong to $\mathring{\mathbf{A}}^q$ and, consequently, for any $\hat{a} = 1, \dots, n$ there hold $e_{\hat{a}}e_{n-1} = e_{\hat{a}}e_n = 0$. Hence for any $c = 0, \dots, n$ the matrix $\|\gamma^c_{ab}\|$ contains zeros everywhere in two last columns (with the numbers $n-1$ and n) except for the first row. Then for each covector (p_c) the matrix $\|q_{ab}\| = \|p_c\gamma^c_{ab}\|$ also contains zeros everywhere in two last columns except the first row, and thus is degenerate. Contradiction. By this reason, $\dim_{\mathbf{R}} \overset{\circ}{\mathbf{A}}{}^q = 1$, which implies $\overset{\circ}{\mathbf{A}}{}^q = \mathbf{I}_n$ or, equivalently, $\overset{\circ}{\mathbf{A}}{}^q = \mathbf{R} \cdot e_n$. $\square$

Lemma 2.2. *For each Frobenius covector p on $\mathbf{A}$, its last component p_n is not zero.*

Proof. From the equalities $1 \cdot e_n = e_0 e_n = e_n$ and $e_{\hat{a}} e_n = 0$ for each $\hat{a} = 1, \dots, n$ it follows, that for each $c = 0, \dots, n-1$ and for each b there holds

$$\gamma^c_{bn} = 0, \qquad \gamma^n_{0n} = 1, \qquad \gamma^n_{\hat{a}n} = 0.$$

Hence, for each $c = 0, \dots, n-1$ the last column of the matrix $\|\gamma^c_{ab}\|$ contains only zeros and the last column of the matrix $\|\gamma^n_{ab}\|$ is

$${}^T(1, 0, \dots, 0).$$

Therefore the last column of $\|q_{ab}\| = \|p_c\gamma^c_{ab}\|$ is

$${}^T(p_n, 0, \dots, 0), \tag{2.1}$$

hence $p_n \neq 0$. $\square$

Remark 2.1. One can prove both lemmas without using coordinates. Indeed, denote

$$\operatorname{Ann} \overset{\circ}{\mathbf{A}} := \{X \in \overset{\circ}{\mathbf{A}} \mid X \cdot \overset{\circ}{\mathbf{A}} = 0\}.$$

Let $0 \neq X \in \operatorname{Ann} \overset{\circ}{\mathbf{A}}$, then for each $Y = y^0 + \overset{\circ}{Y}$ we have $XY = Xy^0$, whence $q(X, Y) = p(XY) = y^0 p(X)$. From the degeneracy of q it follows that $p(X) \neq 0$. Therefore $\operatorname{Ann} \overset{\circ}{\mathbf{A}} \cap \ker p = 0$ which implies that $\dim \operatorname{Ann} \overset{\circ}{\mathbf{A}} \leq 1$. But, clearly, $0 \neq \overset{\circ}{\mathbf{A}}{}^q \subset \operatorname{Ann} \overset{\circ}{\mathbf{A}}$, hence $\dim \overset{\circ}{\mathbf{A}}{}^q = \dim \operatorname{Ann} \overset{\circ}{\mathbf{A}} = 1$.

Let us denote $\|h_{ab}\| := \|\gamma^n_{ab}\|$.

Lemma 2.3. *The Jordan-Hölder basis (1.1) can be chosen in such a way that the matrix $\|h_{ab}\|$ is nondegenerate.*

Proof. Let us choose any Jordan-Hölder basis (1.1). If $p_{(0)} = (0, \dots, 0, 1)$ is a Frobenius covector, then the matrix $\|h_{ab}\|$ is nondegenerate. Assume the contrary and consider any $p \in \mathbf{A}^*_{Fr}$. Without loss of generality we may assume that $p_n = 1$ (otherwise consider $\frac{1}{p_n} p$).

1) We prove that the first component p_0 of p may be taken to be zero. Indeed, in the matrix $\|q_{ab}\|$ only the element q_{00} depends on p_0:

$$\|q_{ab}\| = \begin{pmatrix} p_0 & * & \dots & * & 1 \\ * & * & \dots & * & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \dots & * & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

($*$ denotes the elements which do not depend on p_0.) The cofactor of $q_{00} = p_0$ contains only zeros in the last column, hence it is zero itself. Thus, the determinant $\det\|q_{ab}\|$ does not depend on p_0 and we may assume that $p_0 = 0$.

2) By the assumption, $p_{(0)} = (0, \dots, 0, 1)$ is not a Frobenius covector, therefore there exists at least one c, $1 \le c \le n-1$, such that $p_c \ne 0$. Consider another basis $\{e'_a\}$ in $\mathbf{A}$:

$$e'_0 = e_0 = e_0 - p_0 e_n, \qquad e'_a = e_a - p_a e_n, \quad a = 1, 2, \dots, n-1, \qquad e'_n = e_n. \tag{2.2}$$

One can easily see that $\{e'_a\}$ is also a Jordan-Hölder basis. Since $e_a e_n = 0$ for each $a \ge 1$, the structural constants γ'^c_{ab} will have the following form with respect to this basis: $\gamma'^c_{ab} = \gamma^c_{ab}$ for $c = 1, 2, \dots, n-1$ and $\gamma'^n_{ab} = \gamma^n_{ab} + \gamma^d_{ab} p_d$, where the summation over d is taken from 1 to $n-1$. Thus, $\|\gamma'^n_{ab}\|$ equals to $\|q_{ab}\|$ and therefore is nondegenerate. □

In what follows we will suppose the Jordan-Hölder basis to be chosen in such a way that $\|h_{ab}\|$ is nondegenerate and we will call $p_{(0)} = (0, \dots, 0, 1)$ the *standard Frobenius covector*.

Remark 2.2. One can also give a noncoordinate proof of Lemma 2.3. Let $\overset{\circ}{p} \in \mathbf{A}^*$ be defined by $\overset{\circ}{p}(X) := x^0$.

1) We show that if $p \in \mathbf{A}^*_{Fr}$ then $\widetilde{p} := p - p(1)\overset{\circ}{p}$ also is a Frobenius covector. Suppose the contrary. Then there exists $X \in \mathbf{A}$ such that $\widetilde{p}(XY) = 0$ for any $Y \in \mathbf{A}$. This means that $x^0 p(\overset{\circ}{Y}) + y^0 p(\overset{\circ}{X}) + p(\overset{\circ}{X}\overset{\circ}{Y}) = 0$. Let $Z \in \overset{\circ}{\mathbf{A}}{}^q$ be an element such that $p(Z) = 1$ (in terms of the Jordan-Hölder basis, $Z = e_n$) and let $\widetilde{X} = X - x^0 Z$. Then for any $Y \in \mathbf{A}$ one has $\widetilde{X}Y = XY - x^0 y^0 Z = x^0 y^0 + x^0 \overset{\circ}{Y} + y^0 \overset{\circ}{X} + \overset{\circ}{X}\overset{\circ}{Y} - x^0 y^0 Z$. One can easily see that $p(\widetilde{X}Y) = 0$, which contradicts to the fact that p is a Frobenius covector.

This means that we can deform any $p \in \mathbf{A}^*_{Fr}$ in such a way that $\mathbf{R} \subset \ker p$.

2) Now, by Lemma 2.2 or Remark 2.1, one has $\mathbf{A} = \ker p \oplus \overset{\circ}{\mathbf{A}}{}^q$. We define a bilinear form $h(X,Y)$ on $\mathbf{A}$ to be the projection of XY onto $\overset{\circ}{\mathbf{A}}{}^q$ along $\ker p$. This form is nondegenerate. Indeed, let $X \in \mathbf{A}$ be such that $h(X,Y) = 0$ for any $Y \in \mathbf{A}$. We write $XY = U + Z$, where $U \in \ker p$, $Z \in \overset{\circ}{\mathbf{A}}{}^q$. Then $0 = h(X,Y) = Z$, hence $p(XY) = p(Z) = 0$ which contradicts to Lemma 2.2.

Thus, without loss of generality, we may assume the matrix $\|h_{ab}\| = \|\gamma^n_{ab}\|$ to be nondegenerate. This matrix has the following form:

$$\|h_{ab}\| = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & & & & 0 \\ \vdots & & B & & \vdots \\ 0 & & & & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{2.3}$$

where B denotes the nonsingular square block. The inverse matrix is of the same form:

$$\|h^{ab}\| = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & & & & 0 \\ \vdots & & B^{-1} & & \vdots \\ 0 & & & & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{2.4}$$

This allows us to introduce another basis in $\mathbf{A}$: we put $\overline{e}^a = h^{ab}e_b$, then $e_a = h_{ab}\overline{e}^b$. Denote by $\overline{\gamma}^{ab}_c$ the structural constants of $\mathbf{A}$ with respect to the basis $\{\overline{e}^a\}$, i.e. $\overline{e}^a\overline{e}^b = \overline{\gamma}^{ab}_c\overline{e}^c$. Clearly,

$$\overline{\gamma}^{ab}_s = h^{ak}h^{b\ell}h_{cs}\gamma^c_{k\ell} \tag{2.5}$$

and

$$\gamma^c_{k\ell} = h_{ak}h_{b\ell}h^{cs}\overline{\gamma}^{ab}_s. \tag{2.6}$$

Since $\delta^a = \delta^a_0$ with respect to the basis (1.1), we have $h_{cs}\delta^s = \gamma^n_{cs}\delta^s = \gamma^n_{c0}$, which is not zero only for $c = n$ and $\gamma^n_{n0} = 1$. Therefore $\overline{\gamma}^{ab}_0 = \overline{\gamma}^{ab}_s\delta^s = h^{ak}h^{b\ell}\gamma^c_{k\ell}h_{cs}\delta^s = h^{ak}h^{b\ell}\gamma^n_{k\ell} = h^{ak}h^{b\ell}h_{k\ell} = h^{ak}\delta^b_k = h^{ab}$. Thus,

