Mathematical and Numerical Analysis of Dielectric Waveguides by the Integral Equation Method

E. Karchevskiy[1] **and Y. Shestopalov**[2]

[1]Kazan Federal University, Russia
[2]Karlstad University, Sweden

Abstract— The eigenvalue problems for generalized natural modes of an inhomogeneous dielectric waveguide without a sharp boundary and a step-index dielectric waveguide with a smooth boundary of cross-section are formulated as problems for the set of time-harmonic Maxwell equations with partial radiation conditions at infinity in the cross-sectional plane. The original problems are reduced by the integral equation method to nonlinear spectral problems with Fredholm integral operators. Properties of the spectrum are investigated. The Galerkin and collocation methods for the calculations of generalized natural modes are proposed and convergence of the methods is proved. Some results of numerical experiments are discussed.

1. INTRODUCTION

Many different numerical techniques are applied for computing eigenmodes of dielectric waveguides [1, 2]; namely, Finite-element, Finite-difference, beam propagation, and spline collocation methods, as well as multidomain spectral approach. Often the authors concentrate on the algorithm's features and physical interpretation of the numerical results rather than on fundamental mathematical aspects including the existence, properties, and distribution of the spectra on the complex plane of the spectral parameter. In this study, we propose a new approach to mathematical and numerical analysis of dielectric waveguides based on the methods of spectral theory of operator-valued functions [3, 4] and integral equations (IEs) [4–6]. The eigenvalue problems for the determination of natural modes (surface, leaky, and complex eigenmodes) of inhomogeneous optical waveguides and step-index optical waveguides with the smooth cross-sectional boundary are formulated [3–5] for the time-harmonic Maxwell equations with partial radiation conditions at infinity in the cross-sectional plane. The initial problems are reduced with the aid of the integral equation (IE) method (using appropriate Green functions) to nonlinear spectral problems with Fredholm integral operators. Theorems on the spectrum localization are proved. It is shown that the sets of all eigenvalues of the initial problems may consist of isolated points on the Reimann surface of the spectral parameter (longitudinal wavenumber) and each eigenvalue depends continuously on the frequency and permittivity and can appear or disappear only at the boundary of the Reimann surface. The initial problems for surface waves are reduced to linear eigenvalue problems for integral operators with real-valued symmetric weakly singular kernels. The existence, localization, and dependence of the spectrum on parameters are investigated. The collocation and Galerkin methods for the calculation of natural modes are proposed, the convergence of the methods is proved, and some results of numerical experiments are discussed.

2. GENERALIZED NATURAL MODES OF A STEP-INDEX DIELECTRIC WAVEGUIDE

Let the three-dimensional space be occupied by an isotropic source-free medium, and let the permittivity be prescribed as a positive real-valued function $\varepsilon = \varepsilon(x)$ independent of the longitudinal coordinate and equal to a constant $\varepsilon_\infty > 0$ outside a cylinder. In this section, we consider the generalized natural modes of a step-index optical fiber and suppose that the permittivity is equal to a constant $\varepsilon_+ > \varepsilon_\infty$ inside the cylinder. The axis of the cylinder is parallel to the longitudinal coordinate and its cross section is a bounded domain Ω_i with a twice continuously differentiable boundary γ (see Fig. 1). The domain Ω_i is a subset of a circle with radius R_0. Denote by Ω_e the unbounded domain $\Omega_e = \mathbb{R}^2 \setminus \bar{\Omega}_i$, by U the space of complex-valued continuous and continuously differentiable in $\bar{\Omega}_i$ and $\bar{\Omega}_e$, twice continuously differentiable in Ω_i and Ω_e functions, and by Λ the Riemann surface of the function $\ln \chi_\infty(\beta)$, where $\chi_\infty = \sqrt{k^2\varepsilon_\infty - \beta^2}$. Here $k^2 = \omega^2\varepsilon_0\mu_0$, ω is a given radian frequency and ε_0, μ_0 are the free-space dielectric and magnetic constants, respectively. Denote by Λ_0 the principal ("proper") sheet of this Riemann surface specified by the condition $\text{Im}\chi_\infty(\beta) \geq 0$.

