Generic semisimplicity of the reduced enveloping algebras

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Dedicated to Helmut Strade on his 70th birthday

Let $\mathfrak{g}$ be a finite dimensional restricted Lie algebra over an algebraically closed field of characteristic $p > 0$. With each linear function $\xi \in \mathfrak{g}^*$ one associates the reduced enveloping algebra $U_\xi(\mathfrak{g})$ (see [14]). These finite dimensional associative algebras arise naturally in the study of representations of $\mathfrak{g}$.

According to a fundamental idea the representation theory of $\mathfrak{g}$ should be related to the coadjoint action of $\mathfrak{g}$ in the dual space $\mathfrak{g}^*$. Especially, Kac and Weisfeiler conjectured a formula for the maximum dimension of simple $\mathfrak{g}$-modules in terms of the minimum dimension of stabilizers of linear functions [15]. Another conjecture put forward in [10] says that the family of reduced enveloping algebras of $\mathfrak{g}$ is generically semisimple if and only if there exists a linear function on $\mathfrak{g}$ with a toral stabilizer. Those $\xi \in \mathfrak{g}^*$ for which $U_\xi(\mathfrak{g})$ is semisimple always form a Zariski open subset of $\mathfrak{g}^*$. When this subset is nonempty, the family of reduced enveloping algebras is called generically semisimple. Both conjectures are still open.

Our aim is to confirm the second conjecture in the case when $\mathfrak{g}$ is solvable and $p > 2$. This is done in Theorem 2.3. The class of solvable Lie algebras is the easiest to analyze since there is a precise description of irreducible representations obtained by Helmut Strade in 1978. Given any simple $\mathfrak{g}$-module M, it was shown in [13] that there exist a subalgebra $\mathfrak{p} \subset \mathfrak{g}$ and a linear function $\lambda \in \mathfrak{p}^*$ such that

$$M \cong U_\xi(\mathfrak{g}) \otimes_{U_\xi(\mathfrak{p})} 1_\lambda$$

where 1_λ is the one-dimensional $\mathfrak{p}$-module on which $\mathfrak{p}$ operates via λ. The subalgebra $\mathfrak{p}$ is obtained as the final term of a chain of subalgebras $\mathfrak{g} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots$ where each $\mathfrak{p}_i$ for $i > 0$ is defined as the stabilizer of a suitable one-dimensional representation of an ideal in $\mathfrak{p}_{i-1}$. Then M is induced from a simple $U_\xi(\mathfrak{p}_1)$-module, and so on. By the explicit construction $\mathfrak{p}$ turns out to be a polarization of any linear function $\eta \in \mathfrak{g}^*$ extending λ. Therefore

$$\dim M = p^{(\dim \mathfrak{g} - \dim \mathfrak{z}(\eta))/2}$$

where $\mathfrak{z}(\eta)$ stands for the stabilizer of η in $\mathfrak{g}$. In particular, Helmut's paper gave a solution of the Kac-Weisfeiler conjecture for solvable Lie algebras of characteristic $p > 2$. In [15] the desired formula was verified only for completely solvable Lie algebras.

Essentially the same reduction of representations is useful when dealing with the second of the aforementioned conjectures. We start with a semisimple algebra $U_\xi(\mathfrak{g})$ and define $\mathfrak{p}_1$ as above with respect to an arbitrarily chosen simple $U_\xi(\mathfrak{g})$-module. Then we consider a certain factor algebra $S_{\xi,\lambda}$ of the reduced enveloping algebra $U_\xi(\mathfrak{p}_1)$ whose simple modules are in a bijective correspondence with a subset of the set of all simple $U_\xi(\mathfrak{g})$-modules. By comparing the dimension of $S_{\xi,\lambda}$ with the dimensions of its simple modules we conclude that $S_{\xi,\lambda}$ is semisimple. Furthermore, $S_{\xi,\lambda}$ can be described as the factor algebra $U_0(\mathfrak{h})/(z-1)$ where $\mathfrak{h}$ is another restricted Lie algebra of dimension smaller than that of $\mathfrak{g}$, and z is a central toral element of $\mathfrak{h}$. At this point we want to argue by induction on the dimension of the Lie algebra. For this we have to be sure that the semisimplicity of $U_0(\mathfrak{h})/(z-1)$ implies that the family of reduced enveloping algebras of $\mathfrak{h}$ is generically semisimple. Here some geometric arguments are involved. I don't know whether Theorem 2.3 can be proved in a way which avoids such methods.

A more complete form of the fact just discussed is presented in section 3. It is shown in Theorem 3.6 that the family of reduced enveloping algebras of an arbitrary finite dimensional restricted Lie algebra $\mathfrak{g}$ is generically semisimple provided that at least one of these algebras has a simple projective module. The main intermediate step consists in checking that the dimension of any projective $U_\xi(\mathfrak{g})$-module is always divisible by the maximum dimension of simple $\mathfrak{g}$-modules. Here we need sophisticated, but well-known, technique based on representability of certain functors by smooth commutative algebras over a commutative ring.

These ideas are developed further in section 4. The results there are no longer concerned with either Lie algebras or even the generic semisimplicity, but they have common features with the previous results. We consider a flat family $(A_\xi)_{\xi\in X}$ of finite dimensional associative algebras over an arbitrary algebraically closed field parameterized by an irreducible algebraic variety X. By Theorem 4.6 the algebras corresponding to points of a nonempty Zariski open subset of X all have the same dimensions of simple modules and the same Cartan invariants. By the numeric type of a finite dimensional associative algebra we understand the collection of these numeric invariants. It turns out that the numeric type of an arbitrary algebra in the family never exceeds the generic numeric type with respect to a certain partial ordering on the set T of all possible numeric types. This means that the numeric invariants of two algebras are related by a set of linear equations with integer coefficients. Finally, Theorem 4.7 establishes the openness of the locus of those $\xi \in X$ for which a fixed semisimple algebra B is isomorphic to a direct factor of A_ξ. This may be viewed as the rigidity of semisimple blocks.

Reverting to the reduced enveloping algebras of a restricted Lie algebra $\mathfrak{g}$, it should be mentioned that the set of p-characters $\xi \in \mathfrak{g}^*$ defining semisimple algebras $U_\xi(\mathfrak{g})$ is in general different from the set of those $\xi \in \mathfrak{g}^*$ for which $\mathfrak{z}(\xi)$ are tori. An easy example is given in [12]. However, for some Lie algebras the two sets are the same. One could ask whether this always holds when $\mathfrak{g}$ is the Lie algebra of an algebraic group G. If G is semisimple and p is good for the root system of G, then the answer is positive since $U_\xi(\mathfrak{g})$ is semisimple if and only if ξ is regular semisimple [5, Cor. 3.6]. The case of simple generic algebras is rather special in this respect. When $\mathfrak{g}$ is solvable and $p > 2$ it was proved in [12] that $U_\xi(\mathfrak{g})$ is simple if and only if $\mathfrak{z}(\xi) = 0$ (equivalently, the alternating bilinear form on $\mathfrak{g}$ associated with ξ is nondegenerate).

1. Some facts related to generic semisimplicity

We will be dealing with families $(A_\xi)_{\xi \in X}$ of finite dimensional associative algebras over an algebraically closed field k parameterized by points of an algebraic variety X. We say that such a family is *flat* if there exists a sheaf of associative $\mathcal{O}_X$-algebras $\mathcal{A}$, coherent locally free as an $\mathcal{O}_X$-module, such that $A_\xi \cong \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_\xi/\mathfrak{m}_\xi$ for each $\xi \in X$ where $\mathcal{O}_X$ is the structure sheaf, $\mathcal{O}_\xi$ the local ring of ξ, and $\mathfrak{m}_\xi$ the maximal ideal of $\mathcal{O}_\xi$. Alternatively this can be rephrased by saying that the algebras A_ξ are fibers of an algebraic vector bundle over X with total space $|A|$, and the multiplication in each A_ξ is induced by a morphism of algebraic varieties $|A| \times_X |A| \to |A|$ (a correspondence between vector bundles and locally free sheaves is explained, e.g., in [7, Ch. II, Exercise 5.18]). The term "flat family" refers to the fact that for coherent sheaves on a variety flatness is equivalent to local freeness. It is a more precise substitute for the term "continuous family" used in [10].

Suppose that $(A_\xi)_{\xi \in X}$ is a flat family of finite dimensional algebras parameterized by X and $f : Y \to X$ is a morphism of algebraic varieties. Setting $A'_\eta = A_{f(\eta)}$ for $\eta \in Y$, we get a flat family of finite dimensional algebras parameterized by Y. The latter corresponds to the pullback sheaf $f^*\mathcal{A}$.

Let B be a not necessarily finite dimensional associative algebra over k. A family $(M_\xi)_{\xi \in X}$ of finite dimensional B-modules is *flat* if there exist a coherent locally free $\mathcal{O}_X$-module $\mathcal{M}$ and an algebra homomorphism $B \to \operatorname{End}_{\mathcal{O}_X} \mathcal{M}$ such that M_ξ is isomorphic as a B-module with $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_\xi/\mathfrak{m}_\xi$ for each $\xi \in X$. This also can be reformulated in terms of vector bundles, as in [10, 4.3].

Lemma 1.1. *Let $(A_\xi)_{\xi \in X}$ be a flat family of finite dimensional associative algebras parameterized by an algebraic variety X. For any fixed semisimple finite dimensional algebra S the subset $\{\xi \in X \mid A_\xi \cong S\}$ is open in X.*

This lemma presents one possible formulation of the rigidity of semisimple associative algebras. Its proof reduces to [9, Th. 22.1]. In the special case of the reduced enveloping algebras details of the argument are given in [10, 4.2].

For any commutative ring R we denote by $\operatorname{Specm} R$ the spectrum of its maximal ideals. If R is a finitely generated k-algebra then we may view $\operatorname{Specm} R$ as an affine algebraic variety with the coordinate ring R/N where N is the nilradical of R.

Lemma 1.2. *Let A be any associative algebra over k, module-finite over a finitely generated central subalgebra R. Let A^{op} be A taken with the opposite multiplication and $m = \min\{\dim A/\mathfrak{m}A \mid \mathfrak{m} \in \operatorname{Specm} R\}$. Then:*

(i) *The subset $W_1 = \{\mathfrak{m} \in \operatorname{Specm} R \mid \dim A/\mathfrak{m}A = m\}$ is open in* $\operatorname{Specm} R$.