$$h^{ab} = \overline{\gamma}^{ab}_0.$$

Moreover, it is clear from (2.4) that $\overline{e}^n = e_0$, hence $\overline{e}^a\overline{e}^n = \overline{e}^a$, which implies that

$$\overline{\gamma}^{an}_c = \delta^a_c. \tag{2.7}$$

From the formula (1.2) it follows that $\gamma^c_{sr}h_{cd} = \gamma^c_{sr}\gamma^n_{cd} = \gamma^c_{ds}\gamma^n_{cr} = \gamma^c_{ds}h_{cr} = \gamma^c_{dr}\gamma^n_{cs} = \gamma^c_{dr}h_{cs}$. Thus,

$$\gamma^c_{sr}h_{cd} = \gamma^c_{ds}h_{cr} = \gamma^c_{dr}h_{cs}. \tag{2.8}$$

The tensors γ^c_{ab} and $\overline{\gamma}^{ab}_c$ are also related with the following formulas. We have $\gamma^c_{ka}h^{k\ell} = h^{sc}h_{ba}h_{dk}h^{k\ell}\overline{\gamma}^{db}_s$ (by (2.6)) $= h^{sc}h_{ba}\overline{\gamma}^{\ell b}_s$, hence $h_{ba}\overline{\gamma}^{\ell b}_s = h_{sc}\gamma^c_{ka}h^{k\ell} = \gamma^c_{sa}h_{ck}h^{k\ell}$ (by (2.8)) $= \gamma^\ell_{sa}$. Thus,

$$\gamma^\ell_{sa} = h_{ba}\overline{\gamma}^{\ell b}_s, \tag{2.9}$$

whence

$$\overline{\gamma}^{bc}_s = h^{ab}\gamma^c_{sa}. \tag{2.10}$$

Let p ($p_n \neq 0$) be an arbitrary Frobenius covector on $\mathbf{A}$ and $\|q_{ab}\| = \|\gamma^c_{ab}p_c\|$. Let us find the explicit form of the inverse matrix $\|q^{ab}\|$. Denote $\|\overline{q}^{ab}\| = \|\overline{\gamma}^{ab}_c t^c\|$, where (t^c) are to be defined later.

From (2.7) it follows that the last column of $\|\overline{q}^{ab}\|$ is

$${}^T(t^0, t^1, \dots, t^n). \tag{2.11}$$

The system of linear equations $q_{ab}x^b = \delta^n_a$ on (x^b) has the unique solution. We define (t^b) to be this solution:

$$q_{ab}t^b = \gamma^k_{ab}p_k t^b = \delta^n_a. \tag{2.12}$$

Equivalently, $t = (t^b)$ is defined by $q(t, \cdot\,) = p_{(0)}(\,\cdot\,)$. Therefore, by (2.11), the last column of the matrix $\|q_{ab}\overline{q}^{bc}\|$ coincides with the last column of the unit matrix.

Let us show that this is true for any other column, i.e., that for each $c = 0, 1, \dots, n-1$,

$$q_{ab}\overline{q}^{bc} = \delta^c_a. \tag{2.13}$$

We need to check that $p_k\gamma^k_{ab}t^s\overline{\gamma}^{bc}_s = p_k\gamma^k_{ab}\gamma^c_{sr}h^{br}t^s = \delta^c_a$. Contracting the left-hand side with h_{cd} and using (2.8) yields $p_k\gamma^k_{ab}\gamma^c_{sr}h^{br}t^s h_{cd} = p_k t^s h^{br}\gamma^k_{ab}\gamma^c_{ds}h_{cr} = p_k t^s\gamma^k_{ab}\gamma^b_{ds} = p_k t^s\gamma^k_{bs}\gamma^b_{ad} = \delta^n_b\gamma^b_{ad} = \gamma^n_{ad} = h_{ad}$. Since the contraction of δ^c_a with h_{cd} also gives h_{ad} and h_{ad} is nondegenerate, the relation (2.13) holds true. Thus, the inverse matrix $\|q^{ab}\|$ has the form $\|\overline{\gamma}^{ab}_c t^c\|$, where $t = (t^c)$ is defined by (2.12).

We also show that (p_a) is defined uniquely by (t^c). It is sufficient to prove that $\det\|\gamma^c_{ab}t^b\| \neq 0$. Contracting with h^{ad} gives $t^b\gamma^c_{ab}h^{ad} = t^b\overline{\gamma}^{cd}_b = \overline{q}^{cd}$. The last matrix is nondegenerate, hence the result follows. It follows also that if $\det\|\gamma^c_{ab}t^b\| \neq 0$ for some $t = (t^b)$ then the corresponding covector $p \in \mathbf{A}^*_{Fr}$. Indeed, in this case $\|\overline{q}^{cd}\| = \|q^{cd}\|$ is nondegenerate, hence $\|q_{cd}\|$ is also nondegenerate.

The last row of $\|q_{ab}\|$ has the form $(p_n, 0, \dots, 0)$. Its product with the last column of $\|q^{bc}\|$ equals 1 by the definition of the inverse matrix and also equals $p_n t^0$ by (2.1). Therefore, $p_n t^0 = 1$, in particular, $t^0 \neq 0$.

Thus, we proved the following

Proposition 2.1. *For any Frobenius covector p ($p_n \neq 0$) on $\mathbf{A}$ the matrix $\|q^{ab}\|$ is of the form $q^{ab} = \overline{\gamma}^{ab}_c t^c$, where the vector $t = (t^c)$ and covector p uniquely define each other by (2.12), moreover, $t^0 p_n = 1$.*

In particular, if $p = p_{(0)}$ is the standard Frobenius covector, i.e., $p_a = \delta^n_a$, then $t^c = \delta^c_0$. Indeed, in this case $\gamma^c_{ab} p_c = \gamma^n_{ab} = h_{ab}$, hence, $h_{ab} t^b = \delta^a_n$ by (2.12). Denote by t' the column ${}^T(t^1, \dots, t^{n-1})$. From (2.3) we obtain $t^n = 0$, $Bt' = 0$, $t^0 = 1$. Since $\det B \neq 0$, each of t^1, …, t^{n-1} is equal to zero.

One can represent the "multiplication table" of $\mathbf{A}$ with respect to the basis (1.1) as follows. By $\mathring{\mathbf{A}}^s$, $s = 1, 2, \dots, q-1$, in the first column and the first row we denote the $d_s(\mathbf{A})$ elements of (1.1) which lie in $\mathring{\mathbf{A}}^s \setminus \mathring{\mathbf{A}}^{s+1}$ (or, equivalently, project to the basis of $\mathring{\mathbf{A}}^s / \mathring{\mathbf{A}}^{s+1}$ under the natural epimorphism $\mathring{\mathbf{A}}^s \to \mathring{\mathbf{A}}^{s+1}$). The product of two such elements of (1.1), one of them lying in the $\mathring{\mathbf{A}}^k$-column, another in the $\mathring{\mathbf{A}}^\ell$-row, belongs to $\mathring{\mathbf{A}}^{k+\ell}$, thus we write $\mathring{\mathbf{A}}^{k+\ell}$ in the intersection of $\mathring{\mathbf{A}}^k$-column and $\mathring{\mathbf{A}}^\ell$-row. The whole table now has the following block structure:

	1	$\mathring{\mathbf{A}}$	$\mathring{\mathbf{A}}^2$	$\dots$	$\mathring{\mathbf{A}}^k$	$\dots$	$\mathring{\mathbf{A}}^{q-2}$	$\mathring{\mathbf{A}}^{q-1}$	e_n
1	1	$\mathring{\mathbf{A}}$	$\mathring{\mathbf{A}}^2$	$\dots$	$\mathring{\mathbf{A}}^k$	$\dots$	$\mathring{\mathbf{A}}^{q-2}$	$\mathring{\mathbf{A}}^{q-2}$	$\mathring{\mathbf{A}}^q$
$\mathring{\mathbf{A}}$	$\mathring{\mathbf{A}}$	$\mathring{\mathbf{A}}^2$	$\mathring{\mathbf{A}}^3$	$\dots$	$\mathring{\mathbf{A}}^{k+1}$	$\dots$	$\mathring{\mathbf{A}}^{q-1}$	$\mathring{\mathbf{A}}^q$	0
$\mathring{\mathbf{A}}^2$	$\mathring{\mathbf{A}}^2$	$\mathring{\mathbf{A}}^3$	$\mathring{\mathbf{A}}^4$	$\dots$	$\mathring{\mathbf{A}}^{k+2}$	$\dots$	$\mathring{\mathbf{A}}^q$	0	0
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\iddots$	$\iddots$	$\iddots$	$\vdots$
$\mathring{\mathbf{A}}^{q-k}$	$\mathring{\mathbf{A}}^{q-k}$	$\mathring{\mathbf{A}}^{q-k+1}$	$\mathring{\mathbf{A}}^{q-k+2}$	$\dots$	$\mathring{\mathbf{A}}^q$	$\iddots$	$\iddots$	$\dots$	$\vdots$
$\vdots$	$\dots$	$\dots$	$\dots$	$\iddots$	$\iddots$	$\iddots$	$\dots$	$\dots$	$\vdots$
$\mathring{\mathbf{A}}^{q-2}$	$\mathring{\mathbf{A}}^{q-2}$	$\mathring{\mathbf{A}}^{q-1}$	$\mathring{\mathbf{A}}^q$	0	$\iddots$	$\dots$	0	0	0
$\mathring{\mathbf{A}}^{q-1}$	$\mathring{\mathbf{A}}^{q-1}$	$\mathring{\mathbf{A}}^q$	0	$\iddots$	$\dots$	$\dots$	0	0	0
e_n	$\mathring{\mathbf{A}}^q$	0	0	$\dots$	$\dots$	$\dots$	0	0	0

The secondary diagonal of this table contains blocks $\overset{\circ}{\mathbf{A}}{}^q$ consisting of real multiples of e_n. Hence all the matrices $\|\gamma^c_{ab}\|$, $c = 0, 1, \dots, n-1$, contain zeros in these blocks and the matrix $\|\gamma^n_{ab}\| = \|h_{ab}\|$ has the following block structure:

	1	$\overset{\circ}{\mathbf{A}}$	$\overset{\circ}{\mathbf{A}}{}^2$	$\dots$	$\overset{\circ}{\mathbf{A}}{}^{q-2}$	$\overset{\circ}{\mathbf{A}}{}^{q-1}$	e_n
1	0	0	0	$\dots$	0	0	1
$\overset{\circ}{\mathbf{A}}$	0	$*$	$*$	$\dots$	$*$	$B_{1,q-1}$	0
$\overset{\circ}{\mathbf{A}}{}^2$	0	$*$	$*$	$\iddots$	$B_{2,q-2}$	0	0
$\vdots$	$\vdots$	$\dots$	$\iddots$	$\iddots$	$\iddots$	$\vdots$	$\vdots$
$\overset{\circ}{\mathbf{A}}{}^{q-2}$	0	$*$	$B_{q-2,2}$	$\iddots$	0	0	0
$\overset{\circ}{\mathbf{A}}{}^{q-1}$	0	$B_{q-1,1}$	0	$\dots$	0	0	0
e_n	1	0	0	$\dots$	0	0	0

Definition. We will say that the Frobenius Weil algebra $\mathbf{A}$ of height q is *weakly symmetric*, if $d_k(\mathbf{A}) = d_{q-k}(\mathbf{A})$ for each $k = 1, 2, \dots, q-1$.