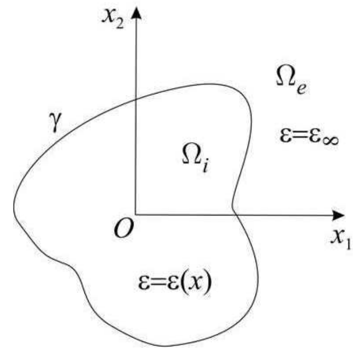

Figure 1: A schematic waveguide's cross-section.

A nonzero vector $\{\mathrm{E}, \mathrm{H}\} \in U^6$ is referred to as a generalized eigenvector (or eigenmode) of the problem corresponding to an eigenvalue $\beta \in \Lambda$ if the following relations are valid [7]:

$$\mathrm{rot}_\beta \mathrm{E} = i\omega\mu_0 \mathrm{H}, \quad \mathrm{rot}_\beta \mathrm{H} = -i\omega\varepsilon_0\varepsilon \mathrm{E}, \quad x \in \mathbb{R}^2 \setminus \gamma, \tag{1}$$

$$\nu \times \mathrm{E}^+ = \nu \times \mathrm{E}^-, \quad x \in \gamma, \tag{2}$$

$$\nu \times \mathrm{H}^+ = \nu \times \mathrm{H}^-, \quad x \in \gamma, \tag{3}$$

$$\begin{bmatrix} \mathrm{E} \\ \mathrm{H} \end{bmatrix} = \sum_{l=-\infty}^{\infty} \begin{bmatrix} \mathrm{A}_l \\ \mathrm{B}_l \end{bmatrix} H_l^{(1)} (\chi_\infty r) \exp(il\varphi), \quad r \geq R_0. \tag{4}$$

Here differential operator rot_β is obtained from the standard operator by replacing the generating waveguide line derivative with $i\beta$ multiplication and $H_l^{(1)}(z)$ is the Hankel function of the first kind and index l. The conditions (4) are the the partial radiation conditions.

Theorem 1 (see [7]). *The imaginary axis $\mathbb{I}$ and the real axis $\mathbb{R}$ of the sheet Λ_0 except the set $G = \{\beta \in \mathbb{R}\colon k^2\varepsilon_\infty < \beta^2 < k^2\varepsilon_+\}$ are free of the eigenvalues of the problem* (1)–(4)*. Surface and complex eigenmodes correspond to real eigenvalues $\beta \in G$ and complex eigenvalues $\beta \in \Lambda_0$, respectively. Leaky eigenmodes correspond to complex eigenvalues β belonging to an "improper" sheet of Λ for which* $\mathrm{Im}\chi_\infty(\beta) < 0$.

Theorem 1 generalizes the well-known results on the spectrum localization of a step-index circular dielectric waveguide which were obtained by the separation of variables method (see, for example [8]).

We use representation of the eigenvectors of problem (1)–(4) in the form of single-layer potentials u and v:

$$\mathrm{E}_1 = \frac{i}{k^2\varepsilon - \beta^2}\left(\mu_0\omega\frac{\partial v}{\partial x_2} + \beta\frac{\partial u}{\partial x_1}\right), \quad \mathrm{E}_2 = \frac{-i}{k^2\varepsilon - \beta^2}\left(\mu_0\omega\frac{\partial v}{\partial x_1} - \beta\frac{\partial u}{\partial x_2}\right), \quad \mathrm{E}_3 = u, \tag{5}$$

$$\mathrm{H}_1 = \frac{i}{k^2\varepsilon - \beta^2}\left(\beta\frac{\partial v}{\partial x_1} - \varepsilon_0\varepsilon\omega\frac{\partial u}{\partial x_2}\right), \quad \mathrm{H}_2 = \frac{i}{k^2\varepsilon - \beta^2}\left(\beta\frac{\partial v}{\partial x_2} + \varepsilon_0\varepsilon\omega\frac{\partial u}{\partial x_1}\right), \quad \mathrm{H}_3 = v, \tag{6}$$