(ii) *The algebras $A/\mathfrak{m}A$ with $\mathfrak{m} \in W_1$ form a flat family parameterized by W_1 which may also be regarded as a flat family of $A \otimes A^{\mathrm{op}}$-modules.*

Proof. Assertion (i) follows from the well-known semicontinuity of the integer-valued function $\mathfrak{m} \mapsto \dim A/\mathfrak{m}A$ (cf. [7, Ch. II, Exercise 5.8]). In (ii) we may assume R to be reduced by passing to factor algebras of R and A. Then the finitely generated

R-module A gives rise to a coherent sheaf on $\operatorname{Specm} R$ whose restriction to W_1 is locally free of rank m, again by [7, Ch. II, Exercise 5.8]. Reformulated algebraically, this means that for each $\mathfrak{m} \in W_1$ there exists $s \in A$ such that $s \notin \mathfrak{m}$ and the algebra $A_s = A \otimes_R R_s$ is a free module of rank m over the localization R_s of R with respect to the multiplicatively closed set of powers of s. The assignment $\mathfrak{n} \mapsto \mathfrak{n}R_s$ gives a bijection $X_s \to \operatorname{Specm} R_s$ where $X_s = \{\mathfrak{n} \in \operatorname{Specm} R \mid s \notin \mathfrak{n}\}$ is an open neighborhood of $\mathfrak{m}$ in $\operatorname{Specm} R$, and $A_s/\mathfrak{n}A_s \cong A/\mathfrak{n}A$ for all $\mathfrak{n} \in X_s$. □

Lemma 1.3. *Let $f : X \to Y$ be a finite morphism of irreducible algebraic varieties. If U is any nonempty open subset of X, then there exists a nonempty open subset V of Y such that $f^{-1}(V) \subset U$.*

Proof. A finite morphism takes closed subsets to closed ones and preserves the dimensions. Applying this to the complement U^c of U in X, we deduce that $f(U^c)$ is a proper closed subset of Y. So we may take $V = Y \setminus f(U^c)$. □

Let $\mathfrak{g}$ be a finite dimensional p-Lie algebra over an algebraically closed field k of characteristic $p > 0$. We will denote by $Z(\mathfrak{g})$ the center of the universal enveloping algebra $U(\mathfrak{g})$ and by $Z_p(\mathfrak{g})$ the subalgebra of $Z(\mathfrak{g})$ generated by all elements $x^p - x^{[p]}$ with $x \in \mathfrak{g}$. There is a bicontinuous bijective morphism of algebraic varieties

$$F : \mathfrak{g}^* \to \operatorname{Specm} Z_p(\mathfrak{g})$$

given by the rule $\xi \mapsto \mathfrak{m}_\xi$ for $\xi \in \mathfrak{g}^*$ where $\mathfrak{m}_\xi$ is the maximal ideal of $Z_p(\mathfrak{g})$ generated by all elements $x^p - x^{[p]} - \xi(x)^p\,1$ with $x \in \mathfrak{g}$. In fact $\operatorname{Specm} Z_p(\mathfrak{g})$ is isomorphic with the Frobenius twist $\mathfrak{g}^{*(1)}$ of $\mathfrak{g}^*$ (see [5, p. 1058]). Now $U_\xi(\mathfrak{g})$ is just the factor algebra $U(\mathfrak{g})/\mathfrak{m}_\xi U(\mathfrak{g})$.

We say that $\xi \in \mathfrak{g}^*$ is the *p-character* of any $\mathfrak{g}$-module which can be realized as a module over $U_\xi(\mathfrak{g})$, and for a p-Lie subalgebra $\mathfrak{p} \subset \mathfrak{g}$ we will denote by $U_\xi(\mathfrak{p})$ the reduced enveloping algebra of $\mathfrak{p}$ defined with respect to the linear function $\xi|_{\mathfrak{p}}$.

Lemma 1.4. *A finite dimensional $U_\xi(\mathfrak{g})$-module V is projective if and only if so is the $U_0(\mathfrak{g})$-module $\operatorname{End}_k V \cong V \otimes V^*$.*

This lemma just recalls a part of [5, Prop. 6.2]. A shorter proof of the "if" statement would be to observe that V is always a direct summand of $V \otimes V^* \otimes V$.

Given a $\mathfrak{g}$-module V, we will denote by x_V the linear transformation of V associated with an element $x \in \mathfrak{g}$. Put

$$\operatorname{md}(\mathfrak{g}) = \max\{\dim V \mid V \text{ is a simple } U(\mathfrak{g})\text{-module}\}.$$

Lemma 1.5. *Suppose that for some $\xi \in \mathfrak{g}^*$ the algebra $U_\xi(\mathfrak{g})$ has a simple projective module V such that* $\dim V = \operatorname{md}(\mathfrak{g})$. *Then:*

(i) *The family of reduced enveloping algebras of $\mathfrak{g}$ is generically semisimple.*

(ii) *If V' is any simple projective $U_{\xi'}(\mathfrak{g})$-module of dimension equal to* $\operatorname{md}(\mathfrak{g})$ *for some $\xi' \in \mathfrak{g}^*$, then the $U_0(\mathfrak{g})$-module* $\operatorname{End}_k V'$ *is isomorphic with* $\operatorname{End}_k V$.

(iii) *The set* $\mathfrak{z}_V = \{x \in \mathfrak{g} \mid x_V \text{ is a scalar transformation of } V\}$ *coincides with the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ and consists of $[p]$-semisimple elements.*

Proof. As follows from [17, Th. 5 and 6], there exists a nonempty Zariski open subset $W \subset \operatorname{Specm} Z(\mathfrak{g})$ such that the factor algebra $U(\mathfrak{g})/\mathfrak{m}U(\mathfrak{g})$ is either simple with a simple module of dimension equal to $\operatorname{md}(\mathfrak{g})$ or has all simple modules of smaller dimension depending on whether $\mathfrak{m} \in W$ or not. Thus W coincides with the set of annihilators in $Z(\mathfrak{g})$ of all simple $\mathfrak{g}$-modules of dimension $\operatorname{md}(\mathfrak{g})$. We have $\dim U(\mathfrak{g})/\mathfrak{m}U(\mathfrak{g}) = \operatorname{md}(\mathfrak{g})^2$ for all $\mathfrak{m} \in W$.

Applying Lemma 1.2 with $A = U(\mathfrak{g})$ and $R = Z(\mathfrak{g})$, we conclude that $W \subset W_1$. Hence the algebras $U(\mathfrak{g})/\mathfrak{m}U(\mathfrak{g})$ with $\mathfrak{m} \in W$ form a flat family parameterized by W which we may also view as a flat family of $U_0(\mathfrak{g})$-modules with respect to the adjoint action. In fact the adjoint action of $U(\mathfrak{g})$ on $U_\xi(\mathfrak{g})$ comes from an embedding of $U(\mathfrak{g})$ in $U(\mathfrak{g}) \otimes U(\mathfrak{g})^{\mathrm{op}}$ such that $x \mapsto x \otimes 1 - 1 \otimes x$ for $x \in \mathfrak{g}$, and this action factors through $U_0(\mathfrak{g})$.

By the rigidity of projective modules (see [10, 4.3]) the subset

$$W_P = \{\mathfrak{m} \in W \mid U(\mathfrak{g})/\mathfrak{m}U(\mathfrak{g}) \text{ is a projective } U_0(\mathfrak{g})\text{-module}\}$$

is open in W, hence open in $\operatorname{Specm} Z(\mathfrak{g})$. Moreover, since W is an irreducible variety, all these projective modules corresponding to various $\mathfrak{m} \in W_P$ are isomorphic to each other. Now $U(\mathfrak{g})/\mathfrak{m}U(\mathfrak{g}) \cong \operatorname{End}_k V_\mathfrak{m}$ where $V_\mathfrak{m}$ is the simple $U(\mathfrak{g})$-module annihilated by $\mathfrak{m}$. So the isomorphism class of the $U_0(\mathfrak{g})$-module $\operatorname{End}_k V_\mathfrak{m}$ with $\mathfrak{m} \in W_P$ does not depend on $\mathfrak{m}$. By Lemma 1.4 the maximal ideals of $Z(\mathfrak{g})$ corresponding to V and V' lie in W_P, whence (ii) follows.

Since $Z(\mathfrak{g})$ is module-finite over $Z_p(\mathfrak{g})$, the morphism of algebraic varieties

$$\pi : \operatorname{Specm} Z(\mathfrak{g}) \to \operatorname{Specm} Z_p(\mathfrak{g}), \qquad \mathfrak{m} \mapsto \mathfrak{m} \cap Z_p(\mathfrak{g}),$$

is finite. By Lemma 1.3 there exists a nonempty open subset $W'_P \subset \operatorname{Specm} Z_p(\mathfrak{g})$ such that $\pi^{-1}(W'_P) \subset W_P$. We claim that the algebra $U_\eta(\mathfrak{g})$ is semisimple for all η in the open subset $F^{-1}(W'_P)$ of $\mathfrak{g}^*$. Indeed, suppose that $\mathfrak{m}' = F(\eta) \in W'_P$. The annihilators of simple $\mathfrak{g}$-modules in $Z(\mathfrak{g})$ are maximal ideals since $Z(\mathfrak{g})$ operates in simple modules via scalar transformations by Schur's Lemma. If $\mathfrak{m} \in \operatorname{Specm} Z(\mathfrak{g})$ is the annihilator of a simple $U_\eta(\mathfrak{g})$-module V_1, then $\mathfrak{m}' \subset \mathfrak{m}$, whence $\pi(\mathfrak{m}) = \mathfrak{m}'$. But then $\mathfrak{m} \in W_P$, so that $\dim V_1 = \operatorname{md}(\mathfrak{g})$ and $\operatorname{End}_k V_1$ is a projective $U_0(\mathfrak{g})$-module. By Lemma 1.4 V_1 is a projective $U_\eta(\mathfrak{g})$-module. Thus all simple $U_\eta(\mathfrak{g})$-modules are projective, which implies the semisimplicity of $U_\eta(\mathfrak{g})$. This proves (i).

The set $\mathfrak{z}_V$ consists precisely of those $x \in \mathfrak{g}$ which annihilate the $U_0(\mathfrak{g})$-module $\operatorname{End}_k V$. Hence $\mathfrak{z}_V = \mathfrak{z}_{V'}$ for any V' as in (ii). If $\eta \in F^{-1}(W'_P)$, then

$$U_\eta(\mathfrak{g}) \cong \operatorname{End}_k V_1 \times \cdots \times \operatorname{End}_k V_s$$

where $V_1, \ldots, V_s$ are pairwise nonisomorphic simple $U_\eta(\mathfrak{g})$-modules. Since each V_i satisfies the hypothesis of item (ii), we see that $\mathfrak{z}_V$ annihilates $U_\eta(\mathfrak{g})$ in the adjoint representation. But $\mathfrak{g}$ embeds in $U_\eta(\mathfrak{g})$ as a $U_0(\mathfrak{g})$-submodule. Hence $[\mathfrak{z}_V, \mathfrak{g}] = 0$, that is, $\mathfrak{z}_V \subset \mathfrak{z}(\mathfrak{g})$. The inverse inclusion is immediate from Schur's Lemma.

The reduced enveloping algebra $U_\eta\big(\mathfrak{z}(\mathfrak{g})\big)$ embeds in the center of $U_\eta(\mathfrak{g})$. Since this center is a direct product of several copies of the ground field, the algebra $U_\eta\big(\mathfrak{z}(\mathfrak{g})\big)$

must be semisimple. But this implies that $\mathfrak{z}(\mathfrak{g})$ cannot contain nonzero $[p]$-nilpotent elements, i.e., $\mathfrak{z}(\mathfrak{g})$ is a torus. Now (iii) is also proved. □

Given a p-ideal $\mathfrak{a}$ of $\mathfrak{g}$, we put

$$X(\mathfrak{a},\mathfrak{g}) = \{(\lambda,\xi)\in\mathfrak{a}^*\times\mathfrak{g}^* \mid \lambda([\mathfrak{a},\mathfrak{a}])=0 \quad \text{and} \\ \lambda(x)^p-\lambda(x^{[p]})=\xi(x)^p \text{ for all } x\in\mathfrak{a}\}.$$

The two conditions in the definition of $X(\mathfrak{a},\mathfrak{g})$ mean precisely that λ defines a one-dimensional representation of the p-Lie algebra $\mathfrak{a}$ with p-character $\xi|_{\mathfrak{a}}$.