For a weakly symmetric algebra all the blocks $B_{k,q-k}$, $k = 1, 2, \dots, q-1$, are squares. One can easily see that in this case

$$\det \|q_{ab}\| = \det \|p_n \cdot \gamma^n_{ab}\| = p_n^{n+1} \det B_{1,q-1} \cdot \dots \cdot \det B_{q-1,1}, \qquad (2.14)$$

therefore, all the blocks $B_{k,q-k}$, $k = 1, 2, \dots, q-1$, are nondegenerate. From (2.14) it follows that for this algebra each $p \in \mathbf{A}^*$ such that $p_n \neq 0$, is a Frobenius covector.

Example 2.1. The simplest example of weakly symmetric Weil algebra is the algebra of plural numbers $\mathbf{R}(\varepsilon^n)$. For this algebra $q = n$ and the Jordan-Hölder basis is $e_a = \varepsilon^a$, $a = 0, 1, \dots, n$. Therefore $d_s(\mathbf{R}(\varepsilon^n)) = 1$ for each $s = 1, \dots, n$ and all the blocks $B_{k,n-k}$ consist of one element each. Clearly,

$$\|\gamma^n_{ab}\| = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}.$$

Example 2.2. It is clear that every Frobenius Weil algebra $\mathbf{A}$ of height $q = 2$ is weakly symmetric. For the elements $\{e_a\}$, $a = 0, \dots, n$, of Jordan-Hölder basis we have $e_1 \dots, e_{n-1} \in \overset{\circ}{\mathbf{A}}$, $e_n \in \overset{\circ}{\mathbf{A}}{}^2$. Therefore, $e_a e_b = \lambda_{ab} e_n$, $a, b = 1, \dots, n-1$, and $e_a e_n = 0$, $a = 1, \dots, n$. Hence, for any Frobenius covector $p = (p_a)$ $(p_n \neq 0)$

$$\|q_{ab}\| = \begin{pmatrix} p_0 & p_1 & p_2 & \dots & p_{n-1} & p_n \\ p_1 & \lambda_{11}\, p_n & \lambda_{12}\, p_n & \dots & \lambda_{1,n-1}\, p_n & 0 \\ p_2 & \lambda_{21}\, p_n & \lambda_{22}\, p_n & \dots & \lambda_{2,n-1}\, p_n & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{n-1} & \lambda_{n-1,1}\, p_n & \lambda_{n-1,2}\, p_n & \dots & \lambda_{n-1,n-1}\, p_n & 0 \\ p_n & 0 & \dots & \dots & 0 & 0 \end{pmatrix}$$

and $\|q_{ab}\|$ is nondegenerate if and only if $\Lambda = \|\lambda_{ab}\|$ is nondegenerate. We will denote this Weil algebra by $\mathbf{A}(\Lambda, n)$.

Proposition 2.2. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two weakly symmetric Frobenius Weil algebras. Then* $\mathbf{A} \otimes \mathbf{B}$ *is also weakly symmetric.*

Proof. If $\{e_a\}$ and $\{f_\alpha\}$ are Jordan-Hölder bases of $\mathbf{A}$ and $\mathbf{B}$ respectively, then $\{e_a \otimes f_\alpha\}$ is the Jordan-Hölder basis of $\mathbf{A} \otimes \mathbf{B}$. It follows that the height of $\mathbf{A} \otimes \mathbf{B}$ equals $q_1 + q_2$, where q_1 and q_2 are the heights of $\mathbf{A}$ and $\mathbf{B}$, respectively. It can be easily seen that $d_k(\mathbf{A} \otimes \mathbf{B}) = d_k(\mathbf{A}) + d_{k-1}(\mathbf{A}) d_1(\mathbf{B}) + \dots + d_1(\mathbf{A}) d_{k-1}(\mathbf{B}) + d_k(\mathbf{B})$. Moreover, $d_{q_1+q_2-k}(\mathbf{A} \otimes \mathbf{B}) = d_{q_1-k}(\mathbf{A}) + d_{q_1-k+1}(\mathbf{A}) d_{q_2-1}(\mathbf{B}) + \dots + d_{q_1-1}(\mathbf{A}) d_{q_2-k+1}(\mathbf{B}) + d_{q_2-k}(\mathbf{B})$, which coincides with $d_k(\mathbf{A} \otimes \mathbf{B})$. □

3. Complete lifts of tensor fields

Let $\mathbf{A}$ be a Weil algebra of height q and let $n = \dim \overset{\circ}{\mathbf{A}}$. In what follows we will assume that $\mathbf{A}$ is a Frobenius algebra. As before, we denote the Frobenius covector by $p = (p_c)$ and the Frobenius form by $q = (q_{ab})$.

Let M be an m-dimensional smooth manifold. Then $T^{\mathbf{A}}M$ is an m-dimensional $\mathbf{A}$-smooth manifold and for each $x \in T^{\mathbf{A}}M$ the tangent space $T_x T^{\mathbf{A}}M$ is an m-dimensional $\mathbf{A}$-module. Thus, we can consider $\mathbf{A}$-tensors at any point $x \in T^{\mathbf{A}}M$ and $\mathbf{A}$-smooth tensor fields on $T^{\mathbf{A}}M$ (see [19]). In what follows, we assume all the manifolds and the maps between manifolds to be of class C^∞.

We denote the algebra of smooth functions on M by $C^\infty(M)$, the space of covariant tensors on M by $\mathcal{T}^*(M)$ and the space of skew-symmetric contravariant tensors (multivector fields) on M by $\mathcal{V}^*(M)$. By $|\cdot|$ we denote the degree of a tensor field, i.e., $|\xi| = s$ if $\xi \in \mathcal{T}^s(M)$ and $|u| = s$ if $u \in \mathcal{V}^s(M)$.

In this part of the paper we construct complete lifts of covariant and contravariant tensor fields from M to the Weil bundle $T^{\mathbf{A}}M$.

Let $(x^i) = (x^1, \ldots, x^m)$ be local coordinates on M. We will enumerate the corresponding local coordinates on $T^{\mathbf{A}}M$ by the double index ia: (x^{ia}), $i = 1, \ldots, m$, $a = 0, \ldots, n$, where we identify $x^{i0} = x^i$.

Let $\xi \in \mathcal{T}^k(M)$ be a tensor field of type $(k, 0)$ on M. In local coordinates

$$\xi = \xi_{i_1 \ldots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}.$$

Let $\Xi_{i_1 \ldots i_k} = (\xi_{i_1 \ldots i_k})^{\mathbf{A}}$ be analytic prolongations of the functions $\xi_{i_1 \ldots i_k}$. We multiply these $\mathbf{A}$-valued functions by $e_{a_1} \ldots e_{a_k}$, where $\{e_a\}$ is a Jordan-Hölder basis (1.1). Let $\Xi_{i_1 \ldots i_k} e_{a_1} \ldots e_{a_k} = \Xi^b_{i_1 a_1 \ldots i_k a_k} e_b$, where $\Xi^b_{i_1 a_1 \ldots i_k a_k} \in \mathbf{R}$. Denote

$$\xi_{i_1 a_1 \ldots i_k a_k} := \Xi^b_{i_1 a_1 \ldots i_k a_k} p_b. \tag{3.1}$$

We define the *complete lift* ξ^C of ξ by

$$\xi^C = \xi^C_{\mathbf{A}} := \xi_{i_1 a_1 \ldots i_k a_k} dx^{i_1 a_1} \otimes \cdots \otimes dx^{i_k a_k}.$$

If $\{e_{a'}\}$ is another basis in $\mathbf{A}$ and $e_{a'} = \tau^a_{a'} e_a$, then, obviously, $dx^{ia'} = \tau^{a'}_a dx^{ia}$. Thus, our definition does not depend on a choice of a basis in $\mathbf{A}$.

