Gain-Frequency Characteristics for Several Models of Stratified Geological Medium

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Abstract: - Gain-frequency characteristics of displacements of daylight area at forced oscillations are studied. Several models utilizing a layer sitting on top of a basement are considered. Together with a classical model (considering two homogeneous media), investigated are a model for the layer with absorption and a model for the layer having continuous variation of elasticity parameters at transition toward the basement. The case of linear variation of density and a constant velocity is considered in detail. For all structures under consideration, analytical expressions for displacements of daylight area are obtained.

Keywords: - gain-frequency characteristics, gradient medium, displacements of daylight area, elastic oscillation.

1. INTRODUCTION

A physical model for the sedimentary cover represents a stratified elastic medium each layer of which is characterized by a known velocity of longitudinal (and transverse) waves, density, dissipative parameter and thickness [1, 2].

The upper boundary of the sedimentary cover is a daylight area while the lower boundary is a surface of the crystalline basement. Measurements conducted at the daylight area during both active and passive seismic surveys allow investigating frequency-selective properties of the sediment cover and obtaining data regarding its structure [3, 4]. The most detailed data regarding the structure can be obtained through using gain-frequency characteristics (GFC) of the normal (vertical) constituent of the wave process.

The paper resolves questions of modeling (calculation) of GFC, calculating parameters at essential peaks (frequency, extrema, width) for simple models of medium, propagation and wave processes. Several models for stratified media are considered including a homogeneous layer and a dissipative layer on top of a geological basement as well as a homogeneous layer with continuous variation of elasticity parameters at transition to the basement. Obtained are analytical and semi-empirical expressions describing GFC parameters as well as parameters of propagation medium. Conclusions are made with regards to similarities and differences of GFC for the considered models.

2. STATEMENT OF THE PROBLEM

We consider a homogeneous isotropic layer (medium 1) of thickness L, density ρ_1 and elastic wave propagation velocity v_1. The layer lies on the homogeneous isotropic substrate (medium S with parameters ρ_S and v_S). We will investigate forced oscillations of the daylight area (of a free surface layer), when stress $\sigma_0(\omega)$ is applied to some point x.

Elastic harmonic oscillations of frequency ω, longitudinal velocity v in homogeneous isotropic medium are described by the equation:

$$u''(x) + k^2 u(x) = 0, \qquad (1)$$

where k is the wave number defined by $k = \omega / v$ for media without absorption. For the fixed value of ω and constant k, we can solve equation (1) explicitly. Then for the layer we have

$$u_1(x) = A_1 e^{-ik_1 x} + B_1 e^{ik_1 x}, \qquad (2)$$

and for the substrate we have

$$u_S(x) = A_S e^{-ik_S x}. \qquad (3)$$

Note that we only consider waves within the substrate propagating toward infinity.

We allocate the supplementary layer (medium 2) of thickness 2ε under the substrate. The layer has a gradient structure. Density $\rho_2(x)$ continuously varies from value ρ_1 to value ρ_S, whereas velocity v_2 is constant. Propagation of oscillations within the like media is described by the equation:

$$(\rho_2(x)u_2'(x))' + \rho_2(x)k_2^2 u_2(x) = 0, \quad L-\varepsilon < x < L+\varepsilon. \tag{4}$$

Equation (4) usually does not have an analytical solution [5].

Displacement $u(x)$ and stress $\sigma(x)$ are continuous functions over the entire region $x > 0$. Stress and displacement satisfy to Hooke's law: $\sigma(x) = \rho v^2 u'(x)$.

The continuity of displacements and stresses provide conditions at the media joint:

$$\begin{aligned} &u_1(L-\varepsilon) = u_2(L-\varepsilon),\\ &u_S(L+\varepsilon) = u_2(L+\varepsilon),\\ &\rho_1 v_1^2 u_1'(L-\varepsilon) = \rho_2(L-\varepsilon)v_2^2 u_2'(L-\varepsilon),\\ &\rho_S v_S^2 u_S'(L+\varepsilon) = \rho_2(L+\varepsilon)v_2^2 u_2'(L+\varepsilon). \end{aligned} \tag{5}$$

Let us locate the source at $x = 0$. Then we obtain the boundary condition in the form $\rho_1 v_1^2 u_1'(0) = \sigma_0$. This condition combined with (2)–(5) is a mathematical statement of the boundary value problem for elastic oscillations within our structure.