Lemma 1.6. *Suppose that $(\lambda,\xi)\in X(\mathfrak{a},\mathfrak{g})$ and $\lambda([\mathfrak{g},\mathfrak{a}])=0$.*

(i) *Let I be the ideal of $U_\xi(\mathfrak{g})$ generated by all elements $x-\lambda(x)$ with $x\in\mathfrak{a}$. Then $\dim U_\xi(\mathfrak{g})/I = p^{\operatorname{codim}_{\mathfrak{g}}\mathfrak{a}}$.*

(ii) *If $U_\xi(\mathfrak{g})$ is semisimple, then $\mathfrak{a}\subset\mathfrak{z}(\mathfrak{g})$.*

Proof. The annihilator $\mathfrak{n}$ of the one-dimensional $U_\xi(\mathfrak{a})$-module corresponding to λ is generated as an ideal of $U_\xi(\mathfrak{a})$ by the set $\{x-\lambda(x)\mid x\in\mathfrak{a}\}$. This ideal is stable under the adjoint action of $\mathfrak{g}$ since $\lambda([\mathfrak{g},\mathfrak{a}])=0$. It follows that $I=U_\xi(\mathfrak{g})\mathfrak{n}$, and therefore $U_\xi(\mathfrak{g})/I\cong U_\xi(\mathfrak{g})\otimes_{U_\xi(\mathfrak{a})}U_\xi(\mathfrak{a})/\mathfrak{n}$. Now the dimension formula is clear since $U_\xi(\mathfrak{g})$ is free of rank $p^{\operatorname{codim}_{\mathfrak{g}}\mathfrak{a}}$ as a $U_\xi(\mathfrak{a})$-module, while $\dim U_\xi(\mathfrak{a})/\mathfrak{n}=1$.

Suppose that $U_\xi(\mathfrak{g})$ is semisimple. By [10, 4.2] $\dim V=\operatorname{md}(\mathfrak{g})$ for each simple $U_\xi(\mathfrak{g})$-module V. Lemma 1.5 shows that $\mathfrak{z}_V=\mathfrak{z}(\mathfrak{g})$. Since $I\neq U_\xi(\mathfrak{g})$, there exists a simple $U_\xi(\mathfrak{g})$-module V annihilated by I. This module is annihilated by $[\mathfrak{g},\mathfrak{a}]$ since $[\mathfrak{g},\mathfrak{a}]\subset\operatorname{Ker}\lambda\subset I$. Hence x_V is a $\mathfrak{g}$-module endomorphism of V for each $x\in\mathfrak{a}$. Since V is simple, by Schur's Lemma $\mathfrak{a}$ must act in V via scalar transformations. Thus $\mathfrak{a}\subset\mathfrak{z}_V$, and we are done. □

Lemma 1.7. *Let $\mathfrak{t}$ be a torus contained in the center of $\mathfrak{g}$. For $(\lambda,\xi)\in X(\mathfrak{t},\mathfrak{g})$ put $B_{\lambda,\xi}=U_\xi(\mathfrak{g})/I_{\lambda,\xi}$ where $I_{\lambda,\xi}$ is the ideal of $U_\xi(\mathfrak{g})$ generated by $\{x-\lambda(x)\mid x\in\mathfrak{t}\}$.*

(i) *$U_\xi(\mathfrak{g})\cong\prod_{\lambda\in X_\xi}B_{\lambda,\xi}$ where $X_\xi=\{\lambda\in\mathfrak{t}^*\mid(\lambda,\xi)\in X(\mathfrak{t},\mathfrak{g})\}$.*

(ii) *If $B_{\lambda,\xi}$ is a semisimple algebra for at least one pair $(\lambda,\xi)\in X(\mathfrak{t},\mathfrak{g})$, then the family of reduced enveloping algebras of $\mathfrak{g}$ is generically semisimple.*

Proof. As is seen from the definition, $X(\mathfrak{t},\mathfrak{g})$ is a closed subset of the affine algebraic variety $\mathfrak{t}^*\times\mathfrak{g}^*$. Let $\mathfrak{s}$ be any vector subspace of $\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{t}\oplus\mathfrak{s}$. Then the assignment $(\lambda,\xi)\mapsto(\lambda,\xi|_{\mathfrak{s}})$ defines an isomorphism of $X(\mathfrak{t},\mathfrak{g})$ onto $\mathfrak{t}^*\times\mathfrak{s}^*$. In particular, $X(\mathfrak{t},\mathfrak{g})$ is irreducible.

The reduced enveloping algebra $U_\xi(\mathfrak{t})$ is a semisimple central subalgebra of $U_\xi(\mathfrak{g})$. It is isomorphic to a direct product of several copies of the ground field with the factors corresponding to the maximal ideals of $U_\xi(\mathfrak{t})$. Thus $U_\xi(\mathfrak{t})\cong\prod_{\lambda\in X_\xi}U_\xi(\mathfrak{t})/\mathfrak{n}_{\lambda,\xi}$ where $\mathfrak{n}_{\lambda,\xi}$ is the annihilator of the one-dimensional $U_\xi(\mathfrak{t})$-module associated with $\lambda\in X_\xi$. Since $\mathfrak{n}_{\lambda,\xi}$ is generated as an ideal of $U_\xi(\mathfrak{t})$ by $\{x-\lambda(x)\mid x\in\mathfrak{t}\}$, we have $I_{\lambda,\xi}=U_\xi(\mathfrak{g})\,\mathfrak{n}_{\lambda,\xi}$. This leads to the decomposition in (i).

Note that the algebras $B_{\lambda,\xi}$ form a flat family parameterized by $X(\mathfrak{t},\mathfrak{g})$. In fact, let $A=U(\mathfrak{g})$, and let R be its central subalgebra generated by $Z_p(\mathfrak{g})$ and $U(\mathfrak{t})$. For each $(\lambda,\xi)\in X(\mathfrak{t},\mathfrak{g})$ denote by $\mathfrak{m}_{\lambda,\xi}$ the ideal of R generated by all elements $x-\lambda(x)$

with $x \in \mathfrak{t}$ and all elements $y^p - y^{[p]} - \xi(y)^p 1$ with $y \in \mathfrak{g}$. Each maximal ideal of R lies above some maximal ideal $\mathfrak{m}_\xi$ of $Z_p(\mathfrak{g})$. Since the canonical homomorphism $U(\mathfrak{g}) \to U_\xi(\mathfrak{g})$ maps R onto $U_\xi(\mathfrak{t})$, such a maximal ideal of R is the preimage of a maximal ideal of $U_\xi(\mathfrak{t})$. Therefore all maximal ideals of R are of the form described above. We see also that $U(\mathfrak{g})/\mathfrak{m}_{\lambda,\xi}U(\mathfrak{g}) \cong B_{\lambda,\xi}$. By Lemma 1.6 $\dim B_{\lambda,\xi} = p^{\operatorname{codim}_\mathfrak{g} \mathfrak{t}}$ for all $(\lambda,\xi) \in X(\mathfrak{t},\mathfrak{g})$. Thus Lemma 1.2 produces a flat family of algebras parameterized by $\operatorname{Specm} R$. Making base change with respect to the morphism of varieties $X(\mathfrak{t},\mathfrak{g}) \to \operatorname{Specm} R$, $(\lambda,\xi) \mapsto \mathfrak{m}_{\lambda,\xi}$, we get a family parameterized by $X(\mathfrak{t},\mathfrak{g})$.

Lemma 1.1 shows that the subset $W = \{(\lambda,\xi) \in X(\mathfrak{t},\mathfrak{g}) \mid B_{\lambda,\xi} \text{ is semisimple}\}$ is open in $X(\mathfrak{t},\mathfrak{g})$. Suppose that W is nonempty. Since the projection $\pi : X(\mathfrak{t},\mathfrak{g}) \to \mathfrak{g}^*$ is a finite morphism of irreducible algebraic varieties, by Lemma 1.3 there exists a nonempty open subset W' of $\mathfrak{g}^*$ such that $\pi^{-1}(W') \subset W$. If $\xi \in W'$, then we have $(\lambda,\xi) \in W$, and therefore the algebra $B_{\lambda,\xi}$ is semisimple, for each $\lambda \in X_\xi$. Then $U_\xi(\mathfrak{g})$ is semisimple in view of (i). □

Corollary 1.8. *Let $z \in \mathfrak{z}(\mathfrak{g})$ be a nonzero element such that $z^{[p]} = z$, and let I be the ideal of $U_0(\mathfrak{g})$ generated by $z - 1$. If the factor algebra $U_0(\mathfrak{g})/I$ is semisimple, then the family of reduced enveloping algebras of $\mathfrak{g}$ is generically semisimple.*

Proof. Take $\mathfrak{t}$ to be the one-dimensional torus spanned by z. Let $\xi = 0$, and let $\lambda(z) = 1$. In this case $B_{\lambda,\xi} = U_0(\mathfrak{g})/I$, and Lemma 1.7 applies. □

2. Solvable Lie algebras

A basic reduction in the representation theory of Lie algebras exploits the existence of ideals. Recall the definition of $X(\mathfrak{a},\mathfrak{g})$ from section 1.

Lemma 2.1. *Suppose that $\mathfrak{a}$ is a p-ideal of a p-Lie algebra $\mathfrak{g}$ and $(\lambda,\xi) \in X(\mathfrak{a},\mathfrak{g})$. Consider a p-Lie subalgebra $\mathfrak{p} = \{y \in \mathfrak{g} \mid [y,\mathfrak{a}] \subset \operatorname{Ker}\lambda\}$. The assignment*

$$V \mapsto \operatorname{ind}_\xi V = U_\xi(\mathfrak{g}) \otimes_{U_\xi(\mathfrak{p})} V$$

gives a bijection between the isomorphism classes of simple $U_\xi(\mathfrak{p})$-modules on which $\mathfrak{a}$ operates via scalar transformations with the eigenvalue function λ and the isomorphism classes of simple $U_\xi(\mathfrak{g})$-modules which contain a common eigenvector for the elements of $\mathfrak{a}$ with the corresponding eigenvalue function λ.