Proposition 3.1. *ξ^C is a well-defined tensor field of type $(k, 0)$ on $T^{\mathbf{A}}M$. If $\varphi : N \to M$ is a smooth map, then*

$$(T^{\mathbf{A}}\varphi)^*(\xi^C) = (\varphi^* \xi)^C.$$

Proof. Let N be another smooth manifold with local coordinates (y^α) and $\varphi : N \to M$ be a smooth map which has the form $x^i = \varphi^i(y^\alpha)$ with respect to these coordinates. Denote $\theta = \varphi^* \xi$, then

$$\theta_{\alpha_1 \ldots \alpha_k} = \frac{\partial \varphi^{i_1}}{\partial y^{\alpha_1}} \cdots \frac{\partial \varphi^{i_k}}{\partial y^{\alpha_k}} \xi_{i_1 \ldots i_k}.$$

Let $\Theta_{\alpha_1 \ldots \alpha_k}$ be analytic prolongations of the components $\theta_{\alpha_1 \ldots \alpha_k}$ and let Φ^i be the analytic prolongations of the maps φ^i. Note that $\Phi^i(Y^\alpha) = \varphi^{ia}(Y^\alpha) e_a = \varphi^{ia}(y^{\alpha b}) e_a$ is the local representation of the map $T^{\mathbf{A}}\varphi : T^{\mathbf{A}}N \to T^{\mathbf{A}}M$, considered as an $\mathbf{A}$-smooth map and the functions

$\varphi^{ia}(y^{\alpha b})$ are the local representations of $T^{\mathbf{A}}\varphi$, considered as a map between real smooth manifolds. From Proposition 1.1 it follows that

$$\Theta_{\alpha_1\dots\alpha_k} = \frac{\partial \Phi^{i_1}}{\partial Y^{\alpha_1}} \dots \frac{\partial \Phi^{i_k}}{\partial Y^{\alpha_k}} \Xi_{i_1\dots i_k}.$$

Since $\delta^a = \delta^a_0$ with respect to the Jordan-Hölder basis, (1.6) implies that

$$\frac{\partial \Phi^{i_\ell}}{\partial Y^{\alpha_j}} = \frac{\partial \Phi^{i_\ell}}{\partial y^{\alpha_j 0}} = \frac{\partial \Phi^{i_\ell}}{\partial y^{\alpha_j}}.$$

Thus

$$\Theta_{\alpha_1\dots\alpha_k} e_{b_1}\dots e_{b_k} = \frac{\partial \varphi^{i_1 c_1}}{\partial y^{\alpha_1}} e_{c_1} \dots \frac{\partial \varphi^{i_k c_k}}{\partial y^{\alpha_k}} e_{c_k} U_{i_1\dots i_k} e_{b_1}\dots e_{b_k}. \tag{3.2}$$

But $e_{b_1} e_{c_1} = \gamma^{a_1}_{b_1 c_1} e_{a_1}, \dots, e_{b_k} e_{c_k} = \gamma^{a_k}_{b_k c_k} e_{a_k}$, therefore the right-hand side of (3.2) takes the form

$$\frac{\partial \varphi^{i_1 c_1}}{\partial y^{\alpha_1}} \gamma^{a_1}_{b_1 c_1} \dots \frac{\partial \varphi^{i_k c_k}}{\partial y^{\alpha_k}} \gamma^{a_k}_{b_k c_k} \Xi_{i_1\dots i_k} e_{a_1}\dots e_{a_k}.$$

From Scheffers' equations (1.5) it follows that

$$\frac{\partial \varphi^{i_1 c_1}}{\partial y^{\alpha_1}} \gamma^{a_1}_{b_1 c_1} = \frac{\partial \varphi^{i_1 a_1}}{\partial y^{\alpha_1 b_1}}, \qquad \dots, \qquad \frac{\partial \varphi^{i_k c_k}}{\partial y^{\alpha_k}} \gamma^{a_k}_{b_k c_k} = \frac{\partial \varphi^{i_k a_k}}{\partial y^{\alpha_k b_k}},$$

hence,

$$\Theta_{\alpha_1\dots\alpha_k} e_{b_1}\dots e_{b_k} = \frac{\partial \varphi^{i_1 a_1}}{\partial y^{\alpha_1 b_1}} \dots \frac{\partial \varphi^{i_k a_k}}{\partial y^{\alpha_k b_k}} \Xi_{i_1\dots i_k} e_{a_1}\dots e_{a_k}.$$

Then, by (3.1), we have

$$\theta_{\alpha_1 b_1\dots\alpha_k b_k} = \frac{\partial \varphi^{i_1 a_1}}{\partial y^{\alpha_1 b_1}} \dots \frac{\partial \varphi^{i_k a_k}}{\partial y^{\alpha_k b_k}} \xi_{i_1 a_1\dots i_k a_k}, \tag{3.3}$$

i.e. $(T^{\mathbf{A}}\varphi)^*(\xi^C) = \theta^C = (\varphi^*\xi)^C$.

In particular, if φ is a local diffeomorphism (coordinate transformation) of M, i.e., $x^{j'} = \varphi^{j'}(x^i)$, then

$$\xi_{i_1 a_1\dots i_k a_k} = \frac{\partial \varphi^{j'_1 b_1}}{\partial x^{i_1 a_1}} \dots \frac{\partial \varphi^{j'_k b_k}}{\partial x^{i_k a_k}} \xi_{j'_1 b_1\dots j'_k b_k}.$$

This implies that $\xi_{i_1 a_1\dots i_k a_k}$ is a well-defined tensor field of type $(k,0)$ on $T^{\mathbf{A}}M$. □

In order to construct the complete lifts of contravariant tensor fields we need to take another basis in $\mathbf{A}$: $e^a = q^{ab} e_b$, then $e_a = q_{ab} e^b$.

Let now $u \in \mathcal{V}^k(M)$ be a skew-symmetric contravariant tensor field on M; with respect to the local coordinates,

$$u = u^{i_1\dots i_k} \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_k}}.$$

Let $U^{i_1\dots i_k} = (u^{i_1\dots i_k})^{\mathbf{A}}$ be the analytic prolongations of $u^{i_1\dots i_k}$ and let $U^{i_1\dots i_k}e^{a_1}\dots e^{a_k} = U_b^{i_1a_1\dots i_ka_k}e^b$, where $U_b^{i_1a_1\dots i_ka_k} \in \mathbf{R}$. Denote

$$u^{i_1a_1\dots i_ka_k} := U_b^{i_1a_1\dots i_ka_k}\delta^b. \tag{3.4}$$

We define the *complete lift* u^C of u by

$$u^C = u^C_{\mathbf{A}} := u^{i_1a_1\dots i_ka_k}\frac{\partial}{\partial x^{i_1a_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_ka_k}}.$$

One can easily check that this definition also does not depend on the choice of a basis in $\mathbf{A}$.

Proposition 3.2. *u^C is a well-defined skew-symmetric contravariant tensor field of degree k on $T^{\mathbf{A}}M$. If $\varphi : M \to N$ is a smooth map and u is φ-related to a tensor filed v on N, then u^C is $(T^{\mathbf{A}}\varphi)$-related with v^C.*

Proof. Let N be another smooth manifold with local coordinates (y^α) and $\varphi : M \to N$ be a smooth map which is given by $y^\alpha = \varphi^\alpha(x^i)$ with respect to the local coordinates. Let $v \in \mathcal{V}^k(N)$ be another tensor field and let u and v be φ-related ([13, 14]). Then, in local coordinates we have

$$v^{\alpha_1\dots\alpha_k} = \frac{\partial\varphi^{\alpha_1}}{\partial x^{i_1}}\dots\frac{\partial\varphi^{\alpha_k}}{\partial x^{i_k}}u^{i_1\dots i_k}$$

Denote the analytic prolongations of $v_{\alpha_1\dots\alpha_k}$ by $V_{\alpha_1\dots\alpha_k}$ and the analytic prolongations of the maps φ^i by Φ^i. Then

$$V^{\alpha_1\dots\alpha_k} = \frac{\partial\Phi^{\alpha_1}}{\partial X^{i_1}}\dots\frac{\partial\Phi^{\alpha_k}}{\partial X^{i_k}}U^{i_1\dots i_k},$$

and

$$V^{\alpha_1\dots\alpha_k}e^{b_1}\dots e^{b_k} = \frac{\partial\varphi^{\alpha_1c_1}}{\partial x^{i_1}}e_{c_1}\dots\frac{\partial\varphi^{\alpha_kc_k}}{\partial x^{i_k}}e_{c_k}U^{i_1\dots i_k}e^{b_1}\dots e^{b_k}.$$

We have $e^{b_r}e_{c_r} = q^{b_rs_r}e_{s_r}e_{c_r} = q^{b_rs_r}\gamma^{d_r}_{s_rc_r}e_{d_r} = q^{b_rs_r}\gamma^{d_r}_{s_rc_r}q_{a_rd_r}e^{a_r}$ for each $r = 1,\dots,k$. Let us show that $q^{bs}\gamma^d_{sc}q_{ad} = \gamma^b_{ac}$. Indeed, the contraction of the left-hand side with q_{bf} gives $q^{bs}\gamma^d_{sc}q_{ad}q_{bf} = \gamma^d_{fc}q_{ad}$, which coincides with $\gamma^b_{ac}q_{bf}$ by (1.9). Thus,

$$\begin{aligned} V^{\alpha_1\dots\alpha_k}e^{b_1}\dots e^{b_k} &= \frac{\partial\varphi^{\alpha_1c_1}}{\partial x^{i_1}}e_{c_1}\dots\frac{\partial\varphi^{\alpha_kc_k}}{\partial x^{i_k}}e_{c_k}U^{i_1\dots i_k}e^{b_1}\dots e^{b_k} = \\ &= \gamma^{b_1}_{a_1c_1}\frac{\partial\varphi^{\alpha_1c_1}}{\partial x^{i_1}}\dots\gamma^{b_k}_{a_kc_k}\frac{\partial\varphi^{\alpha_kc_k}}{\partial x^{i_k}}U^{i_1\dots i_k}e^{a_1}\dots e^{a_k} = \\ &= \frac{\partial\varphi^{\alpha_1b_1}}{\partial x^{i_1a_1}}\dots\frac{\partial\varphi^{\alpha_kb_k}}{\partial x^{i_ka_k}}U^{i_1\dots i_k}e^{a_1}\dots e^{a_k}, \end{aligned}$$

hence, by (3.4),

$$v^{\alpha_1 b_1 \dots \alpha_k b_k} = \frac{\partial \varphi^{\alpha_1 b_1}}{\partial x^{i_1 a_1}} \cdots \frac{\partial \varphi^{\alpha_k b_k}}{\partial x^{i_k a_k}} u^{i_1 a_1 \dots i_k a_k}.$$

Therefore, u^C is $(T^{\mathbf{A}}\varphi)$-related to v^C.