$$\begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = \frac{i}{4}\int_\gamma H_0^{(1)}\left(\sqrt{k^2\varepsilon_{+/\infty} - \beta^2}\,|x-y|\right)\begin{bmatrix} f_{+/\infty}(y) \\ g_{+/\infty}(y) \end{bmatrix} dl(y), \quad x \in \Omega_{i/e}, \tag{7}$$

where unknown densities $f_{+/\infty}$ and $g_{+/\infty}$ belong to the space of Hölder continuous functions $C^{0,\alpha}$. The original problem (1)–(4) is reduced [7] by single-layer potential representation (5)–(7) to a nonlinear eigenvalue problem for a set of singular integral equations on boundary γ. This problem has the operator form

$$A(\beta)w \equiv (I + B(\beta))w = 0, \tag{8}$$

where I is the identical operator in the Banach space $W = (C^{0,\alpha})^4$ and $B(\beta)\colon W \to W$ is a compact operator consisting particularly of the following boundary singular integral operators:

$$Lp = -\frac{1}{2\pi}\int_0^{2\pi} \ln\left|\sin\frac{t-\tau}{2}\right| p(\tau)d\tau, \quad t \in [0, 2\pi], \quad L\colon C^{0,\alpha} \to C^{1,\alpha}, \tag{9}$$

$$Sp = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{ctg}\frac{\tau - t}{2} p(\tau)d\tau + \frac{i}{2\pi}\int_0^{2\pi} p(\tau)d\tau, \quad t \in [0, 2\pi], \quad S\colon C^{0,\alpha} \to C^{0,\alpha}. \tag{10}$$

The original problem (1)–(4) is spectrally equivalent [7] to the problem (8). Namely, suppose that $w \in W$ is an eigenvector of the operator-valued function $A(\beta)$ corresponding to an eigenvalue $\beta \in \Lambda_0 \setminus D$, $D = \{\beta \in \mathbb{I}\} \bigcup \{\beta \in \mathbb{R}\colon \beta^2 < k^2\varepsilon_\infty\}$. Then using this vector we can construct the densities of the single-layer potential representation (5)–(7) of an eigenmode $\{\mathrm{E}, \mathrm{H}\} \in U^6$ of the problem (1)–(4) corresponding to the same eigenvalue β. On the other side, any eigenmode of (1)–(4) corresponding to an eigenvalue $\beta \in \Lambda_0 \setminus D$ can be represented in the form of single-layer potentials. The densities of these potentials constitute an eigenvector $w \in W$ of the operator-valued function $A(\beta)$ corresponding to the same eigenvalue β.

Theorem 2 (see [7]). *For each $\beta \in \{\beta \in \mathbb{R}\colon \beta^2 \geq k^2\varepsilon_+\}$ the operator $A(\beta)$ has a bounded inverse operator. The set of all eigenvalues β of the operator-valued function $A(\beta)$ can be only a set of isolated points on Λ. Each eigenvalue β depends continuously on $\omega > 0$, $\varepsilon_+ > 0$, and $\varepsilon_\infty > 0$ and can appear and disappear only at the boundary of Λ, i.e., at $\beta = \pm k\sqrt{\varepsilon_\infty}$ and at infinity.*

Theorem 2 generalizes the known results on the dependence of the propagation constants β of a step-index circular dielectric waveguide on wavenumber k and permittivity ε (see, for example [8]).

The statements similar to Theorems 1 and 2 for the scalar problem in weakly guiding approximation are proved in [9].

