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STRONGLY q-NIL-CLEAN RINGS

A. N. Abyzov UDC 512.55

Abstract: Under study are the rings whose every element is a sum of a nilpotent and a q-potent that commute with one another. We describe the rings whose every element is a sum of k idempotents (for some $k \in \mathbb{N}$) and a nilpotent that commute with one another.

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1. Introduction

All rings are assumed associative with a nonzero unity. A ring is *clean* provided that its every element can be represented as a sum of an invertible element and an idempotent. The study of clean rings was initiated in [1]. In [2], it was shown that the endomorphism ring of every vector space over an arbitrary skew-field is clean. In [3], this result was widely generalized to the case of continuous and discrete modules. A ring is *strongly clean* provided that its every element can be represented as a sum of an invertible element and an idempotent that commute with one another. The notion of strongly clean ring was introduced and studied in [4]. The clean and strongly clean rings were studied by many mathematicians in the last decade.

A ring is *nil-clean* if its every element is a sum of a nilpotent and an idempotent. If every element of a ring is a sum of a nilpotent and an idempotent that commute with one another then such ring is called *strongly nil-clean*. The nil-clean and strongly nil-clean rings were introduced and studied in [5]. Every (strongly) nil-clean ring is (strongly) clean. In [6], a field P, which is isomorphic to $\mathbb{F}_2$, was characterized as a field for which the ring $M_n(P)$ is nil-clean for every $n \in \mathbb{N}$. A generalization of this result to the case of an arbitrary finite field was obtained in [7]. It was shown in [8] that a ring R is strongly nil-clean exactly when $J(R)$ is a nil-ideal and $R/J(R)$ is a Boolean ring.

Let q be a natural, $q > 1$. An element r in a ring R is *q-potent* provided that $r^q = r$. A ring R is *strongly q-nil-clean* if each element in R is a sum of q-potent and nilpotent elements that commute with one another. The strongly 3-nil-clean rings have been studied recently in [9–11].

In Section 2 we study the strongly q-nil-clean rings and the rings whose every element is a sum of an idempotent, a q-potent, and a nilpotent that commute with one another. In Section 3 we investigate the rings whose every element is a sum of some idempotents and a nilpotent that commute with one another. In Section 4, as applications of general results, we obtain description of the strongly 5-nil-clean rings and the strongly 7-nil-clean rings.

In this paper we use the standard notions and notations of ring theory (for example, see [12, 13]).

2. Rings Whose Every Element Is Represented as a Sum of an Idempotent, a q-Potent, and a Nilpotent That Commute with One Another

Denote by $[m_1, \dots, m_k]$ the least common multiple of integer numbers $m_1, \dots, m_k$.

Lemma 1. *Let $q > 1$ be a natural. Assume that every element in a ring R is a sum of an idempotent, a q-potent, and a nilpotent that commute with one another. Then*

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(1) $[n^q - n, (n-1)^q - (n-1)] \cdot 1_R$ *is a nilpotent for every natural* $n > 2$;

(2) *if* q *is odd and* R *is a field then* $\mid R \mid < \infty$ *and* $\mid R \mid -1 \mid q-1$;

(3) *if* q *is odd and* $p^n \cdot 1_R = 0$, *where* p *is prime and* $n \in \mathbb{N}$, *then* $p - 1 \mid q - 1$;

(4) *if* $p = \mathrm{Char}(R)$ *is prime then there are some naturals* k_1 *and* k_2 $(k_1 > k_2)$ *such that for an arbitrary* $r \in R$ *there exists a natural* t, *for which* $r^{p^{k_1+t}} = r^{p^{k_2+t}}$.

PROOF. (1) Let $n \in \mathbb{N}$ and $n > 2$. Then $n \cdot 1_R = e + f + \alpha$, where $e^2 = e \in R$, $f^q = f \in R$, α is nilpotent, and e, f, α commute with one another. Then $e(n-1) = e(f+\alpha)$ implies $e((n-1)^q - (n-1)) = e((f+\alpha)^q - (f+\alpha)) = e\beta$, and $(1-e)(n^q - n) = (1-e)((f+\alpha)^q - (f+\alpha)) = (1-e)\beta$, where β is nilpotent. For some natural N, we see that $\beta^N = 0$. Then $e((n-1)^q - (n-1))^N = (1-e)(n^q - n)^N = 0$, and, consequently, $[n^q - n, (n-1)^q - (n-1)]^N \cdot 1_R = 0$.