In particular, if φ is a local diffeomorphism of M, i.e. $x^{j'} = \varphi^{j'}(x^i)$, then

$$u^{j'_1 b_1 \dots j'_k b_k} = \frac{\partial \varphi^{j'_1 b_1}}{\partial x^{i_1 a_1}} \cdots \frac{\partial \varphi^{j'_k b_k}}{\partial x^{i_k a_k}} u^{i_1 a_1 \dots i_k a_k}.$$

This means that $u_{i_1 a_1 \dots i_k a_k}$ is a well-defined tensor field of type $(0, k)$ on $T^{\mathbf{A}}M$. Since u is skew-symmetric and the multiplication in $\mathbf{A}$ is commutative, u^C is also skew-symmetric. □

Let

$$[\cdot, \cdot] : \mathcal{V}^k(M) \times \mathcal{V}^\ell(M) \to \mathcal{V}^{k+\ell-1}(M),$$

be the *Schouten-Nijenhuis bracket* on the multivector fields (a generalization of Lie bracket of vector fields [9, 10, 14, 18]).

Proposition 3.3. *The complete lift is compatible with the Schouten-Nijenhuis bracket, i.e.,*

$$[u, v]^C = [u^C, v^C]. \tag{3.5}$$

Proof. First let us derive an auxiliary formula. Let f be a real-valued function on M and $F = F^a e_a = F_b e^b$ be its analytic prolongation. Then $F_b = q_{ab} F^a$. From the Scheffers' conditions (1.4) it follows that $\partial_c F_b = \gamma^a_{cd} q_{ab} \delta^g \partial_g F^d = \gamma^a_{cd} q_{ab} \delta^g q^{dr} \partial_g F_r$. Contracting with δ^b, by (1.11), we obtain $\partial_c(\delta^b F_b) = \delta^b q_{ab} \delta^g \gamma^a_{cd} q^{dr} \partial_g F_r = p_a \gamma^a_{cd} q^{dr} \delta^g \partial_g F_r = q_{cd} q^{dr} \delta^g \partial_g F_r = \delta^g \partial_g F_c$. Thus,

$$\partial_c(\delta^b F_b) = \delta^g \partial_g F_c. \tag{3.6}$$

Let $u \in \mathcal{V}^g(M)$, $v \in \mathcal{V}^h(M)$ be multivector fields. With respect to the local coordinates,

$$\begin{aligned} [u, v]^{k_2 \dots k_{g+h}} = \; & \frac{1}{(g-1)!h!} \varepsilon^{k_2 \dots k_{g+h}}_{i_2 \dots i_g j_1 \dots j_h} u^{r i_2 \dots i_g} \frac{\partial}{\partial x^r} v^{j_1 \dots j_h} + \\ & + \frac{(-1)^g}{g!(h-1)!} \varepsilon^{k_2 \dots k_{g+h}}_{i_1 \dots i_g j_2 \dots j_h} v^{r j_2 \dots j_h} \frac{\partial}{\partial x^r} u^{i_1 \dots i_g}, \end{aligned} \tag{3.7}$$

where $\varepsilon^{j_1\dots j_s}_{i_1\dots i_s} = \delta^{j_1}_{i_1}\dots\delta^{j_s}_{i_s}$ (see, e.g., [9]). By Theorem 1.1, the same formula holds for the analytic prolongations:

$$[U,V]^{k_2\dots k_{g+h}} = \frac{1}{(g-1)!h!}\varepsilon^{k_2\dots k_{g+h}}_{i_2\dots i_g i_{g+1}\dots i_{g+h}} U^{ri_2\dots i_g}\frac{\partial}{\partial X^r}V^{i_{g+1}\dots i_{g+h}} + \\ + \frac{(-1)^g}{g!(h-1)!}\varepsilon^{k_2\dots k_{g+h}}_{i_1\dots i_g i_{g+2}\dots i_{g+h}} V^{ri_{g+2}\dots i_{g+h}}\frac{\partial}{\partial X^r}U^{i_1\dots i_g}, \tag{3.8}$$

Let us multiply both sides of (3.8) by $e^{a_2}\dots e^{a_{g+h}}$ and then contract with δ^m. In the left-hand side we get $([u,v]^C)^{k_2 a_2\dots k_{g+h}a_{g+h}}$. Consider the first summand in the right-hand side (without ε):

$$\frac{1}{(g-1)!h!}U^{ri_2\dots i_g}e^{a_2}\dots e^{a_g}\frac{\partial}{\partial X^r}V^{i_{g+1}\dots i_{g+h}}e^{a_{g+1}}\dots e^{a_{g+h}} =$$
$$\frac{1}{(g-1)!h!}U^{ri_2\dots i_g}e^{a_2}\dots e^{a_g}\delta^b\frac{\partial}{\partial x^{rb}}V^{a_{g+1}i_{g+1}\dots a_{g+h}i_{g+h}} \quad \text{by (1.6)} =$$
$$\frac{1}{(g-1)!h!}U^{ri_2\dots i_g}e^a e^{a_2}\dots e^{a_g}\delta^b\frac{\partial}{\partial x^{rb}}V_a^{a_{g+1}i_{g+1}\dots a_{g+h}i_{g+h}} =$$
$$\frac{1}{(g-1)!h!}U^{ri_2\dots i_g}e^a e^{a_2}\dots e^{a_g}\frac{\partial}{\partial x^{ra}}(\delta^b V_b^{a_{g+1}i_{g+1}\dots a_{g+h}i_{g+h}}) \quad \text{by (3.6)} =$$
$$\frac{1}{(g-1)!h!}U_m^{ra\,i_2a_2\dots i_ga_g}e^m\frac{\partial}{\partial x^{ra}}((v^C)^{a_{g+1}i_{g+1}\dots a_{g+h}i_{g+h}}).$$

Contracting with δ^m we get

$$\frac{1}{(g-1)!h!}(u^C)^{ra\,i_2a_2\dots i_ga_g}\frac{\partial}{\partial x^{ra}}\Big((v^C)^{a_{g+1}i_{g+1}\dots a_{g+h}i_{g+h}}\Big).$$

The commutativity of multiplication in $\mathbf{A}$ yields that

$$\varepsilon^{k_2\dots k_{g+h}}_{i_2\dots i_{g+h}} = \varepsilon^{k_2a_2\dots k_{g+h}a_{g+h}}_{i_2a_2\dots i_{g+h}a_{g+h}}.$$

In the same manner we deal with the second summand in (3.8) and then obtain. Therefore

$$([u,v]^C)^{k_2a_2\dots k_{g+h}a_{g+h}} =$$
$$\tfrac{1}{(g-1)!h!}\varepsilon^{k_2a_2\dots k_{g+h}a_{g+h}}_{i_2a_2\dots i_ga_gi_{g+1}a_{g+1}\dots i_{g+h}a_{g+h}}(u^C)^{ra\,i_2a_2\dots i_ga_g}\tfrac{\partial}{\partial x^{ra}}\Big((v^C)^{a_{g+1}i_{g+1}\dots a_{g+h}i_{g+h}}\Big)+$$
$$+\tfrac{(-1)^g}{g!(h-1)!}\varepsilon^{k_2a_2\dots k_{g+h}a_{g+h}}_{i_1a_1\dots i_ga_gi_{g+2}a_{g+2}\dots i_{g+h}a_{g+h}}(v^C)^{ra\,i_{g+2}a_{g+2}\dots i_{g+h}a_{g+h}}\tfrac{\partial}{\partial x^{ra}}\Big((u^C)^{i_1a_1\dots i_ga_g}\Big),$$

which coincides with (3.5). □

The construction of the vertical lift of multivector fields to the tangent bundle (see [23]) also may be generalized to Weil bundles in the following way. Let $u = u^{i_1\dots i_k}\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_k}} \in \mathcal{V}^k(M)$. We define the *vertical lift* $u^V \in \mathcal{V}^k(T^{\mathbf{A}}M)$ of u by

$$u^V = u^V_{\mathbf{A}} := u^{i_1\dots i_k}\frac{\partial}{\partial x^{i_1 n}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_k n}}. \tag{3.9}$$

Proposition 3.4. *The vertical lift u^V is a well-defined multivector field on $T^{\mathbf{A}}M$.*

Proof. Let $x^{i'} = x^{i'}(x^i)$ be a coordinate change on M, then

$$u^{i'_1\dots i'_k} = \frac{\partial x^{i'_1}}{\partial x^{i_1}}\cdots\frac{\partial x^{i'_k}}{\partial x^{i_k}}u^{i_1\dots i_k}.$$

Thus it suffices to prove that

$$\frac{\partial}{\partial x^{i'_s n}} = \frac{\partial x^{i_s}}{\partial x^{i'_s}}\frac{\partial}{\partial x^{i_s n}}. \tag{3.10}$$

Let us determine the corresponding change of coordinates $x^{i'a} = x^{i'a}(x^{ib})$ on $T^{\mathbf{A}}M$. By (1.7),

$$X^{i'} = x^{i'} + x^{i'\hat{a}}e_{\hat{a}} = x^{i'} + \sum_{|p|=1}^{q}\frac{1}{p!}\frac{D^p x^{i'}}{Dx^p}\overset{\circ}{X}{}^p. \tag{3.11}$$

Hence, for $T^{\mathbf{A}}M$ we have

$$x^{i'} = x^{i'}(x^i),$$
$$x^{i'\hat{a}} = f^{i'\hat{a}}(x^{ib}), \qquad \hat{a} = 1,\dots,n.$$

Let us show that

$$\frac{\partial f^{i'\hat{a}}}{\partial x^{ib}} = 0 \quad \text{for} \quad b > \hat{a}.$$