Describe a projection method for numerical solution of the problem (8). Denote by N the set of integers. We represent the approximate eigenvector of the operator-valued function $A(\beta)$ in the form

$$w_n = \left(w_n^{(j)}\right)_{j=1}^4, \quad w_n^{(j)}(t) = \sum_{k=-n}^{n} \alpha_k^{(j)} \exp(ikt), \quad n \in N, \quad j = 1, 2, 3, 4,$$

and look for unknown coefficients $\alpha_k^{(j)}$ by the Galerkin method

$$\int_0^{2\pi} (Aw_n)^{(j)}(t) \exp(-ikt) dt = 0, \quad k = -n, \ldots, n, \quad j = 1, 2, 3, 4.$$

$\exp(ikt)$ are orthogonal eigenfunctions of the singular integral operators $L\colon C^{0,\alpha} \to C^{1,\alpha}$ and $S\colon C^{0,\alpha} \to C^{0,\alpha}$ corresponding to the following eigenvalues:

$$\begin{aligned}\lambda_m^{(L)} &= \{\ln 2 \quad \text{if} \quad m = 0, \quad (2|m|)^{-1} \quad \text{if } m \neq 0\},\\ \lambda_m^{(S)} &= \{i \quad \text{if} \quad m = 0, \quad i\,\mathrm{sign}(m) \quad \text{if} \quad m \neq 0\}\end{aligned}$$

for the operators L and S respectively. Hence, the action of the main (singular) parts of the integral operators in (8) on the basis functions is expressed explicitly.

Denote by W_n^T the set of all trigonometric polynomials of the orders up to n. Denote by $W_n \subset W$ the space of the elements $w_n = (w_n^{(j)})_{j=1}^4$ where $w_n^{(j)} \in W_n^T$. Using the Galerkin method for numerical solution of the problem (8), we get a finite-dimensional nonlinear spectral problem

$$A_n(\beta)w_n = 0, \quad A_n\colon \quad W_n \to W_n. \tag{11}$$

Theorem 3 (see [10]). *If β_0 belongs to the spectrum $\sigma(A)$ of the operator-valued function $A(\beta)$, then there exists a sequence $\{\beta_n\}_{n\in N}$ with $\beta_n \in \sigma(A_n)$ such that $\beta_n \to \beta_0$, $n \in N$. If $\{\beta_n\}_{n\in N}$ is a sequence such that $\beta_n \in \sigma(A_n)$ and $\beta_n \to \beta_0 \in \Lambda$, then $\beta_0 \in \sigma(A)$. If $\beta_n \in \sigma(A_n)$, $A_n(\beta_n)w_n = 0$, and $\beta_n \to \beta_0 \in \Lambda$, $w_n \to w_0$, $n \in N$, $\|w_n\| = 1$, then $\beta_0 \in \sigma(A)$ and $A(\beta_0)w_0 = 0$, $\|w_0\| = 1$.*

Figure 2 shows (a) the dispersion curves for complex modes and (b) surface guided modes of step-index waveguides of circular and square cross-sections. The numerical results obtained by the Galerkin method are marked by circles and squares in Fig. 2(a). The dispersion curves for the circular waveguide are plotted by a solid line, $\widetilde{\beta} = \beta/(k\sqrt{\varepsilon_\infty})$ and $V = kR\sqrt{\varepsilon_+ - \varepsilon_\infty}$. Fig. 2(b) compares the experimental data [11] for surface waves of a square waveguide (marked by squares) with our numerical results (solid lines). Here a is one half of the square's side.

The statement similar to Theorem 3 for a scalar problem in weakly guiding approximation is proved in [12].

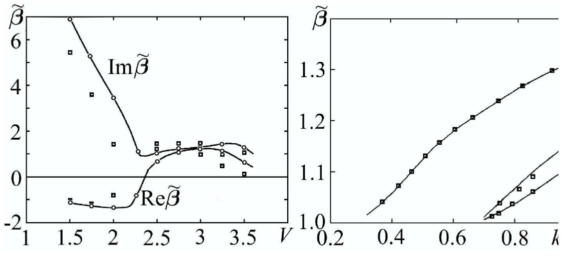

Figure 2: The dispersion curves for (a) the complex modes and (b) surface guided modes of the step-index waveguides of circular and square cross-section.