(2) By Item (1) $p = \mathrm{Char}(R)$ is prime. Let $A = \{a \in R^* \mid a^{q-1} = 1\}$. Then $R = A \cup (1 + A)$, and, consequently, $|R| < \infty$. Assume that $A \neq R^*$. Since R^* is a disjunction union of the cosets of R^* by A; therefore, $|R^* \setminus A| \geq |A|$. The inclusion $r - 1 \in A$, which holds for an arbitrary $r \in R^* \setminus A$, implies $R^* \setminus A \subseteq A + 1$. Since $0 \in A + 1$, $|A| = |A + 1| > |R^* \setminus A|$; a contradiction. Thus, $A = R^*$. Since R^* is cyclic, $|R| - 1 \mid q - 1$.

(3) Consider the quotient ring R/pR, which can be considered as an algebra over the field $\mathbb{F}_p$. If $n \in \mathbb{N}$ and $n > 2$ then Item (1) implies $(n^q - n)((n-1)^q - (n-1))1_{R/pR} = 0$. Then either $(n^q - n)1_{R/pR} = 0$ or $((n-1)^q - (n-1))1_{R/pR} = 0$. Thus, every element in $\mathbb{F}_p$ is represented as a sum of an idempotent and a q-potent, and so $p - 1 \mid q - 1$ by Item (2).

(4) There are some naturals k_1 and k_2 $(k_1 > k_2)$ such that $q - 1 \mid p^{k_1} - p^{k_2}$. Take $r \in R$. The equality $r = e + f + \alpha$ holds, where $e^2 = e \in R, f^q = f \in R$, α is nilpotent, and e, f, α commute with one another. Since f is a q-potent, $f^{p^{k_1}} = f^{p^{k_2}}$. For some natural t, $\alpha^{p^{k_1+t}} = \alpha^{p^{k_2+t}} = 0$. Then $r^{p^{k_1+t}} = e + f^{p^{k_1+t}} = e + f^{p^{k_2+t}} = r^{p^{k_2+t}}$. □

A ring R is *strongly π-regular* provided that for every $r \in R$ there is $s \in R$ such that $r^n = sr^{n+1}$ for some $n \in \mathbb{N}$.

Lemma 2. *Let* $q > 1$ *be a natural. Assume that every element in* R *is represented as a sum of an idempotent, a nilpotent, and a* q*-potent that commute with one another. Then*

(1) R *is a strongly* π*-regular ring*;

(2) R *is a strongly clean ring*;

(3) $J(R)$ *is a nil-ideal.*

PROOF. (1) By Lemma 1 there exists a least natural N, for which $N \cdot 1_R = 0$. Let $N = p_1^{\alpha_1} \dots p_m^{\alpha_m}$, where $p_1, \dots, p_m$ are pairwise different primes. By the Chinese Remainder Theorem, we have the ring isomorphism $R \cong R_1 \times \dots \times R_m$, where $p_i^{\alpha_i} \cdot 1_{R_i} = 0$ for all $1 \leq i \leq m$. Show that R_i is strongly π-regular for an arbitrary $1 \leq i \leq m$. Take $r \in R_i$. By Lemma 1 $\bar{r}^{n_1} = \bar{r}^{n_2}$ for some naturals n_1 and n_2 $(n_1 > n_2)$, where $\bar{r} = r + p_i R_i \in R_i / p_i R_i$. Then $(r^{n_1} - r^{n_2})^{\alpha_i} = 0$, and, consequently, $r^{n_2 \alpha_i} = r^{n_2 \alpha_i + 1} f(r)$ for some $f(x) \in \mathbb{Z}[x]$. Since the finite direct product of strongly π-regular rings is a strongly π-regular ring, R is a strongly π-regular ring.

(2) It follows from Item (1) and [14, Proposition 2.6].

(3) It follows from the fact that the Jacobson radical of every strongly π-regular ring is a nil-ideal. □

The multiplicative group of invertible elements in R will be denoted by $U(R)$.

Lemma 3. *Let* R *be a ring without zero divisors, let* $q > 1$ *be an integer number, and let, for some natural* n, *every matrix of the form* $\mathrm{diag}(r, r, \dots, r)$ $(r \in R)$ *in* $M_n(R)$ *is represented as a sum of idempotent, nilpotent and* q*-potent elements that commute with one another. Then*

(1) R *is a finite field*;

(2) *if* q *is an odd integer number then* $|R| - 1 \mid q - 1$.

PROOF. It follows from the proof of Item (1) of Lemma 1 that there is a least natural p, for which $p \cdot 1_{M_n(R)} = 0$. Since R is a ring without zero divisors, p is prime. A ring R can be considered as an algebra over $\mathbb{F}_p$. Let $r \in R$ be nonzero. Consider a diagonal $(n \times n)$-matrix A of the form $\mathrm{diag}(r, r, \dots, r)$. Then