Indeed, the coefficients $\frac{1}{p!}\frac{D^p x^{i'}}{Dx^p}$ in (3.11) depend only on $x^i = x^{i0}$, while the expression $(\overset{\circ}{X}{}^1)^{p_1}\dots(\overset{\circ}{X}{}^m)^{p_m}$ contains x^{ib} only as a coefficient at e_b in $\overset{\circ}{X}{}^i$. Since $\gamma^s_{c\hat{d}} = 0$, for $c > s$ the coefficient at $e_{\hat{a}}$ which appears when we expand brackets in $(\overset{\circ}{X}{}^1)^{p_1}\dots(\overset{\circ}{X}{}^m)^{p_m}$ does not depend on x^{ib}. Moreover, it can depend on $x^{i\hat{a}}$ only for the summand corresponding to $|p| = 1$ (because for $|p| \geq 2$ the expression $x^{i\hat{a}}e_{\hat{a}}$ will be multiplied by some element of $\overset{\circ}{\mathbf{A}}$ and thus after the expanding brackets the coefficient at $e_{\hat a}$ does not depend on $x^{i\hat a}$). For the case $|p| = 1$ we obtain the only summand $\frac{\partial x^{i'}}{\partial x^i} x^{i\hat a} e_{\hat a}$ depending on $x^{i\hat a}$. Therefore,

$$\frac{\partial x^{i'\hat a}}{\partial x^{i\hat a}} = \frac{\partial x^{i'}}{\partial x^i} \quad \text{for any} \quad \hat a = 1, \dots, n.$$

Hence, the Jacobi matrix of the coordinate transformation $x^{i'a} = x^{i'a}(x^{ib})$ on $T^{\mathbf{A}}M$ has the following block structure:

$$\begin{array}{|c|c|c|c|c|c|}
\hline
\frac{\partial x^{i'}}{\partial x^i} & 0 & 0 & \cdots & 0 & 0 \\
\hline
* & \frac{\partial x^{i'}}{\partial x^i} & 0 & \cdots & 0 & 0 \\
\hline
* & * & \ddots & \ddots & \vdots & \vdots \\
\hline
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\hline
* & \cdots & \ddots & * & \frac{\partial x^{i'}}{\partial x^i} & 0 \\
\hline
* & * & \cdots & * & * & \frac{\partial x^{i'}}{\partial x^i} \\
\hline
\end{array},$$

where $*$ denotes the blocks which are unessential for our consideration. Now (3.10) is obvious. □

Proposition 3.5. *For any $u, v \in \mathcal{V}^*(M)$ there holds*

$$[u^V, v^V] = 0. \tag{3.12}$$

Proof. This follows easily from (3.7). Indeed,

$$(u^V)^{i_1 a_1 \dots i_k a_k} = 0,$$

if at least one of indices $a_1, \dots, a_k$ is not equal to n. But then

$$\frac{\partial v^{j_1 b_1 \dots j_\ell b_\ell}}{\partial x^{rn}} = 0.$$

Thus, all the summands in the right-hand side of (3.7) are zero. □

4. Poisson structures on $T^{\mathbf{A}}M$

Recall that a *Poisson bracket* on a smooth manifold M is a bilinear skew-symmetric mapping $\{\,,\,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, satisfying the Leibniz rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

and the Jacobi identity

$$\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}=0.$$

A *Poisson manifold* is a smooth manifold M endowed with a Poisson bracket. The Poisson bracket on M uniquely defines a bivector field

$$w=w^{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}\in\mathcal{V}^2(M) \tag{4.1}$$

such that

$$\{f,g\}=i(w)(df\wedge dg) \tag{4.2}$$

for any $f,g\in C^\infty(M)$, where $i(w):\Omega^m(M)\to\Omega^{m-2}(M)$ is the inner product; in local coordinates

$$(i(w)\alpha)_{i_1\dots i_{m-2}}=w^{jk}\alpha_{jki_1\dots i_{m-2}}.$$

This bivector field is usually called the *Poisson bivector.* It is known (see, e.g., [7, 9, 18]) that the bracket (4.2) on $C^\infty(M)$ satisfies the Jacobi identity if and only if

$$[w,w]=0. \tag{4.3}$$

In local coordinates this is written as

$$w^{js}\frac{\partial w^{k\ell}}{\partial x^s}+w^{ks}\frac{\partial w^{\ell j}}{\partial x^s}+w^{\ell s}\frac{\partial w^{jk}}{\partial x^s}=0.$$

In what follows we will denote the Poisson manifold by (M,w).

To each function $f\in C^\infty(M)$ there corresponds the *Hamiltonian vector field* $X_f=X_f^w\in\mathcal{V}^1(M)$ defined by $X_f(g):=\{f,g\}$. Locally, Hamiltonian vector fields on (M,w) have the form [14, 11, 18]

$$X_f^w=\{f,\cdot\}_w=\frac{\partial f}{\partial x^i}w^{ij}\frac{\partial}{\partial x^j}\qquad\Big(f\in C^\infty(M)\Big). \tag{4.4}$$

For a Poisson manifold (M,w), the Lichnerowicz-Poisson coboundary operator

$$\sigma=\sigma_w:\mathcal{V}^k(M)\to\mathcal{V}^{k+1}(M),$$

is defined by $\sigma u:=[w,u]$. Because the Schouten-Nijenhuis bracket satisfies the super-Jacobi identity (see [9, 14])

$$(-1)^{|u|\cdot|v|}[[v,y],u]+(-1)^{|v|\cdot|y|}[[y,u],v]+(-1)^{|y|\cdot|u|}[[u,v],y]=0,$$

one has $\sigma\circ\sigma=0$. Thus the cohomology spaces

$$H^k_{LP}(M,w):=\frac{\ker\sigma:\mathcal{V}^k(M)\to\mathcal{V}^{k+1}(M)}{\mathrm{im}\ \sigma:\mathcal{V}^{k-1}(M)\to\mathcal{V}^k(M)},$$

called the *Lichnerowicz-Poisson cohomology spaces* of Poisson manifold (M,w) are defined. The problem of computing this cohomology is very difficult (see, e.g., [3, 17, 22]).

One can easily see that $[w, f] = X_f$ for any $f \in C^\infty(M)$, hence Hamiltonian vector fields form the space of 1-coboundaries of σ [14].

Let (M, w) be a Poisson manifold and $\mathbf{A}$ be a Frobenius Weil algebra. Consider the complete lift w^C of w to $T^{\mathbf{A}}M$. By (3.5) and (4.3), w^C is a Poisson bivector on $T^{\mathbf{A}}M$.

Proposition 4.1. *The complete lift of multivector fields induces a homomorphism of Lichnerowicz-Poisson cohomology*

$$[u] \in H^*_{LP}(M, w) \longmapsto [u^C] \in H^*_{LP}(T^{\mathbf{A}}M, w^C). \tag{4.5}$$

Proof. From (3.5) it follows that

$$(\sigma_w u)^C = \sigma_{w^C}\, u^C,$$

which implies that (4.5) is a homomorphism. □

Let us find how the complete lift w^C does depend on the choice of a Frobenius covector p on $\mathbf{A}$. We denote by t the corresponding vector defined by (2.12). Let w^{ij}_s be the components of the analytic prolongation $W^{ij} = (w^{ij})^{\mathbf{A}}$, i.e., $W^{ij} = \sum_{s\geq 0} w^{ij}_s e_s$. In what follows we will omit the sign of summation over s. Then $W^{ij}e^a e^b = w^{ij}_s e_s e^a e^b = w^{ij}_s q^{ac} q^{bk} e_s e_c e_d = w^{ij}_s q^{ac} q^{bk} \gamma^d_{sc} \gamma^\ell_{kd} e_\ell = w^{ij}_s q^{ac} q^{bk} \gamma^d_{sc} \gamma^\ell_{kd} q_{\ell r} e^r$. Contracting with δ^r, we obtain

$$w^{iajb} = w^{ij}_s q^{ac} q^{bk} \gamma^d_{sc} \gamma^\ell_{kd} q_{\ell r} \delta^r = w^{ij}_s q^{ac} q^{bk} \gamma^d_{sc} \gamma^\ell_{kd} p_\ell =$$

$$w^{ij}_s \overline{\gamma}^{ac}_r t^r \overline{\gamma}^{bk}_f t^f \gamma^d_{sc} \gamma^\ell_{kd} p_\ell = w^{ij}_s t^r t^f h^{ag} \gamma^c_{gr} h^{eb} \gamma^k_{fe} \gamma^d_{sc} \gamma^\ell_{kd} p_\ell \text{ (by (2.5))} =$$

$$w^{ij}_s t^r t^f h^{ag} h^{eb} \gamma^c_{gr} \gamma^k_{de} \gamma^\ell_{kf} \gamma^d_{sc} p_\ell \text{ (by (1.2))} = w^{ij}_s t^r h^{ag} h^{eb} \gamma^c_{gr} \gamma^k_{de} \delta^n_k \gamma^d_{sc} =$$

(by (2.12)) =

$$w^{ij}_s t^r h^{ag} h^{eb} \gamma^c_{gr} h_{de} \gamma^d_{sc} = w^{ij}_s t^r h^{ag} \delta^b_d \gamma^c_{gr} \gamma^d_{sc} = w^{ij}_s t^r h^{ag} \gamma^c_{gr} \gamma^b_{sc} =$$

$$w^{ij}_s t^r \overline{\gamma}^{ac}_r \gamma^b_{sc}.$$

Thus, for any Frobenius covector p, the complete lift w^C is the linear combination

$$w^C = t^k w^C_k, \tag{4.6}$$

where

$$w^C_k = \sum_{s=0}^{n} w^{ij}_s \overline{\gamma}^{ac}_k \gamma^b_{cs} \frac{\partial}{\partial x^{ia}} \wedge \frac{\partial}{\partial x^{jb}}. \tag{4.7}$$