We will investigate the dependence $|u_1(0)|$ on frequency ω for the cases $\varepsilon = 0$ and $\varepsilon > 0$. Thus, we evaluate the range of amplitude leaps for different cases of elastic structures.

3. STEPWISE DISTRIBUTION OF MEDIUM PARAMETERS

We consider the simple case of the uniform layer 1 placed onto the substrate S. Amplitudes A_1, B_1 and A_S can be found in the analytical form. The displacement of the daylight area ($x = 0$) can be written in the form:

$$u_1(0) = \frac{\cos Lk_1 + it\sin Lk_1}{t\cos Lk_1 + i\sin Lk_1}\frac{i\sigma_0}{k_1\rho_1 v_1^2},$$

where $t = (\rho_S v_S)/(\rho_1 v_1)$. Thus, the absolute value of displacement takes the form

$$|u_1(0)| = \frac{\sqrt{t^2 + \frac{(t^2-1)^2}{4}\sin^2\frac{2L\omega}{v_1}}}{\sin^2\frac{L\omega}{v_1} + t^2\cos^2\frac{L\omega}{v_1}}\frac{|\sigma_0|}{\rho_1 v_1\omega}.$$

We define the source in the form $\sigma_0(\omega) = A_0\rho_1 v_1\omega$, where A_0 is a parameter (in meters) which is independent on frequency.

Then, extrema of $|u_1(0)|$ occur for frequencies

$$\omega = \frac{\pi n v_1}{L} \text{ и } \omega = \frac{\pi(2n-1)v_1}{2L}. \tag{6}$$

Values of $|u_1(0)|$ in these points are $\rho_1 v_1/(\rho_S v_S)|A_0|$ and $\rho_S v_S/(\rho_1 v_1)|A_0|$ (see the graph of $|u_1(0)|$ in Fig. 1).

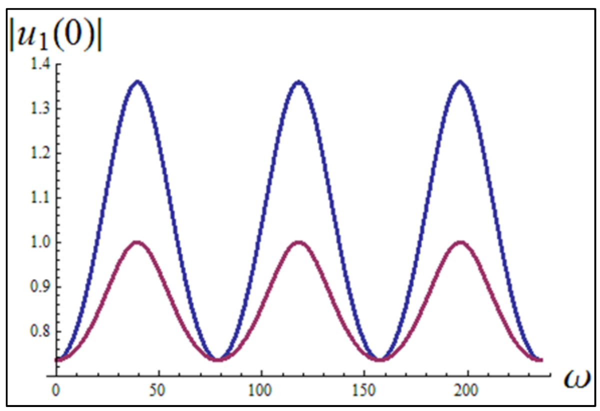

Figure 1. GFC of displacements $|u_1(0)|$ for source $|\sigma_0(\omega)| = \rho_1 v_1\omega|A_0|$. Blue line is for the top source; red line is for the bottom source. Parameters of media: $\rho_S/\rho_1 = 1700/1250$, $L = 100$ m, $A_0 = 1$ m.

Next, the daylight area is a free surface, and the wave in the form $u_0(x) = A_0\rho_1 v_1/(\rho_S v_S)e^{ik_S(x-L)}$ comes from infinity. Then we have the source $\sigma_0(\omega) = iA_0\rho_1 v_1\omega$ in point $x = L$.