3. GENERALIZED NATURAL MODES OF AN INHOMOGENEOUS WAVEGUIDE

In this section, we consider the generalized natural modes of an inhomogeneous optical fiber without a sharp boundary. Let the permittivity ε belong to the space $C^2(\mathbb{R}^2)$ of twice continuously differentiable in $\mathbb{R}^2$ functions. Denote by ε_+ the maximum of the function ε in the domain Ω_i and let $\varepsilon_+ > \varepsilon_\infty > 0$. A nonzero complex vector $\{\mathrm{E},\mathrm{H}\} \in (C^2(\mathbb{R}^2))^6$ is referred to as a generalized eigenvector (or eigenmode) of the problem corresponding to an eigenvalue $\beta \in \Lambda$ if the following relations are valid [5]:

$$\mathrm{rot}_\beta \mathrm{E} = i\omega\mu_0 \mathrm{H}, \quad \mathrm{rot}_\beta \mathrm{H} = -i\omega\varepsilon_0\varepsilon \mathrm{E}, \quad x \in \mathbb{R}^2, \tag{12}$$

$$\begin{bmatrix} \mathrm{E} \\ \mathrm{H} \end{bmatrix} = \sum_{l=-\infty}^{\infty} \begin{bmatrix} \mathrm{A}_l \\ \mathrm{B}_l \end{bmatrix} H_l^{(1)}(\chi_\infty r) \exp(il\varphi), \quad r \geq R_0. \tag{13}$$

Theorem 4 (see [5]). *The imaginary axis $\mathbb{I}$ and the real axis $\mathbb{R}$ of the sheet Λ_0 except the set $G = \{\beta \in \mathbb{R}\colon k^2\varepsilon_\infty < \beta^2 < k^2\varepsilon_+\}$ are free of eigenvalues of the problem* (12), (13). *Surface and complex eigenmodes correspond to real eigenvalues $\beta \in G$ and complex eigenvalues $\beta \in \Lambda_0$, respectively. Leaky eigenmodes correspond to complex eigenvalues β belonging to an "improper" sheet of Λ for which* $\mathrm{Im}\chi_\infty(\beta) < 0$.

If vector $\{\mathrm{E},\mathrm{H}\} \in (C^2(\mathbb{R}^2))^6$ is an eigenvector of problem (12), (13) corresponding to an eigenvalue $\beta \in \Lambda$, then (see [5])

$$\mathrm{E}(x) = k^2 \int_{\Omega_i} (\varepsilon(y) - \varepsilon_\infty)\Phi(\beta;x,y)\mathrm{E}(y)dy + \mathrm{grad}_\beta \int_{\Omega_i} \left(\mathrm{E}, \varepsilon^{-1}\mathrm{grad}\varepsilon\right)(y)\Phi(\beta;x,y)dy, \quad x \in \mathbb{R}^2, \tag{14}$$

$$\mathrm{H}(x) = -i\omega\varepsilon_0 \mathrm{rot}_\beta \int_{\Omega_i} (\varepsilon(y) - \varepsilon_\infty)\Phi(\beta;x,y)\mathrm{E}(y)dy, \quad x \in \mathbb{R}^2. \tag{15}$$

Using the integral representation (14) for $x \in \Omega_i$ we obtain a nonlinear eigenvalue problem for an IE in Ω_i which can be written in the operator form

$$A(\beta)\mathrm{F} \equiv (I - B(\beta))\mathrm{F} = 0, \tag{16}$$

where the operator $B(\beta)$: $(L_2(\Omega_i))^3 \to (L_2(\Omega_i))^3$ corresponds to the right side of the integral representation (14) for $x \in \Omega_i$. For any $\beta \in \Lambda$ the operator $B(\beta)$ is compact [5].

It was proved in [5] that the original problem (12), (13) is spectrally equivalent to problem (16). Namely, suppose that $\{\mathrm{E},\mathrm{H}\} \in (C^2(\mathbb{R}^2))^6$ is the eigenmode of problem (12), (13) corresponding to an eigenvalue $\beta \in \Lambda$. Then $\mathrm{F} = \mathrm{E} \in [L_2(\Omega_i)]^3$ is an eigenvector of the operator-valued function $A(\beta)$ corresponding to the same eigenvalue β. Suppose that $\mathrm{F} \in [L_2(\Omega_i)]^3$ is an eigenvector of the operator-valued function $A(\beta)$ corresponding to an eigenvalue $\beta \in \Lambda$ and that the same number β is not an eigenvalue of the following problem:

$$\left[\Delta + \left(k^2\varepsilon - \beta^2\right)\right]u = 0, \quad x \in \mathbb{R}^2, \quad u \in C^2\left(\mathbb{R}^2\right), \tag{17}$$

$$u = \sum_{l=-\infty}^{\infty} a_l H_l^{(1)}(\chi_\infty r) \exp(il\varphi), \quad r \geq R_0. \tag{18}$$