This is a reformulation of [14, Th. 5.7.7]. The inverse correspondence is given by the assignment $M \mapsto M^\lambda$ where

$$M^\lambda = \{v \in M \mid xv = \lambda(x)v \ \text{ for all } x \in \mathfrak{a}\}.$$

Note that the subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ coincides with the stabilizer (as defined in [2]) of the one-dimensional representation of $\mathfrak{a}$ given by $\lambda \in \mathfrak{a}^*$. Therefore the irreducibility of the induced representations in Lemma 2.1 is nothing else but the characteristic p analog of Blattner's criterion [2, Th. 3] proved in zero characteristic (see [14, Cor. 5.7.5] for details). Given two simple $U_\xi(\mathfrak{p})$-modules V and V' on which $\mathfrak{a}$ operates via scalar transformations with the eigenvalue function λ, the canonical map

$$\operatorname{Hom}_\mathfrak{p}(V,V') \to \operatorname{Hom}_\mathfrak{g}(\operatorname{ind}_\xi V, \operatorname{ind}_\xi V')$$

is bijective by the characteristic p version of [4, 5.3.7]. In particular, $\operatorname{ind}_\xi V \cong \operatorname{ind}_\xi V'$ if and only if $V \cong V'$. The next lemma is a reformulation of [14, Cor. 5.8.2]:

Lemma 2.2. *Suppose that $\mathfrak{g}$ is solvable and $p > 2$. If $\mathfrak{g}$ is not abelian, then $\mathfrak{g}$ has an abelian ideal not contained in the center $\mathfrak{z}(\mathfrak{g})$.*

For $\xi \in \mathfrak{g}^*$ we will denote by $\mathfrak{z}(\xi)$ its stabilizer in $\mathfrak{g}$, that is,

$$\mathfrak{z}(\xi) = \{x \in \mathfrak{g} \mid \xi\big([x, \mathfrak{g}]\big) = 0\}.$$

Theorem 2.3. *Suppose that $\mathfrak{g}$ is solvable and $p > 2$. The family of reduced enveloping algebras of $\mathfrak{g}$ is generically semisimple if and only if there exists a linear function on $\mathfrak{g}$ with a toral stabilizer.*

Proof. If $\mathfrak{g}$ admits a linear function with a toral stabilizer then the family of reduced enveloping algebras is generically semisimple by [10, 4.4]. So it remains to verify the converse. Proceeding by induction on $\dim \mathfrak{g}$, we may assume that the converse statement holds for any solvable p-Lie algebra of smaller dimension. If $\mathfrak{g}$ is abelian, then the semisimplicity of $U_\xi(\mathfrak{g})$ implies that $\mathfrak{g}$ is a torus, in which case the stabilizers of all linear functions are toral. So we may assume $\mathfrak{g}$ to be nonabelian.

Using Lemma 2.2, we find an abelian ideal $\mathfrak{a}$ of $\mathfrak{g}$ such that $\mathfrak{a} \not\subset \mathfrak{z}(\mathfrak{g})$. Replacing $\mathfrak{a}$ with its p-envelope, we may assume that $\mathfrak{a}$ is a p-ideal. There is a decomposition $\mathfrak{a} = \mathfrak{a}_s \oplus \mathfrak{a}_n$ where $\mathfrak{a}_s$ is a torus and $\mathfrak{a}_n$ is a subspace consisting of $[p]$-nilpotent elements. Since $[\mathfrak{a}_s, [\mathfrak{a}_s, \mathfrak{g}]] \subset [\mathfrak{a}, \mathfrak{a}] = 0$ and the adjoint action of $\mathfrak{a}_s$ on $\mathfrak{g}$ is semisimple, we have $\mathfrak{a}_s \subset \mathfrak{z}(\mathfrak{g})$.

Take any $\xi \in \mathfrak{g}^*$ such that the algebra $U_\xi(\mathfrak{g})$ is semisimple. For $(\lambda, \xi) \in X(\mathfrak{a}_s, \mathfrak{g})$ put $B_{\lambda,\xi} = U_\xi(\mathfrak{g})/I_{\lambda,\xi}$ where $I_{\lambda,\xi}$ is the ideal of $U_\xi(\mathfrak{g})$ generated by all elements $x - \lambda(x)$ with $x \in \mathfrak{a}_s$. By Lemma 1.7

$$U_\xi(\mathfrak{g}) \cong \prod_{\lambda \in X_\xi} B_{\lambda,\xi} \quad \text{where } X_\xi = \{\lambda \in \mathfrak{a}_s^* \mid (\lambda, \xi) \in X(\mathfrak{a}_s, \mathfrak{g})\}.$$

Since $U_\xi(\mathfrak{g})$ is semisimple, so too are all blocks $B_{\lambda,\xi}$ in this decomposition. Each simple $U_\xi(\mathfrak{g})$-module M is a $B_{\lambda,\xi}$-module for exactly one $\lambda \in X_\xi$. Since $\mathfrak{a}$ is abelian, there exists a common eigenvector $v \in M$ for the elements of $\mathfrak{a}$. Let $\mu \in \mathfrak{a}^*$ be the corresponding eigenvalue function, so that $xv = \mu(x)v$ for all $x \in \mathfrak{a}$. Then $\mu|_{\mathfrak{a}_s} = \lambda$. Since v spans a one-dimensional $U_\xi(\mathfrak{a})$-submodule of M, we have $(\mu, \xi) \in X(\mathfrak{a}, \mathfrak{g})$. Note that μ is the unique extension of λ to a linear function on $\mathfrak{a}$ with the previous property. Indeed, if $x \in \mathfrak{a}_n$, then $x^{[p]^r} = 0$ for sufficiently large $r > 0$, and the equalities $\mu\big(x^{[p]^i}\big)^p - \mu\big(x^{[p]^{i+1}}\big) = \xi\big(x^{[p]^i}\big)^p$ for $i = 0, \dots, r-1$ force

$$\mu(x) = \sum_{i=0}^{r-1} \xi\big(x^{[p]^i}\big)^{p^{-i}}.$$

Thus μ is completely determined by λ and ξ.

From now on we fix some $\lambda \in X_\xi$ and extend it to a linear function on $\mathfrak{a}$ as above. This extension will be denoted by the same letter λ, so that $(\lambda, \xi) \in X(\mathfrak{a}, \mathfrak{g})$. Now define $\mathfrak{p}$ with respect to this $\lambda \in \mathfrak{a}^*$ as in Lemma 2.1. Since $\mathfrak{a} \not\subset \mathfrak{z}(\mathfrak{g})$, we have $\lambda([\mathfrak{g}, \mathfrak{a}]) \ne 0$ in view of Lemma 1.6. Hence $\mathfrak{p} \ne \mathfrak{g}$ and $\lambda(\mathfrak{a}) \ne 0$.

Put $S_{\lambda,\xi} = U_\xi(\mathfrak{p})/J_{\lambda,\xi}$ where $J_{\lambda,\xi}$ is the ideal of $U_\xi(\mathfrak{p})$ generated by all elements $x - \lambda(x)$ with $x \in \mathfrak{a}$. Since $\lambda([\mathfrak{p}, \mathfrak{a}]) = 0$, Lemma 1.6 yields

$$\dim S_{\lambda,\xi} = p^{\,\mathrm{codim}_{\mathfrak p}\,\mathfrak a}.$$

Note that simple $B_{\lambda,\xi}$-modules may be identified with those simple $U_\xi(\mathfrak g)$-modules which contain a common eigenvector for the elements of $\mathfrak a$ with the eigenvalue function λ. The $S_{\lambda,\xi}$-modules may be identified with those $U_\xi(\mathfrak p)$-modules on which $\mathfrak a$ operates via scalar transformations with the eigenvalue function λ. Let $V_1,\dots,V_n$ be a full set of pairwise nonisomorphic simple $S_{\lambda,\xi}$-modules. By Lemma 2.1 the induced modules $\mathrm{ind}_\xi V_1,\dots,\mathrm{ind}_\xi V_n$ give a full set of pairwise nonisomorphic simple $B_{\lambda,\xi}$-modules. We have

$$\dim \mathrm{ind}_\xi V_i = p^{\,\mathrm{codim}_{\mathfrak g}\,\mathfrak p}\cdot \dim V_i$$

for each i. Since the algebra $B_{\lambda,\xi}$ is semisimple,

$$\dim B_{\lambda,\xi} = \sum_{i=1}^n (\dim \mathrm{ind}_\xi V_i)^2 = p^{\,2\,\mathrm{codim}_{\mathfrak g}\,\mathfrak p}\cdot \sum_{i=1}^n (\dim V_i)^2.$$

On the other hand, $\dim B_{\lambda,\xi} = p^{\,\mathrm{codim}_{\mathfrak g}\,\mathfrak a_s}$ by Lemma 1.6. Hence

$$\begin{aligned}\sum_{i=1}^n (\dim V_i)^2 = p^{\,\mathrm{codim}_{\mathfrak g}\,\mathfrak a_s - 2\,\mathrm{codim}_{\mathfrak g}\,\mathfrak p} &= p^{\,\mathrm{codim}_{\mathfrak p}\,\mathfrak a_s - \mathrm{codim}_{\mathfrak g}\,\mathfrak p}\\ &= p^{\,\mathrm{codim}_{\mathfrak a}\,\mathfrak a_s - \mathrm{codim}_{\mathfrak g}\,\mathfrak p}\cdot \dim S_{\lambda,\xi}.\end{aligned}$$

Define a bilinear pairing $\beta:\mathfrak g\times\mathfrak a\to k$ by the rule $\beta(x,y)=\lambda([x,y])$ for $x\in\mathfrak g$ and $y\in\mathfrak a$. The left kernel of β coincides with $\mathfrak p$ by the definition of $\mathfrak p$. Consider the right kernel

$$\mathfrak r = \{y\in\mathfrak a \mid \lambda([x,y])=0 \text{ for all } x\in\mathfrak g\}.$$

Note that $\mathfrak a_s\subset\mathfrak r$ since $\mathfrak a_s\subset\mathfrak z(\mathfrak g)$. Since the pairing $\mathfrak g/\mathfrak p\times\mathfrak a/\mathfrak r\to k$ induced by β is nondegenerate, we deduce that $\mathrm{codim}_{\mathfrak g}\,\mathfrak p = \mathrm{codim}_{\mathfrak a}\,\mathfrak r$. Hence

$$\mathrm{codim}_{\mathfrak a}\,\mathfrak a_s - \mathrm{codim}_{\mathfrak g}\,\mathfrak p = \mathrm{codim}_{\mathfrak r}\,\mathfrak a_s \ge 0.$$

We always have $\dim S_{\lambda,\xi}\ge \sum(\dim V_i)^2$, and the equality is attained here precisely when the algebra $S_{\lambda,\xi}$ is semisimple. Comparing this with the earlier formula for the sum $\sum(\dim V_i)^2$, we conclude that $S_{\lambda,\xi}$ is indeed semisimple and we must have $\mathrm{codim}_{\mathfrak r}\,\mathfrak a_s = 0$, so that $\mathfrak r=\mathfrak a_s$.