Extrema of the function $|u_1(0)|$ occur at frequencies $\omega = \pi n v_1/L$ and $\omega = \pi(2n-1)v_1/(2L)$. Their values equal to

$\rho_1 v_1/(\rho_S v_S)|A_0|$ and $|A_0|$. If $\rho_S v_S > \rho_1 v_1$, then value $|A_0|$ corresponds to maximum of the above function. The absolute value of displacement of daylight area becomes

$$\left|u_1(0)\right| = \frac{1}{\sqrt{\sin^2 \frac{h\omega}{v_1} + t^2 \cos^2 \frac{h\omega}{v_1}}} |A_0|.$$

The graph of $\left|u_1(0)\right|$ is presented in Fig. 1 (red line). The pattern of oscillations for the top source is similar to that for the bottom source.

Now, we consider the dissipative layer. In this case we have $k^2 = k_0^2/(1+i\chi)$ in equation (1), where

$$k_0 = \frac{\omega}{v},\ \chi = \frac{\omega}{\omega_0},\ \omega_0 = \frac{\rho v^2}{b},$$

b is the dissipation parameter.

If all coefficients are constant, then the general solution can be written in the form:

$$u(x) = A_1 \exp[-(\alpha + i\tilde{k})x] \\ + B_1 \exp[(\alpha + i\tilde{k})x].$$

Here, expressions for acoustical absorption coefficient and for wave number are given as

$$\alpha^2 = \frac{2\alpha_0^2}{\chi^2\left(1+\chi^2\right)}\left(\sqrt{1+\chi^2} - 1\right),$$

$$\tilde{k}^2 = \frac{k_0^2}{2\left(1+\chi^2\right)}\left(\sqrt{1+\chi^2} + 1\right),$$

where $\alpha_0 = (b\omega^2)/(2\rho v^3)$ is the acoustical absorption coefficient for low-frequency approximation.

We make several observations regarding an asymptotic behavior of α and $\tilde{k}$. For $\chi << 1$, wave number $\tilde{k}$ tends to the limit equal to k_0; for $\chi >> 1$: $\tilde{k}^2 \to (\omega\omega_0)/(2v^2)$. The absorption coefficient α tends to α_0 for small values of χ. For $\chi >> 1$: $\alpha^2 \to (\omega\omega_0)/(2v^2)$. In other words, the absorption coefficient increases proportionally to ω^2 for small frequencies, whereas for high frequencies it increases proportionally to $\sqrt{\omega}$. Velocity increases proportionally to $\sqrt{\omega}$ for high frequencies (dispersion of velocity is characteristic of a dissipative medium).

Stress and displacement are related via the law: $\sigma(x) = \rho v^2 u'(x) + i\omega b u'(x)$. Accordingly, the conjugate condition of required functions at the joint (at $x = L$) takes another form in this case. The condition for the daylight area becomes $\left(\rho_1 v_1^2 + i\omega b_1\right) u'_1(0) = \sigma_0(\omega)$.

Oscillations of the daylight area take

$$u(0) = -\frac{\left(1+\frac{\tilde{z}_1}{z_S}\right)e^{2(\alpha_1 + i\tilde{k}_1)L}}{1 - \frac{\tilde{z}_1}{z_S} + \left(1+\frac{\tilde{z}_1}{z_S}\right)e^{2(\alpha_1 + i\tilde{k}_1)L}} \\ \times \frac{2\sigma_0(\omega)}{z_S},$$

where $z_S = ik_S \rho_S v_S^2$ and

$$\tilde{z}_1 = \left(\alpha_1 + ik_1\right)\left(\rho_1 v_1^2 + i\omega b_1\right).$$

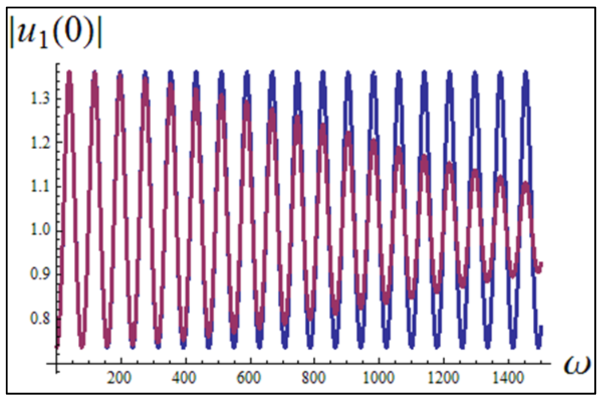

Figure 2. GFC of displacements $\left|u_1(0)\right|$ for source $|\sigma_0(\omega)| = \rho_1 v_1 \omega |A_0|$. Blue line is for medium without absorption, red line is for medium with absorption. Parameters of media: $b = 10^5$, $L = 100$ m, $\rho_1 = 1250$ kg/m³, $\rho_S = 1700$ kg/m³, $v_n = 2500$ m/s.