Let us show that each of w^C_k is a multivector field and that

$$[w^C_k, w^C_k] = [w^C_k, w^C_\ell] = 0 \tag{4.8}$$

for any $k, \ell = 0, \dots, n$. To this end we will express each of w_k^C in terms of complete lifts w^C corresponding to different Frobenius covectors. We will assume the basis (1.1) to be chosen in such a way that $p_{(0)} = (0, \dots, 0, 1)$ is a Frobenius covector. Recall that the corresponding vector $t_{(0)}$ is $(1, 0, \dots, 0)$. Since the determinant $\det \|\gamma_{ab}^c t^b\|$ is a continuous function in t and does not vanish at $t = t_{(0)}$, this determinant does not vanish in a neighborhood of this point. Therefore one can find $\varepsilon > 0$ such that this determinant is nonzero for each of the vectors $t_{(k)} = (1, 0 \dots, 0, \varepsilon, 0, \dots, 0)$ (ε at the kth place), $t_{(k-)} = (1, 0 \dots, 0, -\varepsilon, 0, \dots, 0)$ ($-\varepsilon$ at the kth place) and

$$t_{(k,\ell)} = (1, 0 \dots, 0, \varepsilon, 0, \dots, 0, \varepsilon, 0, \dots, 0)$$

(ε at the kth and ℓth places), $k, \ell = 1, \dots, n$. Hence the corresponding covectors $p_{(k)}$ $p_{(k,\ell)}$ are Frobenius covectors. For each vector t under consideration the complete lift w^C is a Poisson bivector, which implies

$$[w^C, w^C] = [t^k w_k^C, t^\ell w_\ell^C] = 0. \tag{4.9}$$

Substituting $t = t_{(0)}$ into (4.6) and (4.9), we see that $w_0^C \in \mathcal{V}^2(T^{\mathbf{A}}M)$ and $[w_0^C, w_0^C] = 0$. Now substitute $t = t_{(k)}$ and $t = t_{(k-)}$, which gives $w_0^C + \varepsilon w_k^C \in \mathcal{V}^2(T^{\mathbf{A}}M)$, therefore $w_k^C \in \mathcal{V}^2(T^{\mathbf{A}}M)$ and

$$[w_0^C + \varepsilon w_k^C, w_0^C + \varepsilon w_k^C] = [w_0^C - \varepsilon w_k^C, w_0^C - \varepsilon w_k^C] = 0.$$

Expanding these equations yields $2\varepsilon[w_k^C, w_0^C] + \varepsilon^2[w_k^C, w_k^C] = -2\varepsilon[w_k^C, w_0^C] + \varepsilon^2[w_k^C, w_k^C] = 0$, which implies $[w_0^C, w_k^C] = [w_k^C, w_k^C] = 0$. Finally, substituting $t = t_{(k,\ell)}$ into (4.9) one gets $[w_k^C, w_\ell^C] = 0$. Thus, the following theorem is valid.

Theorem 4.1. *Let (M, w) be a Poisson manifold and $T^{\mathbf{A}}M$ be its Weil bundle for a Frobenius Weil algebra $\mathbf{A}$. Then for each Frobenius covector p on $\mathbf{A}$ the complete lift w^C of Poisson bivector w to $T^{\mathbf{A}}M$ is of the form*

$$w^C = t^k w_k^C,$$

where $t = (t^k)$ and w_k^C are defined by (2.12) and (4.7) respectively. Moreover,

$$[w_k^C, w_k^C] = [w_k^C, w_\ell^C] = 0$$

for any $k, \ell = 0, \dots, n$.

Remark 4.1. One can easily see that $w_n^C = w^V$. Indeed, in this case $k = n$ in (4.7). But (2.4) implies that $\overline{e}^n = e_0$ and that each of $\overline{e}^a$, $a = 0, 1, \dots, n-1$, does not contain e_0 in its decomposition. Hence the only nonzero component of $\overline{\gamma}_n^{ac}$ is $\overline{\gamma}_n^{nn} = 1$. Then γ_{nd}^b is 1 only for $b = n$, $d = 0$ and $\gamma_{nd}^b = 0$ for any other values of b and d. Consequently,

$$w_n^C = w^{ij} \frac{\partial}{\partial x^{in}} \wedge \frac{\partial}{\partial x^{jn}}.$$

Thus, Theorem 4.1 (as well as Proposition 3.5) implies that w^V is also a Poisson bivector on $T^{\mathbf{A}}M$.

Example 4.1. Let $\mathbf{A}$ be the algebra of plural numbers $\mathbf{R}(\varepsilon^n)$. Choose a Jordan-Hölder basis $e_0 = 1$, $e_a = \varepsilon^a$, $a = 1, \ldots, n$ in it. The explicit form of the analytic prolongations $W^{ij} = \sum_{s \geq 0} w_s^{ij} \varepsilon^s$ can be found by (1.7), for instance,

$$\begin{aligned}
w_0^{ij} &= w^{ij}, \\
w_1^{ij} &= x^{k1} \frac{\partial w^{ij}}{\partial x^k}, \\
w_2^{ij} &= x^{k1} x^{\ell 1} \frac{\partial^2 w^{ij}}{\partial x^k \partial x^\ell} + x^{k2} \frac{\partial w^{ij}}{\partial x^k}, \quad \text{etc.}
\end{aligned}$$

The corresponding bivectors w_k^C are (here w_k^{ij} means the square block $\|w_k^{ij}\|$)

$$w_0^C = \begin{pmatrix}
0 & 0 & \ldots & 0 & 0 & w^{ij} \\
0 & 0 & \ldots & 0 & w^{ij} & w_1^{ij} \\
\vdots & \vdots & \iddots & \iddots & \iddots & \vdots \\
0 & 0 & \iddots & \iddots & \ldots & \ldots \\
0 & w^{ij} & w_1^{ij} & \ldots & \ldots & w_{n-1}^{ij} \\
w^{ij} & w_1^{ij} & \ldots & \ldots & w_{n-1}^{ij} & w_n^{ij}
\end{pmatrix},$$

$$w_k^C = \begin{pmatrix}
0 & 0 & \ldots & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & \ldots & 0 & 0 & \ldots & \vdots & \vdots \\
\vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \ldots & 0 \\
0 & 0 & \iddots & \iddots & 0 & 0 & \ldots & 0 & w^{ij} \\
0 & 0 & \iddots & \iddots & 0 & \ldots & 0 & w^{ij} & w_1^{ij} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \iddots & \iddots & \vdots \\
0 & 0 & \ldots & \ldots & \iddots & \iddots & \iddots & \ldots & \vdots \\
0 & 0 & \ldots & 0 & w^{ij} & w_1^{ij} & \ldots & \ldots & w_{n-k-1}^{ij} \\
0 & \ldots & 0 & w^{ij} & w_1^{ij} & \ldots & \ldots & w_{n-k-1}^{ij} & w_{n-k}^{ij}
\end{pmatrix}, \quad k = 1, \ldots, n-1,$$

$$w_n^C = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & 0 & w^{ij} \end{pmatrix}.$$

In particular, for the case $\mathbf{A} = \mathbf{R}(\varepsilon)$, which corresponds to the tangent bundle TM and the standard Frobenius covector $p = (0, 1)$, the complete lift w^C is

$$w^C = w^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^j} + \frac{1}{2} y^k \frac{\partial w^{ij}}{\partial x^k} \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j}.$$

This Poisson bivector was studied by many authors, see, e.g., [1, 4, 11].

Example 4.2. Consider the algebra $\mathbf{A}(\Lambda, n)$. We will denote the elements of the inverse matrix Λ^{-1} by λ^{ab}.

By (1.7), denote

$$w_a^{ij} = x^{ka} \frac{\partial w^{ij}}{\partial x^k}, \qquad w_n^{ij} = x^{kn} \frac{\partial w^{ij}}{\partial x^k} + \lambda_{ab} x^{ka} x^{\ell b} \frac{\partial^2 w^{ij}}{\partial x^k \partial x^\ell}.$$

Then, for this algebra (4.7) implies that

$$w_0^C = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & w^{ij} \\ 0 & \lambda^{11} w^{ij} & \lambda^{12} w^{ij} & \dots & \lambda^{1,n-1} w^{ij} & w_1^{ij} \\ 0 & \lambda^{21} w^{ij} & \lambda^{22} w^{ij} & \dots & \lambda^{2,n-1} w^{ij} & w_2^{ij} \\ \vdots & \dots & \dots & \dots & \dots & \vdots \\ 0 & \lambda^{n-1,1} w^{ij} & \lambda^{n-1,2} w^{ij} & \dots & \lambda^{n-1,n-1} w^{ij} & w_{n-1}^{ij} \\ w^{ij} & w_1^{ij} & w_2^{ij} & \dots & w_{n-1}^{ij} & w_n^{ij} \end{pmatrix},$$

$$w_b^C = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \dots & w^{ij} \\ \vdots & \vdots & \dots & \dots & \dots & \vdots \\ 0 & \dots & \dots & \dots & 0 & 0 \\ 0 & \dots & w^{ij} & \dots & 0 & \sum_{a=1}^{n-1} \lambda_{ab} w_a^{ij} \end{pmatrix}, b = 1, \dots, n-1, w^{ij} \text{ at the } b\text{th place}$$

$$w_n^C = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \dots & \dots & \vdots \\ 0 & \dots & \dots & 0 \\ 0 & \dots & 0 & w^{ij} \end{pmatrix}.$$

5. Modular classes of Poisson structures on $T^{\mathbf{A}}M$

In the final part of the present paper we compute the modular classes of Poisson structures w_k^C for the case of weakly symmetric Frobenius Weil algebras.

Recall that if μ is a volume form on the oriented manifold M then the divergence $\mathrm{div}_\mu X$ of a vector field X is defined by

$$\mathcal{L}_X \mu = (\mathrm{div}_\mu X)\mu$$

and one has

$$\mathrm{div}_\mu(fX) = f\,\mathrm{div}_\mu X + Xf, \qquad f \in C^\infty(M).$$

Therefore for a Poisson manifold (M, w) with the volume form μ the operator

$$\Delta_\mu : f \in C^\infty(M) \longmapsto \mathrm{div}_\mu X_f \in C^\infty(M)$$

is defined, where X_f is a Hamiltonian vector field of f. Easy computations show that Δ_μ is a derivation on $C^\infty(M)$ and, hence, a vector field on M [21]. This vector field is called the *modular vector field* of oriented Poisson manifold (M, w, μ).