Let $\pi: U_\xi(\mathfrak p)\to S_{\lambda,\xi}$ be the canonical homomorphism, and let $\mathfrak h=\pi(\mathfrak p)$, which is a Lie subalgebra in $S_{\lambda,\xi}$. Note that

$$\pi(x)=\lambda(x)\,1 \ \text{ for all } x\in\mathfrak a,\qquad \pi(y)^p=\pi(y^{[p]})+\xi(y)^p\,1 \ \text{ for all } y\in\mathfrak p.$$

Since $\lambda\ne 0$, we get $\pi(\mathfrak a)=k$, and so $k\subset\mathfrak h$. We see also that $\mathfrak h$ is closed under the p-powers in the associative algebra $S_{\lambda,\xi}$. This allows us to view $\mathfrak h$ as a p-Lie algebra. The map $\pi|_{\mathfrak p}$ is a homomorphism of Lie algebras $\mathfrak p\to\mathfrak h$ which does not respect the $[p]$-structures. However, π induces a surjective homomorphism of p-Lie algebras $\mathfrak p/\mathfrak a\to\mathfrak h/k$. In particular,

$$\dim\mathfrak{h}\le 1+\operatorname{codim}_{\mathfrak{p}}\mathfrak{a}\le\dim\mathfrak{p}<\dim\mathfrak{g}.$$

Since $\mathfrak{h}$ generates the algebra $S_{\lambda,\xi}$, there is a surjective canonical homomorphism $\varphi: U_0(\mathfrak{h})\to S_{\lambda,\xi}$. The Lie algebra $\mathfrak{h}$ has a distinguished central toral element $z=1$, the identity element of $S_{\lambda,\xi}$. Denote by I the ideal of $U_0(\mathfrak{h})$ generated by $z-1$ where 1 is now the identity element of $U_0(\mathfrak{h})$. Clearly $I\subset\operatorname{Ker}\varphi$. Since

$$\dim U_0(\mathfrak{h})/I=p^{\dim\mathfrak{h}-1}\le p^{\operatorname{codim}_{\mathfrak{p}}\mathfrak{a}}=\dim S_{\lambda,\xi},$$

we conclude that $\dim\mathfrak{h}=1+\operatorname{codim}_{\mathfrak{p}}\mathfrak{a}$ and $U_0(\mathfrak{h})/I\cong S_{\lambda,\xi}$. Then $\mathfrak{p}/\mathfrak{a}\cong\mathfrak{h}/k$ as p-Lie algebras. By Corollary 1.8 the family of reduced enveloping algebras of $\mathfrak{h}$ is generically semisimple. Then, by the induction hypothesis, $\mathfrak{h}$ admits a linear function with a toral stabilizer. By [10, 4.4] the set of all such linear functions is open in $\mathfrak{h}^*$. Hence there exists $\eta\in\mathfrak{h}^*$ such that $\mathfrak{z}(\eta)$ is a torus and $\eta(z)\ne 0$. Replacing η with its scalar multiple, we may assume that $\eta(z)=1$.

Now take any linear function $\zeta\in\mathfrak{g}^*$ such that $\zeta|_{\mathfrak{p}}=\eta\circ\pi|_{\mathfrak{p}}$. It remains to check that $\mathfrak{z}(\zeta)$ is a torus as well. It follows from the construction that $\zeta|_{\mathfrak{a}}=\eta\circ\pi|_{\mathfrak{a}}=\lambda$. If $x\in\mathfrak{z}(\zeta)$, then $\lambda([x,y])=\zeta([x,y])=0$ for all $y\in\mathfrak{a}$, whence $x\in\mathfrak{p}$ by the definition of $\mathfrak{p}$. This shows that $\mathfrak{z}(\zeta)\subset\mathfrak{p}$. Since $\pi(\mathfrak{p})=\mathfrak{h}$ and

$$\eta\big([\pi(x),\pi(y)]\big)=\eta\big(\pi([x,y])\big)=\zeta([x,y])=0$$

for all $x\in\mathfrak{z}(\zeta)$ and $y\in\mathfrak{p}$, we deduce that π maps $\mathfrak{z}(\zeta)$ into $\mathfrak{z}(\eta)$. It follows that the p-Lie algebra $\mathfrak{z}(\zeta)/\mathfrak{a}\cap\mathfrak{z}(\zeta)$ is isomorphic to a p-Lie subalgebra of the torus $\mathfrak{z}(\eta)/k$. Then $\mathfrak{z}(\zeta)/\mathfrak{a}\cap\mathfrak{z}(\zeta)$ is itself a torus. If $x\in\mathfrak{a}\cap\mathfrak{z}(\zeta)$, then $\lambda([x,y])=\zeta([x,y])=0$ for all $y\in\mathfrak{g}$, whence $x\in\mathfrak{r}=\mathfrak{a}_s$, as we have proved earlier. Thus $\mathfrak{a}\cap\mathfrak{z}(\zeta)\subset\mathfrak{a}_s$. The opposite inclusion also holds since $\mathfrak{a}_s\subset\mathfrak{z}(\mathfrak{g})\subset\mathfrak{z}(\zeta)$. We see that $\mathfrak{z}(\zeta)/\mathfrak{a}_s$ is a torus. But so too is $\mathfrak{a}_s$. Then $\mathfrak{z}(\zeta)$ is a torus by [16, Cor. 2.13]. □

3. Recognition of generic semisimplicity by a simple block

Let R be a commutative ring. By a commutative R-algebra we will always mean a commutative, associative and unital algebra. Denote by $Comm_R$ the category of commutative R-algebras. We will write $K\in Comm_R$ to mean that K is an object of this category. We will use the language of R-functors following [3, I.1.6.1]. These are covariant functors from the category $Comm_R$ to the category of sets.

Each commutative R-algebra B gives rise to an R-functor h_B such that $h_B(K)$ is the set of algebra homomorphisms $B\to K$ and the map $h_B(\gamma): h_B(K)\to h_B(K')$, for an algebra homomorphism $\gamma: K\to K'$, is obtained by composing algebra homomorphisms $B\to K$ with γ (in the notation and terminology of [3] h_B is just the affine R-scheme $\mathfrak{S}p_R\,B$). An arbitrary R-functor X is *representable* by B if there exists an isomorphism of functors $X\cong h_B$.

A commutative R-algebra S is said to be *formally smooth* (with respect to the discrete topologies on R and S) if for every commutative R-algebra K and every nilpotent ideal I of K each homomorphism of R-algebras $S\to K/I$ admits a lifting to a homomorphism of R-algebras $S\to K$. If, in addition to the formal smoothness, this algebra is finitely presented, it is called *smooth.*

These definitions of smoothness are due to Grothendieck [6, 17.3.2.ii]. Note, however, that Matsumura [8] does not assume finite presentation when using the term "smooth algebra". We will need the following fundamental properties of smooth algebras (it is essential here that the algebra is finitely presented):

Lemma 3.1. *Suppose that S is a smooth commutative R-algebra. Then S is flat over R and the canonical map of the prime spectra* $\operatorname{Spec} S \to \operatorname{Spec} R$ *is open.*

This lemma records the affine case of two well-known facts. By [6, 17.5.1] any smooth morphism of schemes is flat, and by [6, 2.4.6] any flat, locally finitely presented morphism of schemes is open in the sense that it takes open subsets of one topological space to open subsets of the other.

With each R-module M we associate an R-functor $M_a = M \otimes_R ?$.

Lemma 3.2. *Let M, N be two finitely generated projective R-modules, and let X be the equalizer of a pair of natural transformations $\varphi, \psi : M_a \to N_a$. The R-functor X is representable by a finitely presented commutative R-algebra.*

Proof. The R-functor M_a is representable by the symmetric algebra SM_R^* of the dual R-module $M_R^* = \operatorname{Hom}_R(M, R)$. Indeed, for $K \in Comm_R$ the R-algebra homomorphisms $SM_R^* \to K$ are in a natural bijective correspondence with the R-module homomorphisms $M_R^* \to K$, but $\operatorname{Hom}_R(M_R^*, K) \cong M \otimes_R K$ by projectivity of M. Similarly, N_a is representable by SN_R^*.

By Yoneda's Lemma the natural transformations φ and ψ correspond to some algebra homomorphisms $\varphi^\sharp$, $\psi^\sharp : SN_R^* \to SM_R^*$. Put $Q = SM_R^*/J$ where J is the ideal of SM_R^* generated by the image of the map $\varphi^\sharp - \psi^\sharp$. Consider the algebra homomorphism $\alpha_x : SM_R^* \to K$ corresponding to some $x \in M \otimes_R K$. We have $x \in X(K)$ if and only if $\varphi(x) = \psi(x)$, if and only if $\alpha_x \circ \varphi^\sharp = \alpha_x \circ \psi^\sharp$, if and only if $\alpha_x(J) = 0$, if and only if α_x factors through Q. This shows that $X \cong h_Q$.

The R-algebras SM_R^* and SN_R^* are finitely presented since M_R^* and N_R^* are finitely generated projective R-modules. Let $\{y_1, \dots, y_n\}$ be any finite set generating SN_R^* as an R-algebra. It is easy to see that the ideal J of SM_R^* is generated by the finite set $\{\varphi^\sharp(y_i) - \psi^\sharp(y_i) \mid i = 1, \dots, n\}$. Hence the R-algebra Q is finitely presented too. □

Now let A be any associative R-algebra (with 1). For each $K \in Comm_R$ denote by $E_A(K)$ the set of all idempotent elements of the K-algebra $A \otimes_R K$. Each morphism $\gamma : K \to K'$ in $Comm_R$ induces a ring homomorphism $\mathrm{id} \otimes \gamma : A \otimes_R K \to A \otimes_R K'$ which maps $E_A(K)$ into $E_A(K')$. This makes E_A an R-functor.

For $e \in E_A(K)$ there is a decomposition $A \otimes_R K = (A \otimes_R K)e \oplus (A \otimes_R K)(1-e)$. If the underlying R-module of A is finitely generated projective, then so too is the underlying K-module of $A \otimes_R K$, whence both summands in the above decomposition are finitely generated projective K-modules. For each nonnegative integer d define a subfunctor E_A^d of E_A setting

$$E_A^d(K) = \{e \in E_A(K) \mid (A \otimes_R K)e \text{ is a projective } K\text{-module of constant rank } d\}.$$

Recall that, given any finitely generated projective K-module M, its localization $M_\mathfrak{p}$ at a prime $\mathfrak{p} \in \operatorname{Spec} K$ is a free module over the local ring $K_\mathfrak{p}$. The rank function $\mathfrak{p} \mapsto \operatorname{rank} M_\mathfrak{p}$ is locally constant on $\operatorname{Spec} K$ by [1, Ch. II, §5, Th. 1]. Thus $\operatorname{Spec} K$ is a disjoint union of its open subsets $U_i = \{\mathfrak{p} \in \operatorname{Spec} K \mid \operatorname{rank} M_\mathfrak{p} = i\}$. On the level of R-algebras we get a direct product decomposition $K \cong \prod_{i=0}^n K_i$ where $\operatorname{Spec} K_i$ is identified with U_i for each i. Given a homomorphism of commutative R-algebras $\gamma : K \to K'$, the projective K'-module $M \otimes_K K'$ is of constant rank d if and only if γ factors through K_d.

Lemma 3.3. *Let A be an associative R-algebra with a finitely generated projective underlying R-module. Then the R-functors E_A and E_A^d are representable by smooth commutative R-algebras.*

Proof. The R-functor E_A is a subfunctor of the R-functor $A_a = A \otimes_R ?$. Consider the natural transformation $\varphi : A_a \to A_a$ defined by the formula $\varphi(x) = x^2$ for each $x \in A \otimes_R K$ with $K \in Comm_R$. Thus x is an idempotent if and only if $\varphi(x) = x$. This means that E_A coincides with the equalizer of the couple (φ, ψ) where ψ stands for the identity transformation of A_a. By Lemma 3.2 E_A is representable by a finitely presented R-algebra Q.

Let $K \in Comm_R$, and let I be any nilpotent ideal of K. Since the canonical map $A \otimes_R K \to A \otimes_R K/I$ is a surjective ring homomorphism with a nilpotent kernel, every idempotent of $A \otimes_R K/I$ can be lifted to an idempotent of $A \otimes_R K$ by [11, Cor. 1.1.28]. This means that the map $E_A(K) \to E_A(K/I)$ is surjective. Since E_A is representable by the R-algebra Q, the previous property translates readily to give the required formal smoothness of Q. Thus Q is a smooth R-algebra.