The oscillations have the same phase pattern as oscillations without absorption. However, amplitudes of oscillations decrease exponentially (see Fig. 2).

4. MEDIA WITH THE GRADIENT VARIATION OF MEDIUM PARAMETERS

We make several observations regarding the behavior of displacements for high frequencies in gradient media. We write wave equation in the following form:

$$(G(x)u'(x))' + \rho(x)\omega^2 u(x) = 0\,, \qquad (7)$$

where $G(x)$ is elastic modulus. Equation (7) can be reduced to the form:

$$\begin{aligned} &G'(x)u'(x) + G(x)u''(x) \\ &+\rho(x)\omega^2 u(x) = 0. \end{aligned} \qquad (8)$$

We introduce the characteristic frequency

$$\omega_0(x) = \frac{G'(x)}{\sqrt{\rho(x)G(x)}}\,.$$

Let the velocity of medium $v(x)$ have a constant value. Then equation (8) can be written in the form:

$$k(x)u'(x) + u''(x) + k_0^2(x)u(x) = 0\,, \qquad (9)$$

where $k_0(x) = \omega / c(x)$, $k(x) = \omega_0(x)/c(x)$, $c(x)$ is phase velocity.

Note, for $\omega >> \Omega_0 = \max|\omega_0(x)|$ equation (9) is reduced to the following equation:

$$u''(x) + k_0^2(x)u(x) = 0\,.$$

We consider the case, when the distribution of medium density is given by $\rho(x) = \rho_0(1+\sin Bx)$ and velocity is constant [6]. For this case:

$$\omega_0(x) = \frac{vB\cos Bx}{1+\sin Bx}\,,$$

and $\Omega_0 = vB$.

Solution for the propagating in the positive direction wave can be presented in the form:

$$u(x) = \frac{A}{\sqrt{\rho(x)}}\exp(-ixk_0\sqrt{1+\frac{\Omega_0^2}{4\omega^2}})\,.$$

Phase velocity of this medium is defined by the following relation:

$$c = \frac{v}{\sqrt{1+\Omega_0^2/(4\omega^2)}}$$

Media of this kind are characterized by anomalous dispersion. For $\omega >> \Omega_0$, we obtain $c(x) = v$ and $k(x) = k_0$. For $\omega << 1$, we obtain $c(x) = 2\omega / B$ and $k(x) = B/2$.

In [6] we showed that the elastic wave reflected off the gradient layer having density distribution in the form $\rho_2(x) = A(1+\sin Bx)$ decay according to law $O(\omega^{-1})$.

5. SUPPLEMENTARY LAYER WITH LINEAR DISTRIBUTION OF MEDIUM PARAMETERS

Now, in lieu of stepwise variation of elastic medium parameters at the conjugation of layer ($0 < x < L$) with the substrate ($x > L$), we consider continuous variation of parameters obeying the linear law. In order to accomplish that, we add a gradient layer ($L-\varepsilon < x < L+\varepsilon$) of thickness 2ε in between the layer and the substrate.

In Fig. 3 an example of smoothing of densities of the layer $\rho_1 = 1250$ kg/m³ and of the substrate $\rho_S = 1700$ kg/m³ is shown. For this purpose, a second layer (in this particular case) at $x \in (70$ m 130 m) with the density distribution $\rho_2(x) = 7.5x + 1100$ is introduced.