The modular vector field satisfies $\sigma\Delta_\mu = 0$ [6]. If we replace μ with any other volume form $a\mu$, where $a \in C^\infty(M)$ is a positive function, then the modular vector field changes to $\Delta_{a\mu} = \Delta_\mu + H_{-\log a}$ [21]. As far as Hamiltonian vector fields are 1-coboundaries of σ, this implies that the set of modular vector fields for all volume forms on M is an element of $H^1_{LP}(M, w)$. This cohomology class is called the *modular class* of the Poisson manifold (M, w).

Let $g = g_{ij}dx^i \otimes dx^j$ be a Riemannian metric on an m-dimensional oriented manifold M. Then

$$dV_g = \sqrt{\det\, g}\; dx^1 \wedge \dots \wedge dx^m$$

is a volume form on M. Let g^C be a complete lift of g.

Proposition 5.1. *For a weakly symmetric Frobenius Weil algebra* $\mathbf{A}$ *the complete lift* g^C *is a metric on* $T^{\mathbf{A}}M$*, moreover,*

$$\det\, g^C = M(\det\, g)^{n+1}, \tag{5.1}$$

where M *is some constant (depending only on* $\mathbf{A}$*).*

Proof. We choose the standard Frobenius covector $p_{(0)}$. Let $G_{ij} = g_{ij}^s e_s$ be the analytic prolongations of g_{ij}. Then $G_{ij} e_a e_b = G_{iajb}^c e_c$ and $g_{iajb} = G_{iajb}^c p_c = G_{iajb}^n$. Clearly, $g_{iajb} = g_{jbia}$ for all a, b, i, j. If $s > 0$, then $e_s \in \overset{\circ}{\mathbf{A}}$, hence the component g_{iajb} contains g_{ij}^s only if $e_a e_b \notin \overset{\circ}{\mathbf{A}}{}^q$. Therefore, if $e_a e_b \in \overset{\circ}{\mathbf{A}}{}^q$, then g_{iajb} depends only on $g_{ij}^0 = g_{ij}$ and the matrix $\|g_{iajb}\|$ has the following block structure:

$*$	$*$	$*$	$\cdots$	$*$	$*$	$\|g_{ij}\|$
$*$	$\cdots$	$\cdots$	$\iddots$	$*$	$\widehat{B}_1$	0
$*$	$\cdots$	$\iddots$	$\iddots$	$\widehat{B}_2$	0	0
$\vdots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\vdots$
$*$	$*$	$\widehat{B}_{q-2}$	$\iddots$	$\iddots$	$\cdots$	0
$*$	$\widehat{B}_{q-1}$	0	$\iddots$	$\cdots$	$\cdots$	0
$\|g_{ij}\|$	0	0	$\cdots$	$\cdots$	$\cdots$	0

where $\widetilde{B}_k = B_{q-k,k} \otimes \|g_{ij}\|$ and the symbol $\otimes$ denotes the tensor (Kronecker) product of matrices.

The determinant of this matrix is the product of the determinants of diagonal blocks: $\det \|g_{iajb}\| = \det \|g_{ij}\| \cdot \det(B_{1,q-1} \otimes \|g_{ij}\|) \cdot \dots \cdot \det(B_{q-1,1} \otimes \|g_{ij}\|) \cdot \det \|g_{ij}\|$. For any two matrices S and T of dimensions $k \times k$ and $\ell \times \ell$ respectively, one has

$$\det(S \otimes T) = (\det\, S)^k (\det\, T)^\ell.$$

We have $d_1(\mathbf{A}) + \dots + d_{q-1}(\mathbf{A}) = n-1$. Hence, $\det \|g_{iajb}\| = M(\det\, g)^{n+1}$, where $M = (\det\, B_{1,q-1})^{d_1(\mathbf{A})} \cdot \dots \cdot (\det\, B_{q-1,1})^{d_{q-1}(\mathbf{A})}$. $\square$

Let

$$\Phi = dV_{g^C} = \sqrt{\det\, g^C}\; dx^1 \wedge \dots dx^m \wedge \dots \wedge dx^{1n} \wedge \dots \wedge dx^{mn}$$

be the corresponding volume form on $T^{\mathbf{A}}M$.

Proposition 5.2. *Let* $\mathbf{A}$ $(\dim \overset{\circ}{\mathbf{A}} = n)$ *be a weakly symmetric Frobenius Weil algebra,* (M, w) *a Poisson manifold, and* $T^{\mathbf{A}}M$ *its Weil bundle.*

Then for a Poisson structure w_0^C on $T^{\mathbf{A}}M$ its modular vector field is

$$\Delta_\Phi^{T^{\mathbf{A}}M} = (n+1)(\Delta_{dV_g}^M)^V, \tag{5.2}$$

where V means the vertical lift. For each of the Poisson structures w_1^C, ..., w_n^C, the modular vector fields are zero.

Proof. From (4.4) it follows that the modular vector field of (M, w) is [11]

$$\Delta_{dV_g} = \sum_{j=1}^m \left(\frac{\partial w^{ij}}{\partial x^j} + w^{ij} \frac{\partial \ln \sqrt{\det\, g}}{\partial x^j} \right) \frac{\partial}{\partial x^i}.$$

Then (5.1) implies that

$$\frac{\partial \ln \sqrt{\det\, g^C}}{\partial x^{jb}} = \begin{cases} (n+1)\dfrac{\partial \ln \sqrt{\det\, g}}{\partial x^j}, & b = 0, \\ 0, & b = 1, 2 \dots, n. \end{cases} \tag{5.3}$$

Let w_k^{iajb} denote the components of w_k^C. At first, show that

$$\begin{gathered} \frac{\partial w_0^{iajb}}{\partial x^{jb}} = \begin{cases} \dfrac{\partial w^{ij}}{\partial x^j}, & a = n, \\ 0, & a = 0, 1 \dots, n-1, \end{cases} \\ \frac{\partial w_k^{iajb}}{\partial x^{jb}} = 0, \qquad k = 1, \dots, n. \end{gathered} \tag{5.4}$$

By (4.7), $w_k^{iajb} = \sum_{s=0}^n w_s^{ij} \overline{\gamma}_k^{ac} \gamma_{cs}^b$. The arguments similar to the proof of Proposition 3.4 show that $\frac{\partial w_s^{ij}}{\partial x^{jb}} = 0$ for $s < b$ and that $\frac{\partial w_b^{ij}}{\partial x^{jb}} = \frac{\partial w^{ij}}{\partial x^j}$. Moreover, $\gamma_{cs}^b = 0 \quad s > b$. Hence, the only nonzero summand in $\frac{\partial w_k^{iajb}}{\partial x^{jb}}$ corresponds to $s = b$. Therefore $c = 0$, otherwise $\gamma_{cs}^b = 0$. But $\overline{\gamma}_k^{a0}$ is not zero only for $a = n$ and $k = 0$ by virtue of (2.4) (since $\overline{e}^0 = e_n$, $\overline{e}^n = 1$). Hence $\frac{\partial w_k^{iajb}}{\partial x^{jb}} = 0$ for $k = 1, \dots, n$ and $\frac{\partial w_0^{iajb}}{\partial x^{jb}} = 0$ for $a \neq n$. As for $a = n$, we have

$$\frac{\partial w_0^{injb}}{\partial x^{jb}} = \frac{\partial w_b^{ij}}{\partial x^{jb}} = \frac{\partial w^{ij}}{\partial x^j}.$$

This completes the proof of (5.4).

Now, show that

$$\begin{gathered} w_0^{iaj0} = \begin{cases} w^{ij}, & a = n, \\ 0, & a = 0, 1 \dots, n-1, \end{cases} \\ w_k^{iaj0} = 0, \qquad k = 1, \dots, n. \end{gathered} \tag{5.5}$$

Indeed, $w_k^{iaj0} = \sum_{s=0}^n w_s^{ij} \overline{\gamma}_k^{ac} \gamma_{cs}^0$, thus $c = s = 0$ (otherwise $\gamma_0^{cs} = 0$) which implies $a = n$ and $k = 0$ as before.

It remains to prove (5.2). We have

$$\Delta_\Phi = \sum_{jb} \left(\frac{\partial w^{iajb}}{\partial x^{jb}} + w^{iajb} \frac{\partial \ln \sqrt{\det g^C}}{\partial x^{jb}} \right) \frac{\partial}{\partial x^{ia}}.$$

From (5.4) it follows that

$$\sum_{j=1}^m \frac{\partial w^{iajb}}{\partial x^{jb}} \frac{\partial}{\partial x^{ia}} = (n+1) \frac{\partial w^{ij}}{\partial x^j} \frac{\partial}{\partial x^{in}},$$

since the index b can take $n+1$ distinct values from 0 to n.

In the summand

$$\sum_{jb} w^{iajb} \frac{\partial \ln \sqrt{\det g^C}}{\partial x^{jb}} \frac{\partial}{\partial x^{ia}}$$

the only possibility is $b = 0$ by virtue of (5.3), whence, by (5.5), we obtain

$$\sum_{jb} w^{iajb} \frac{\partial \ln \sqrt{\det g^C}}{\partial x^{jb}} \frac{\partial}{\partial x^{ia}} = (n+1) w^{ij} \frac{\partial \ln \sqrt{\det g}}{\partial x^j} \frac{\partial}{\partial x^{in}}.$$

□

Corollary 5.3. *For a weakly symmetric Frobenius Weil algebra* $\mathbf{A}$ *the modular class of the Poisson manifold* $(T^{\mathbf{A}}M, w^C = w_0^C + \sum_{k=1}^n t^k w_k^C)$ *is represented by* $(n+1)\Delta_\mu^V$*, for every modular vector field* Δ_μ *of the base manifold* (M, w)*.*

Proof. By Proposition 5.2 the result is true for the field Δ_{dV_g}. As in [11], we have

$$(\sigma_w f)^V = \sigma_{w^C}(f \circ \pi_{\mathbf{A}}), \qquad f \in C^\infty(M), \qquad \pi_{\mathbf{A}} : T^{\mathbf{A}}M \to M.$$

This immediately proves the Corollary. □

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