By representability of the functor E_A there are bijections $h_Q(K) \to E_A(K)$, natural in $K \in Comm_R$. Let $u \in A \otimes_R Q$ be the idempotent corresponding to the identity homomorphism $Q \to Q$. If $e \in A \otimes_R K$ is the idempotent corresponding to an arbitrary homomorphism of commutative R-algebras $\beta : Q \to K$, then e coincides with the image of u under the map $\mathrm{id} \otimes \beta : A \otimes_R Q \to A \otimes_R K$.

The direct summand $M = (A \otimes_R Q)u$ of $A \otimes_R Q$ is a finitely generated projective Q-module. Hence there is a decomposition $Q \cong \prod_{i=0}^n Q_i$ such that $M \otimes_Q Q_i$ is a projective Q_i-module of constant rank i for each i. For β and e as above we have

$$(A \otimes_R K)e \cong M \otimes_Q K.$$

It follows that the projective K-module $(A \otimes_R K)e$ has constant rank d if and only if β factors through Q_d. In other words, $E_A^d \cong h_{Q_d}$. The R-algebra Q_d is smooth since Q is smooth, while Q_d is a direct factor of Q. □

Lemma 3.4. *Let $(A_\xi)_{\xi \in X}$ be a flat family of finite dimensional associative algebras over an algebraically closed field k parameterized by an algebraic variety X. Given any nonnegative integer d, the subset of those $\xi \in X$ for which A_ξ has a projective left module of dimension d is open in X.*

Proof. Since the conclusion can be verified locally, we may assume X to be affine. Let R be the ring of regular functions on X. There exists an associative R-algebra A such that its underlying R-module is finitely generated projective and $A_\xi \cong A/\mathfrak{m}_\xi A$ for each $\xi \in X$ where $\mathfrak{m}_\xi$ is the maximal ideal of R consisting of all regular functions on X vanishing at ξ.

We have to prove that, whenever P is a projective A_η-module for some $\eta \in X$, the algebra A_ξ has a projective module of the same dimension for each ξ in a suitable neighborhood of η in X. Since P is a direct sum of indecomposable projectives, it suffices to do this assuming P to be indecomposable. In this case P is isomorphic with a left ideal of A_η generated by an idempotent.

Let $d = \dim P$, and let Q_d be the R-algebra representing the R-functor E_A^d. Since Q_d is smooth by Lemma 3.3, the map $\operatorname{Spec} Q_d \to \operatorname{Spec} R$ has an open image, say U, by Lemma 3.1. For $\xi \in X$ we have $\mathfrak{m}_\xi \in U$ if and only if Q_d has a prime ideal lying over $\mathfrak{m}_\xi$, if and only if Q_d has a maximal ideal lying over $\mathfrak{m}_\xi$. Since R is a finitely generated k-algebra and Q_d a finitely generated R-algebra, the latter is finitely generated also as a k-algebra. By Hilbert's Nullstellensatz $Q_d/\mathfrak{n} \cong k$ for every maximal ideal $\mathfrak{n}$ of Q_d. It follows that $\mathfrak{m}_\xi \in U$ if and only if there exists a homomorphism of R-algebras $Q_d \to R/\mathfrak{m}_\xi$. But such homomorphisms are in a bijective correspondence with the elements of the set $E^d_A(R/\mathfrak{m}_\xi)$, that is, with the idempotents $e \in A_\xi$ such that $\dim A_\xi e = d$ (since $R/\mathfrak{m}_\xi \cong k$, the rank of $A_\xi e$ as a module over $R/\mathfrak{m}_\xi$ is the same as its k-vector space dimension).

Consider the open subset $U' = \{\xi \in X \mid \mathfrak{m}_\xi \in U\}$ of X. We have shown that U' consists precisely of those $\xi \in X$ for which the algebra A_ξ has an idempotent e such that $\dim A_\xi e = d$. In particular, $\eta \in U'$ and the algebra A_ξ has a projective module of dimension d for each $\xi \in U'$. □

Proposition 3.5. *Let $\mathfrak{g}$ be a finite dimensional p-Lie algebra over an algebraically closed field of characteristic $p > 0$, and let $\xi \in \mathfrak{g}^*$.*

(i) *If V is any finite dimensional projective $U_\xi(\mathfrak{g})$-module then* $\mathrm{md}(\mathfrak{g})$ *divides* $\dim V$.

(ii) *If V is a simple projective $U_\xi(\mathfrak{g})$-module then* $\dim V = \mathrm{md}(\mathfrak{g})$.

Proof. Let V be a projective $U_\xi(\mathfrak{g})$-module of finite dimension. By Lemma 3.4 and by [10, 4.2] there exist nonempty open subsets W_1, W_2 of $\mathfrak{g}^*$ such that the algebras $U_\eta(\mathfrak{g})$ with $\eta \in W_1$ all have a projective module of dimension equal to $\dim V$, while the algebras $U_\eta(\mathfrak{g})$ with $\eta \in W_2$ have simple modules only of dimension equal to $\mathrm{md}(\mathfrak{g})$. Since $\mathfrak{g}^*$ is an irreducible algebraic variety, these two subsets have a nonempty intersection. Let $\eta \in W_1 \cap W_2$. Then $U_\eta(\mathfrak{g})$ has a projective module P such that $\dim P = \dim V$. At the same time all composition factors of P are of dimension $\mathrm{md}(\mathfrak{g})$. Therefore $\mathrm{md}(\mathfrak{g})$ divides $\dim P$, and (i) is proved.

Under the hypothesis of (ii) we have $0 < \dim V \le \mathrm{md}(\mathfrak{g})$ since V is simple. Hence the conclusion follows from (i). □

Combining Proposition 3.5(ii) with Lemma 1.5(i), we get

Theorem 3.6. *If for some $\xi \in \mathfrak{g}^*$ the algebra $U_\xi(\mathfrak{g})$ has a simple projective module then the family of reduced enveloping algebras of $\mathfrak{g}$ is generically semisimple.*

Remark 3.7. Since the algebra $U_\xi(\mathfrak{g})$ is Frobenius (see [14, Cor. 5.4.3]), any projective left $U_\xi(\mathfrak{g})$-module V is also injective. If V is simple, the sum of all left ideals of $U_\xi(\mathfrak{g})$ isomorphic to V is therefore a block of $U_\xi(\mathfrak{g})$ isomorphic to $\mathrm{End}_k V$ as an algebra. Thus the hypothesis of Theorem 3.6 means precisely that $U_\xi(\mathfrak{g})$ has a simple block.

4. Variation of the numeric invariants in a family of algebras

The method from the previous section leads to further conclusions. Suppose that A is an associative algebra over a commutative ring R.

Let D^n_A be the R-functor such that $D^n_A(K)$ for $K \in \mathit{Comm}_R$ is the set of all n-tuples $(e_1, \ldots, e_n)$ of pairwise orthogonal idempotents in the K-algebra $A \otimes_R K$ with the property that $\sum_{i=1}^n e_i = 1$ and the map $D^n_A(\gamma)$ for a morphism $\gamma : K \to K'$

in $Comm_R$ is obtained by applying the map $\mathrm{id} \otimes \gamma : A \otimes_R K \to A \otimes_R K'$ to each component of the n-tuples $(e_1, \dots, e_n) \in D_A^n(K)$.

Given an n-tuple $(e_1, \dots, e_n) \in D_A^n(K)$, the K-algebra $A \otimes_R K$ is a direct sum of its K-submodules $e_i(A \otimes_R K)e_j$, each of which has to be projective provided A is projective over R. For any n by n matrix $Z = (z_{ij})$ whose entries are nonnegative integers define a subfunctor D_A^Z of D_A^n by the rule

$$D_A^Z(K) = \{(e_1, \dots, e_n) \in D_A^n(K) \mid \ e_i(A \otimes_R K)e_j \text{ is a projective } K\text{-module of constant rank } z_{ij} \text{ for each pair } i, j\}.$$

Lemma 4.1. *Let A be an associative R-algebra with a finitely generated projective underlying R-module. Then the R-functors D_A^n and D_A^Z are representable by smooth commutative R-algebras.*

Proof. The condition for an n-tuple $(x_1, \dots, x_n) \in (A \otimes_R K)^n$ to consist of pairwise orthogonal idempotents is expressed by n^2 equations $x_i x_j = \delta_{ij} x_i,\ 1 \le i, j \le n$. The n-tuples in $D_A^n(K)$ are distinguished by an additional equation $\sum x_i = 1$. This means that D_A^n coincides with the equalizer of a pair of natural transformations $A_a^n \rightrightarrows A_a^{n^2+1}$. By Lemma 3.2 D_A^n is representable by a finitely presented commutative R-algebra, say Q.

If I is a nilpotent ideal of a commutative R-algebra K, then any n-tuple $(e'_1, \dots, e'_n)$ of pairwise orthogonal idempotents in $A \otimes_R K/I$ can be lifted, by [11, Prop. 1.1.25], to an n-tuple $(e_1, \dots, e_n)$ of pairwise orthogonal idempotents in $A \otimes_R K$. Moreover, the condition $\sum e'_i = 1$ entails $\sum e_i = 1$ since $1 - \sum e_i$ is an idempotent lying in the nilpotent ideal $A \otimes_R I$ of $A \otimes_R K$. In other words, the map $D_A^n(K) \to D_A^n(K/I)$ is surjective. This shows that Q is smooth.

There are bijections $h_Q(K) \to D_A^n(K)$, natural in $K \in Comm_R$. Let $(u_1, \dots, u_n)$ be the n-tuple of pairwise orthogonal idempotents in $A \otimes_R Q$ corresponding to the identity homomorphism $Q \to Q$. There is a decomposition

$$A \otimes_R Q = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n} M_{ij} \quad \text{where } M_{ij} = u_i(A \otimes_R Q)u_j.$$

Since $A \otimes_R Q$ is a finitely generated projective Q-module, so is each direct summand M_{ij}. As the rank functions of these summands are locally constant, $\operatorname{Spec} Q$ is a disjoint union of its open subsets on which each M_{ij} has constant rank. In particular, the subset

$$U_Z = \{\mathfrak{p} \in \operatorname{Spec} Q \mid \operatorname{rank}(M_{ij})_{\mathfrak{p}} = z_{ij} \text{ for each pair } i, j\}$$

is open and closed in $\operatorname{Spec} Q$ simultaneously. Hence U_Z is an affine subscheme of $\operatorname{Spec} Q$ whose coordinate ring, say Q_Z, is a direct factor of Q. In particular, Q_Z is a smooth R-algebra.

Given an arbitrary morphism $\beta : Q \to K$ in $Comm_R$, the corresponding n-tuple $(e_1, \dots, e_n) \in D_A^n(K)$ has components $e_i = (\mathrm{id} \otimes \beta)(u_i)$. Hence

$$e_i(A \otimes_R K)e_j \cong M_{ij} \otimes_Q K$$

for each pair i, j. The projective K-modules $M_{ij} \otimes_Q K,\ 1 \le i, j \le n$, are of constant rank z_{ij}, respectively, if and only if U_Z contains the whole image of the map $\operatorname{Spec} K \to \operatorname{Spec} Q$ associated with β, if and only if β factors through Q_Z. This shows that D_A^Z is representable by the R-algebra Q_Z. □

Lemma 4.2. *Let A be an associative R-algebra, M and N two left A-modules. Suppose that M is projective, and both M and N are finitely generated projective as R-modules. Then*

$$\{\mathfrak{m} \in \operatorname{Specm} R \mid \ M/\mathfrak{m}M \cong N/\mathfrak{m}N \text{ as } A\text{-modules}\}$$

is an open subset of the maximal spectrum $\operatorname{Specm} R$.