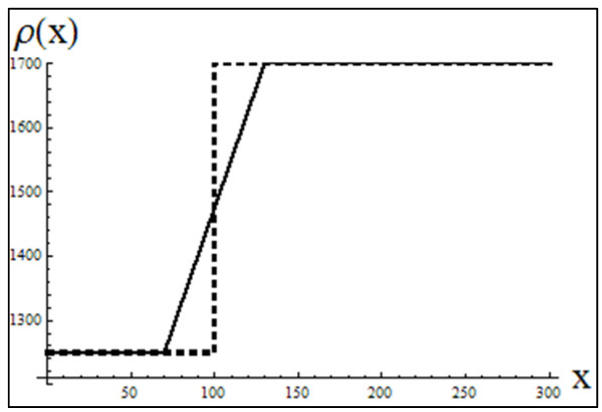

Figure 3. Linear smoothing of the density leap from $\rho_1 = 1250$ kg/m³ to $\rho_S = 1700$ kg/m³.

Next, we assume that propagation speed of elastic waves is uniform everywhere. Thus, due to introducing a new layer, density $\rho(x)$ and elastic modulus $G(x) = \rho(x)v^2(x)$ vary continuously. However, whereever possible, we will leave the notation v_n for the speed of a considered medium.

As to the intermediate layer, we assume that the density distribution is given by $\rho_2(x) = ax + b$, whereas velocity remains constant $v_2(x) = v_2$. Besides, $\rho_2(x)$ changes its values from ρ_1 to ρ_S. Parameters a and b can be expressed through ε:

$$a = \frac{\rho_S - \rho_1}{2\varepsilon},\ b = \frac{\rho_S + \rho_1}{2} - L\frac{\rho_S - \rho_1}{2\varepsilon}.$$

Equation (4) can be solved explicitly:

$$u_2(x) = A_2 J_0(|\frac{\omega}{v_2}x + \frac{b\omega}{av_2}|)$$
$$+B_2 Y_0(|\frac{\omega}{v_2}x + \frac{b\omega}{av_2}|).$$

Then conditions of continuity of displacement and stress lead to a system of linear algebraic equations which can be solved analytically. Hence, displacement of the daylight area takes the form:

$$u_1(0) = \frac{(-1+E)v_2Q_1 + (1+E)iv_1Q_0}{(-1+E)v_1Q_0 - (1+E)iv_2Q_1}\frac{\sigma_0}{\omega\rho_1 v_1}$$

where

$$E = \exp[2i(L-\varepsilon)\frac{\omega}{v_1}],$$

$$Q_n = (J_0(\xi_2)v_3 + iJ_1(\xi_2)v_2)Y_n(\xi_1)$$
$$-(Y_0(\xi_2)v_3 + iY_1(\xi_2)v_2)J_n(\xi_1), \quad n = 0,1$$

and

$$\xi_1 = \left(\frac{b}{a} + L - \varepsilon\right)\frac{\omega}{v_2},$$

$$\xi_2 = \left(\frac{b}{a} + L + \varepsilon\right)\frac{\omega}{v_2}.$$

We investigate behavior of displacement at low frequencies ω. We have

$$\lim_{\omega\to 0} u_1(0) = \frac{i}{\rho_S v_S}\lim_{\omega\to 0}\frac{\sigma_0}{\omega}. \tag{10}$$

Let the velocities in all the media be equal. Then for high frequancies we obtain

$$\lim_{\omega\to\infty} u_1(0) = \frac{i}{\rho_1 v_1}\lim_{\omega\to\infty}\frac{\sigma_0}{\omega}. \tag{11}$$

Here we observe that the value $u_1(0)$ oscillates around the value leaning toward a given limit with increasing ω.

Amplitude of a wave passed into the substrate takes the form

$$A_S = -2\exp[i(\frac{L-\varepsilon}{v_1} + \frac{L+\varepsilon}{v_S})\omega]$$
$$\times\frac{J_1(\xi_2)Y_0(\xi_2) - J_0(\xi_2)Y_1(\xi_2)}{(-1+E)v_1Q_0 - (1+E)iv_2Q_1}\frac{v_2\sigma_0}{\rho_1\omega}.$$

Asymptotic expressions for A_S at low frequencies ω become:

$$\lim_{\omega\to 0} A_S = \frac{i}{\rho_S v_S}\lim_{\omega\to 0}\frac{\sigma_0}{\omega},$$

whereas at high frequencies the corresponding expressions take the form:

$$\lim_{\omega\to\infty} A_S = \frac{i}{\sqrt{\rho_1\rho_S}\,v_1}\lim_{\omega\to\infty}\frac{\sigma_0}{\omega}.$$

Below are given results of calculation of absolute value of displacement of the daylight area and of amplitude of the substrate wave (Fig. 4) for layer of thickness 100 m, density $\rho_1 = 1250$ kg/m^3 sitting on top of the geological basement having density $\rho_S = 1700$ kg/m^3. We assume that speed of propagation of elastic oscillations equals $v_n = 2500$ m/s everywhere. The source is located at the daylight area and $\sigma_0(\omega) = A_0\rho_1 v_1\omega$, where $A_0 = 1$ m.

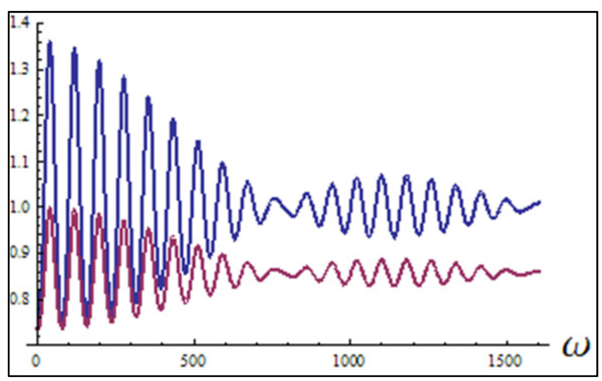

Figure 4. Dependence of absolute value of displacement for daylight area and for substrate on frequency ω at $\varepsilon = 5$ m. Blue line corresponds to displacement for daylight area $|u_1(0)|$; red line corresponds to amplitude of transmitted wave $|A_S|$.

We observe that the layered structures in the performed calculations are those described above. The upper layer is located at $0 < x < L - \varepsilon$, the gradient layer is at $L-\varepsilon<x<L+\varepsilon$, and the substrate is at $x>L+\varepsilon$.

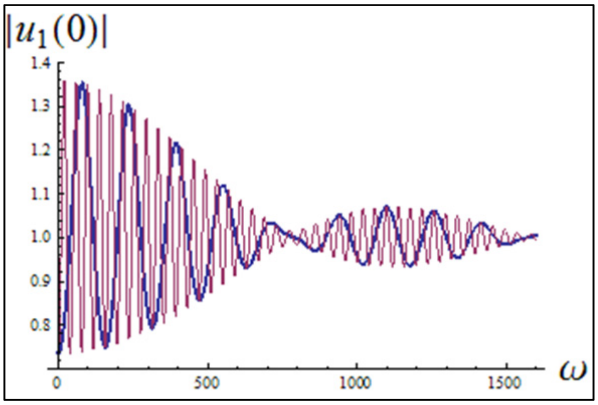

Figure 5. Dependence of absolute value of displacement for daylight area on frequency ω at $\varepsilon=5$ m; for $L=50$ m (blue line) и $L=200$ m (red line).

Fig. 5 shows comparison of GFC $|u_1(0)|$ for $L=50$ m and $L=200$ m. It can be seen that the envelope curves for both graphs coincide. It is also evident that the large layer thickness L leads to more frequent oscillations of the plots.