Proof. Suppose that $M/\mathfrak{m}M \cong N/\mathfrak{m}N$ for some $\mathfrak{m} \in \operatorname{Specm} R$. Since M is a projective A-module, there exists an A-linear map $\varphi : M \to N$ whose reduction modulo $\mathfrak{m}$ is an isomorphism $M/\mathfrak{m}M \to N/\mathfrak{m}N$. Put $K = \operatorname{Ker}\varphi$ and $L = \operatorname{Coker}\varphi$. Then $L/\mathfrak{m}L = 0$. Since L is finitely generated as an R-module, there exists an element $s \in A$ such that $s \notin \mathfrak{m}$ and $sL = 0$. Passing to the localizations with respect to the multiplicatively closed set of powers of s, we get an epimorphism of A_s-modules $\varphi_s : M_s \to N_s$ with $\operatorname{Ker}\varphi_s \cong K_s$. Since M_s and N_s are finitely generated projective R_s-modules, φ_s splits as an epimorphism of R_s-modules. Hence K_s is an R_s-module direct summand of M_s. In particular, K_s is a finitely generated R_s-module. Also, $K_s/\mathfrak{m}K_s$ is isomorphic with the kernel of the map $\varphi_s \otimes_{R_s} R_s/\mathfrak{m}R_s$, which may be identified with the isomorphism $\varphi \otimes_R R/\mathfrak{m}$ since $R_s/\mathfrak{m}R_s \cong R/\mathfrak{m}$. It follows that $K_s/\mathfrak{m}K_s = 0$. By Nakayama's Lemma the annihilator of K_s in R_s is not contained in $\mathfrak{m}R_s$. Hence there exists $u \in R$ such that $u \notin \mathfrak{m}$ and $uK_s = 0$. Replacing s with su, we may assume that $K_s = 0$. Then φ_s is bijective, and therefore $\varphi \otimes_R R/\mathfrak{n}$ is an isomorphism of A-modules $M/\mathfrak{n}M \to N/\mathfrak{n}N$ for all $\mathfrak{n}$ in the open neighborhood $\{\mathfrak{n} \in \operatorname{Specm} R \mid s \notin \mathfrak{n}\}$ of $\mathfrak{m}$ in $\operatorname{Specm} R$. □

Denote by T_{pre} the set of all pairs (r, C) where $r = (r_1, \dots, r_n)$ is an n-tuple of positive integers for some $n > 0$ and $C = (c_{ij})$ is an n by n matrix whose entries are nonnegative integers with $c_{ii} > 0$ for all i. Let (r, C) and (r', C') be two elements of T_{pre}. We say that $(r, C) \ge (r', C')$ if there exists a nonnegative integer matrix D such that

$$r = r'D \quad \text{and} \quad C' = DCD^t$$

where D^t is the transpose of D. Such a matrix D must be of size m by n where n is the length of r and m is the length of r'. It is immediate that the relation $\ge$ defined on T_{pre} is reflexive and transitive.

Lemma 4.3. *Suppose that $(r, C) \ge (r', C')$, and let D be any nonnegative integer matrix such that $r = r'D$ and $C' = DCD^t$. Then $(r, C) \le (r', C')$ if and only if D is a permutation matrix.*

Proof. Let $r = (r_1, \dots, r_n)$, $r' = (r'_1, \dots, r'_m)$, $C = (c_{ij})$, $C' = (c'_{ij})$, and $D = (d_{ij})$. Put $|r| = \sum_{j=1}^n r_j$ and $|r'| = \sum_{i=1}^m r'_i$. The equality $C' = DCD^t$ shows that

$$c'_{ii} = \sum_{j=1}^n \sum_{l=1}^n d_{ij} c_{jl} d_{il} \quad \text{for all } i = 1, \dots, m.$$

Since $c'_{ii} > 0$, the i-th row of the matrix D cannot be zero. So for each i there exists j such that $d_{ij} \geq 1$. Since $r_j = \sum_{i=1}^m r'_i d_{ij}$ for each $j = 1, \ldots, n$, we get

$$|r| = \sum_{i=1}^{m} \big(\sum_{j=1}^{n} d_{ij}\big) r'_i \geq |r'|.$$

Suppose that $(r, C) \leq (r', C')$. Then $|r| \leq |r'|$ by symmetry, whence $|r| = |r'|$. We must have $\sum_{j=1}^n d_{ij} = 1$ for each i, which means that each row of D contains exactly one nonzero entry, and this entry is equal to 1. Since $\sum_{i=1}^m r'_i d_{ij} = r_j > 0$ for all j, the matrix D cannot contain zero columns as well. It follows that $n \leq m$. Then we must have $m \leq n$ by symmetry, whence $m = n$. Any matrix D with these properties is necessarily a permutation matrix.

Conversely, if D is a permutation matrix, then D is invertible with $D^{-1} = D^t$, and the equalities $r' = rD^t$, $C = D^t C' D$ show that $(r, C) \leq (r', C')$. □

Define an equivalence relation on T_{pre} setting $(r, C) \sim (r', C')$ if $(r, C) \geq (r', C')$ and $(r, C) \leq (r', C')$ simultaneously. By Lemma 4.3 this means that r' is obtained from r by a permutation of components, and C' is obtained from C by the same permutation of rows and columns. The preordering $\leq$ on T_{pre} induces a partial ordering on the set of equivalence classes $T = T_{\text{pre}}/\sim$. We will denote by $[r, C] \in T$ the equivalence class of $(r, C) \in T_{\text{pre}}$.

Let A be a finite dimensional associative algebra over an algebraically closed field. We say that A has a *decomposition of type* $[r, C] \in T$ where $r = (r_1, \ldots, r_n)$ and $C = (c_{ij})$ if there exist projective left A-modules $P_1, \ldots, P_n$ such that

$$A \cong \bigoplus_{i=1}^{n} P_i^{r_i} \text{ as left } A\text{-modules} \quad \text{and} \quad \dim \operatorname{Hom}_A(P_i, P_j) = c_{ij} \text{ for all } i, j.$$

Isomorphic modules are allowed among $P_1, \ldots, P_n$. Note that the condition $c_{ii} > 0$ entails $P_i \neq 0$.

We say that $[r, C]$ is the *numeric type* of A, abbreviated as $\operatorname{nt} A$, if there is a decomposition as above in which $P_1, \ldots, P_n$ are pairwise nonisomorphic indecomposable projectives. In this case P_i is the projective cover of a simple A-module S_i and the multiplicity r_i with which P_i occurs in the decomposition of A is equal to the dimension of S_i. The number c_{ij} is equal to the multiplicity of S_i as a composition factor of P_j and the matrix C is known as the Cartan matrix.

Lemma 4.4. *If A has a decomposition of type $[r', C'] \in T$, then $[r', C'] \leq \operatorname{nt} A$.*

Proof. Consider a decomposition $A \cong \bigoplus_{i=1}^m P_i^{r'_i}$ of type $[r', C']$ with

$$\dim \operatorname{Hom}_A(P_i, P_j) = c'_{ij}.$$

Each P_i can be written as a direct sum of indecomposable projective left A-modules $I_1, \ldots, I_n$. Let $P_i \cong \bigoplus_{x=1}^n I_x^{d_{ix}}$. Then $r_x = \sum_{i=1}^m r'_i d_{ix}$ is the multiplicity of I_x as a direct summand of A, and

$$c'_{ij} = \dim \operatorname{Hom}_A\big(\bigoplus_{x=1}^{n} I_x^{d_{ix}}, \bigoplus_{y=1}^{n} I_y^{d_{jy}}\big) = \sum_{x=1}^{n} \sum_{y=1}^{n} d_{ix} d_{jy} c_{xy}$$

where $c_{xy} = \dim \operatorname{Hom}_A(I_x, I_y)$ are the entries of the Cartan matrix C. Since the matrix $D = (d_{ix})$ satisfies $r = r'D$ and $C' = DCD^t$, we have $[r', C'] \le [r, C] = \operatorname{nt} A$. □

Lemma 4.5. *Let $(A_\xi)_{\xi \in X}$ be a flat family of finite dimensional associative algebras over an algebraically closed field k parameterized by an algebraic variety X. Then for any $[r, C] \in T$ the following subsets are open in X:*

(i) $\{\xi \in X \mid$ *the algebra* A_ξ *has a decomposition of type* $[r, C]\}$,

(ii) $\{\xi \in X \mid \operatorname{nt} A_\xi \ge [r, C]\}$.

Proof. As in Lemma 3.4 we may assume X to be affine. Then $X \cong \operatorname{Specm} R$ and there is an associative R-algebra A, finitely generated projective as an R-module, such that $A_\xi \cong A/\mathfrak{m}_\xi A$ for each $\xi \in X$ where $\mathfrak{m}_\xi \in \operatorname{Specm} R$ corresponds to ξ.

Let $r = (r_1, \ldots, r_n)$ and $C = (c_{ij})$. Put $|r| = r_1 + \cdots + r_n$. Denote by Y the subset in (i) and suppose that $\xi \in Y$. Then the algebra A_ξ has a collection of pairwise orthogonal idempotents $e_1, \ldots, e_{|r|}$ such that $\sum e_i = 1$ and for a suitable partitioning of the set $\{1, \ldots, |r|\}$ into n disjoint subsets $\omega_1, \ldots, \omega_n$ of cardinality $r_1, \ldots, r_n$, respectively, the isomorphism class of the left ideal $A_\xi e_i$ depends only on the subset $\omega_{i'}$ containing i with

$$\dim e_i A_\xi e_j = z_{ij} \quad \text{for all } i, j \in \{1, \ldots, |r|\}$$

where $z_{ij} = c_{i'j'}$ whenever $i \in \omega_{i'}$, $j \in \omega_{j'}$ (note that $e_i A_\xi e_j \cong \operatorname{Hom}_{A_\xi}(A_\xi e_i, A_\xi e_j)$, so that the above conditions mean precisely that the direct summands in the decomposition $A_\xi = \bigoplus_{i=1}^{|r|} A_\xi e_i$ break up into n groups of pairwise isomorphic modules with the required dimensions of Hom spaces).

Let Q be the commutative R-algebra representing the R-functor D_A^Z where Z is the $|r|$ by $|r|$ matrix (z_{ij}), and let $(u_1, \ldots, u_{|r|})$ be the $|r|$-tuple of pairwise orthogonal idempotents in $A \otimes_R Q$ corresponding to the identity homomorphism $Q \to Q$. The $|r|$-tuple $(e_1, \ldots, e_{|r|})$ corresponds to a homomorphism of R-algebras $\beta : Q \to R/\mathfrak{m}_\xi$ such that $e_i = (\mathrm{id} \otimes \beta)(u_i)$ for each i.