In the presented figures (Fig. 4 and 5) compression points of oscillations are clearly seen. They correspond to frequencies $\omega\approx 800$ rad and $\omega\approx 1600$ rad. The general expression for these frequencies (as well as for high frequencies) is following:

$$\omega_n \approx \frac{\pi n v_2}{2\varepsilon}, \quad n=1,2,\dots \tag{12}$$

Just as in the case of a single layer (6), extreme values are achieved at frequencies

$$\omega \approx \frac{\pi n v_1}{L} \text{ and } \omega \approx \frac{\pi(2n-1)v_1}{2L}. \tag{13}$$

Let us say a few words regarding the other method for smoothing the density. Let us once again locate the gradient layer at ($L-\varepsilon<x<L+\varepsilon$) but having parameters varying in accordance with the law $\rho_2(x)=A(1+\sin Bx)$. For example, we can choose $\varepsilon=4.8746$ m and

$$\rho_2(x)=\frac{\rho_1+\rho_S}{2}(1-\sin\frac{\pi x}{L}),.$$

Since the value of B is small, the density distribution curve of the “sine” layer is close to the distribution curve for the “linear” layer. The intereference picture practically coincides with that observed in the case of the linear transition.

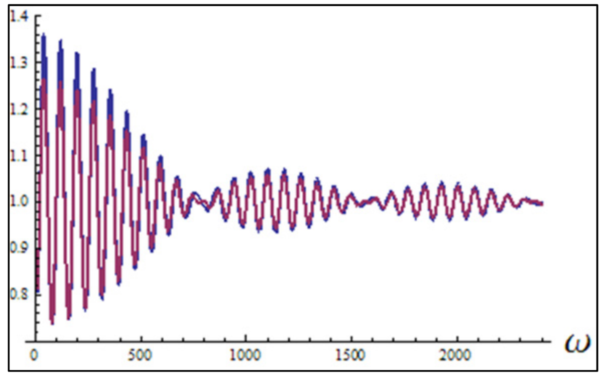

Figure 6. Graphs of displacement $|u_1(0)|$ and its approximation $\tilde{u}_1$.

By analyzing asymptotes (10), (11) and expressions (12), (13), and accounting for conclusions of the previous point regarding the decay of oscillations, let us approximate $|u_1(0)|$ using function $\tilde{u}_1$:

$$\tilde{u}_1 = 1-\frac{\rho_S v_S-\rho_1 v_1}{\rho_S v_S}\frac{v_2}{2\varepsilon\omega}$$

$$\times\sin\frac{2\varepsilon\omega}{v_2}\cos\frac{2L\omega}{v_1}.$$

We observe that $\sin(2\varepsilon\omega/v_2)$ gives beatings of period around 800 rad whereas $\cos(2L\omega/v_1)$ provides high-frequency oscillations. The graphs for $|u_1(0)|$ and $\tilde{u}_1$ for the case $\varepsilon=5$ m are given in Fig. 6. Here $L=100$ m, $\rho_1=1250$ kg/m^3, $\rho_S=1700$ kg/m^3, $v_n=2500$ m/s.

6. CONCLUSIONS

The work considered several models for stratified layers including a homogeneous or dissipative layer sitting on top of a basement as well as a homogeneous layer having continuous variation of elasticity parameters at transition to the basement.

Conclusion was made that in all considered layered structures frequency of extrema of displacements of daylight area remains constant. What changes are amplitudes of the displacements only.

Dispersive dependences for several gradient structures are obtained. The work presented explicit relations for displacements of daylight area at forced oscillations for the case of a source allocated at the daylight area.

It was concluded that in case of a thin gradient layer being present, amplitude of a signal reflected off a substrate decreases inversely proportional to frequency. This knowledge can be used, for example, to explain mechanisms of scattering of high frequency oscillations in models with continuous variation of elasticity parameters as well as to explain dependences of displacements at the daylight area (GFC) on frequency.

The work also showed that presence of a gradient layer on top of a substrate leads to additional complications of a shape of dependences of GFC on frequencies of the “beating” type; the phenomenon can be attributed to complications of processes of reflection off the substrate. Approximate expressions for displacements of daylight area in the case of a linear gradient layer are obtained.

Thus, it was shown that there exist two mechanisms of decrease, on average, of displacements of daylight area with increasing frequency. The first mechanism is related to absorption in dissipative media. The second mechanism is related to specifics of reflections off the gradient layers (it gives dependence inversely proportional to frequency).

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