Let $\mathfrak{n} = \operatorname{Ker} \beta$. Then $\mathfrak{n} \in \operatorname{Specm} Q$ and $Q/\mathfrak{n} \cong R/\mathfrak{m}_\xi$. For each i the left ideal $M_i = (A \otimes_R Q)u_i$ is a direct summand of $A \otimes_R Q$ and

$$M_i/\mathfrak{n}M_i \cong M_i \otimes_Q R/\mathfrak{m}_\xi \cong A_\xi e_i.$$

Hence $M_i/\mathfrak{n}M_i \cong M_j/\mathfrak{n}M_j$ as $A \otimes_R Q$-modules whenever i and j lie in the same subset ω_p. Each M_i is a finitely generated projective $A \otimes_R Q$-module. Since $A \otimes_R Q$ is finitely generated projective over Q, so too is M_i. By Lemma 4.2 the subset

$$U = \{\mathfrak{q} \in \operatorname{Specm} Q \mid M_i/\mathfrak{q}M_i \cong M_j/\mathfrak{q}M_j \text{ for each pair of indices } i, j \text{ lying in the same subset } \omega_p\}$$

is open in $\operatorname{Specm} Q$. We have $\mathfrak{n} \in U$. Since Q is smooth by Lemma 4.1, the canonical map $f : \operatorname{Spec} Q \to \operatorname{Spec} R$ is open by Lemma 3.1. Since Q is finitely generated as a k-algebra, the open subsets of $\operatorname{Spec} Q$ are in a bijective correspondence with the open subsets of $\operatorname{Specm} Q$. Hence $f^{-1}(\mathfrak{m}) \cap \operatorname{Specm} Q$ is dense in $f^{-1}(\mathfrak{m})$ for each

$\mathfrak{m} \in \operatorname{Specm} R$, and it follows that the map $\operatorname{Specm} Q \to \operatorname{Specm} R$ is open as well. In particular, $f(U)$ is an open subset of $\operatorname{Specm} R$.

Now $V = \{\zeta \in X \mid \mathfrak{m}_\zeta \in f(U)\}$ is an open subset of X. Since $\mathfrak{m}_\xi = f(\mathfrak{n})$, we have $\xi \in V$. Suppose that $\zeta \in V$. Then there exists $\mathfrak{q} \in U$ lying over $\mathfrak{m}_\zeta$. By Hilbert's Nullstellensatz $Q/\mathfrak{q} \cong k$. So $Q/\mathfrak{q} \cong R/\mathfrak{m}_\zeta$ as R-algebras. Let $\gamma : Q \to R/\mathfrak{m}_\zeta$ be the homomorphism of R-algebras with kernel $\mathfrak{q}$. Setting $\varepsilon_i = (\mathrm{id} \otimes \gamma)(u_i)$, we get pairwise orthogonal idempotents $\varepsilon_1, \ldots, \varepsilon_r \in A_\zeta$ such that $\sum \varepsilon_i = 1$ and $A_\zeta \varepsilon_i \cong M_i/\mathfrak{q}M_i$ for each i. Moreover, $\dim \varepsilon_i A_\zeta \varepsilon_j = z_{ij}$ since Q represents the functor D_A^Z, and $A_\zeta \varepsilon_i \cong A_\zeta \varepsilon_j$ whenever i and j lie in the same subset ω_p by our choice of U. In other words, the decomposition $A_\zeta = \bigoplus_{i=1}^{|r|} A_\zeta \varepsilon_i$ may be viewed as a decomposition of type $[r, C]$. We conclude that $V \subset Y$. Thus Y contains a suitable neighborhood in X of any of its points.

The conclusion in case (i) is now proved. Then for any $\xi \in X$ there exists a neighborhood W of ξ in X such that for each $\eta \in W$ the algebra A_η has a decomposition of type equal to $\operatorname{nt} A_\xi$, and therefore $\operatorname{nt} A_\eta \geq \operatorname{nt} A_\xi$ by Lemma 4.4. So (ii) is also clear. □

Theorem 4.6. *Let $(A_\xi)_{\xi \in X}$ be a flat family of finite dimensional associative algebras over an algebraically closed field parameterized by an irreducible algebraic variety. Then the set $\{\operatorname{nt} A_\xi \mid \xi \in X\}$ has a largest element, say t_{gen}. Furthermore, $\{\xi \in X \mid \operatorname{nt} A_\xi = t_{\mathrm{gen}}\}$ is an open subset of X.*

Proof. The dimension of algebras in a flat family is a locally constant function of the parameter. Since X is irreducible, all the algebras A_ξ have the same dimension, say d. If $\operatorname{nt} A_\xi = [r, C]$ with $r = (r_1, \ldots, r_n)$ and $C = (c_{ij})$, then j-th indecomposable projective left A_ξ-module I_j has dimension $\sum_{i=1}^m r_i c_{ij}$, whence

$$d = \dim A_\xi = \sum_{j=1}^n r_j \dim I_j = \sum_{j=1}^n \sum_{i=1}^n r_i r_j c_{ij}.$$

Since $r_i > 0$ and $c_{ii} > 0$ for all i, the set T has finitely many elements satisfying this equality. Hence the set $\{\operatorname{nt} A_\xi \mid \xi \in X\}$ is finite, and so it contains a maximal element. If t is such a maximal element, then $\{\xi \in X \mid \operatorname{nt} A_\xi = t\}$ is an open subset of X by Lemma 4.5(ii). Since any two nonempty open subsets of X have a nonempty intersection, there cannot exist another maximal element. □

The approach of this section enables us to derive a stronger rigidity property of semisimple algebras than that stated in Lemma 1.1. We say that an algebra B is a *direct factor* of an algebra A if there is an isomorphism of algebras $A \cong B \times B'$ for a suitable B'.

Theorem 4.7. *Let $(A_\xi)_{\xi \in X}$ be a flat family of finite dimensional associative algebras over an algebraically closed field, and let B be a fixed semisimple associative algebra. Then the subset of those $\xi \in X$ for which B is a direct factor of A_ξ is open in the variety X.*

Proof. Suppose that B is a direct factor of A_ξ for some ξ. We will show that B is a direct factor of A_η for each η in a suitable neighborhood of ξ in X.

Let $[r, C] = \operatorname{nt} A_\xi$ with $r = (r_1, \dots, r_n)$ and $C = (c_{ij})$. Let $S_1, \dots, S_n$ be a full set of pairwise nonisomorphic simple A_ξ-modules. We may assume that they are ordered in such a way that first come all simple B-modules, that is, the modules on which A_ξ operates via the projection $A_\xi \to B$. Suppose that B has q nonisomorphic simple modules; these are $S_1, \dots, S_q$. Then the Cartan matrix of A_ξ has the block form

$$C = \begin{pmatrix} E_q & 0 \\ 0 & M \end{pmatrix} \tag{$*$}$$

where E_q is the identity matrix of order q and M is the Cartan matrix of the direct factor B', complementary to B in A_ξ. By Lemma 4.5 there exists a neighborhood U of ξ in X such that for each $\eta \in U$ the algebra A_η has a decomposition of type $[r, C]$. Let $A_\eta \cong \bigoplus_{i=1}^n P_i^{r_i}$ be such a decomposition. Then

$$\dim \operatorname{Hom}_{A_\eta}(P_i, P_j) = c_{ij} \quad \text{for all } i, j.$$

In particular, $\dim \operatorname{End}_{A_\eta} P_i = 1$ whenever $i \le q$. Note that for any A_η-module V, the dimension of $\operatorname{End}_{A_\eta} V$ is never less than the number of indecomposable summands in a direct sum decomposition of V. Therefore P_i is indecomposable for $i \le q$. It follows also that the simple A_η-module on top of P_i has multiplicity 1 in P_i. Since $\operatorname{Hom}_{A_\eta}(P_j, P_i) = 0$ for $j \ne i$, we have $\operatorname{Hom}_{A_\eta}(I, P_i) = 0$ for each indecomposable projective left A_η-module $I \not\cong P_i$. Hence P_i cannot contain other composition factors. We also have $\operatorname{Hom}_{A_\eta}(P_i, P_j) = 0$ for $j \ne i$, and therefore $\operatorname{Hom}_{A_\eta}(P_i, I) = 0$ for I as above.

In other words, the projective A_η-modules $P_1, \dots, P_q$ are simple, and they do not occur as composition factors in the remaining indecomposable projectives. This property ensures that the algebra $B_\eta = \operatorname{End}_k P_1 \times \cdots \times \operatorname{End}_k P_q$ is a direct factor of A_η. Since

$$P_i \cong \operatorname{Hom}_{A_\eta}(A_\eta, P_i) \cong \bigoplus_{j=1}^n \operatorname{Hom}_{A_\eta}(P_j^{r_j}, P_i),$$

we get $\dim P_i = r_i = \dim S_i$ for each $i = 1, \dots, q$. Hence $B_\eta \cong B$. □

Remark 4.8. One can view the conclusion of Theorem 4.7 as a purely combinatorial consequence of the fact that $\operatorname{nt} A_\eta \ge \operatorname{nt} A_\xi$ for all η in a suitable neighborhood of ξ in X. Indeed, if $[r, C] \le [r', C']$ for two elements of T and the matrix $C = (c_{ij})$ has the block decomposition $(*)$, then the matrix $C' = (c'_{ij})$ always has a similar block decomposition, after a suitable renumbering of its rows and columns. To see this let $D = (d_{ij})$ be an n by m nonnegative integer matrix such that $r' = rD$ and $C = DC'D^t$ where n be the length of r and m the length of r'. We have

$$c_{ij} = \sum_{x=1}^m \sum_{y=1}^m d_{ix} d_{jy} c'_{xy} \ge \sum_{x=1}^m d_{ix} d_{jx} \quad \text{for all } 1 \le i, j \le n.$$

If $i \le q$, then $c_{ii} = 1$, and it follows from the above inequality with $j = i$ that i-th row of D has exactly one nonzero entry, necessarily equal to 1. Let i' be such that $d_{ii'} = 1$. If $i \le q$ and $j \ne i$, then $c_{ij} = 0$, and it follows that $d_{ji'} = 0$, that is, i'-th column of D has exactly one nonzero entry. We also get $c_{ij} = c'_{i'j'}$ and $r'_{i'} = r_i$ whenever $i \le q$ and $j \le q$.

In particular, this argument shows that, whenever C is the identity matrix, any element $[r, C] \in T$ is maximal in T. Note that a finite dimensional associative algebra is semisimple if and only if its Cartan matrix is the identity matrix. Thus, $\operatorname{nt} A$ is a maximal element of T for any finite dimensional semisimple algebra A.

Remark 4.9. Suppose that $(A_\xi)_{\xi \in X}$ is a generically semisimple flat family of finite dimensional associative algebras over an algebraically closed field parameterized by an irreducible algebraic variety. Let C be the Cartan matrix of an arbitrary algebra A_ξ in this family. If A_η is a semisimple algebra in this family, then $\operatorname{nt} A_\eta \geq \operatorname{nt} A_\xi$. Let n be the number of simple A_ξ-modules, m the number of simple A_η-modules. Since the Cartan matrix of A_η is the identity matrix of order m, there exists an n by m nonnegative integer matrix D such that $C = DD^t$. In particular, the matrix C is symmetric. Unlike the well-known decomposition matrix in the modular representation theory of finite groups, the matrix D in the above setup is not defined canonically.

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