Let $\mathrm{E} = B(\beta)\mathrm{F}$ and $\mathrm{H} = (i\omega\mu_0)^{-1}\mathrm{rot}_\beta\mathrm{E}$ for $x \in \mathbb{R}^2$. Then $\{\mathrm{E}, \mathrm{H}\} \in (C^2(\mathbb{R}^2))^6$ and $\{\mathrm{E}, \mathrm{H}\}$ is an eigenvector of the original problem (12), (13) corresponding to the same eigenvalue β.

Theorem 5 (see [5]). *For each $\beta \in \{\beta \in \mathbb{R}\colon \beta^2 \geq k^2\varepsilon_+\}$ the operator $A(\beta)$ has a bounded inverse. The set of all eigenvalues β of the operator-valued function $A(\beta)$ can be only a set of isolated points on Λ. Each eigenvalue β depends continuously on $\omega > 0$, $\varepsilon_+ > 0$, and $\varepsilon_\infty > 0$ and can appear and disappear only at the boundary of Λ, i.e., at $\beta = \pm k\sqrt{\varepsilon_\infty}$ and at infinity.*

Similar results for integrated optical guides are obtained in [13].

The scalar problem (17), (18) is a problem on eigenmodes of a nonhomogeneous optical fiber in weakly guiding approximation. The statements similar to Theorems 4 and 5 for scalar problem (17), (18) are proved in [14].

The initial problem (17), (18) for surface waves is reduced to a linear eigenvalue problem for an integral operator with a real-valued symmetric weakly singular kernel. The existence of the spectrum of this operator are proved in [15].

The collocation method for numerical approximation of weakly singular domain integral operators associated with problem (17), (18) is proposed in [15]. The statement similar to Theorem 3 concerning convergence of the collocation method is proved in [15].

As a numerical example Fig. 3 shows the isolines for real and imaginary parts of the fourth eigenfunction of a unite circular waveguide [15]. Here $\varepsilon = 2$, $x \in \Omega_i$, $\varepsilon_\infty = 1$, $\chi_\infty = 2.039 + i1.003$, $k^2 = 5.025$.

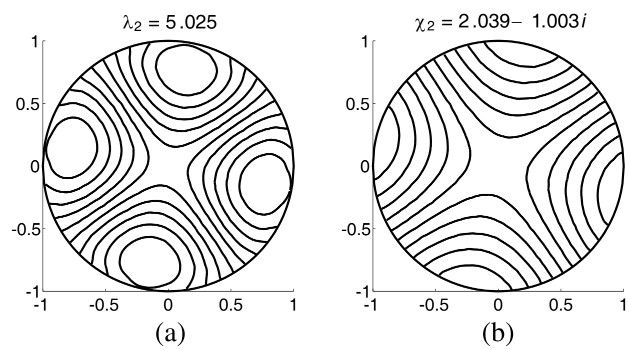

Figure 3: The isolines for (a) real and (b) imaginary part of the fourth eigenfunction of circular waveguide.

ACKNOWLEDGMENT

This work is supported by RFBR 12-01-97012-r-povolzh'e-a and by the Visby program of the Swedish Institute.

REFERENCES

1. Horikis, T. P., "Dielectric waveguides of arbitrary cross sectional shape," *Applied Mathematical Modeling*, Vol. 37, No. 7, 5080–5091, 2013.
2. Xiao, J. and X. Sun, "Full-vectorial mode solver for anisotropic optical waveguides using multidomain spectral collocation method," *Optics Communications*, Vol. 283, No. 14, 2835–2840, 2010.
3. Shestopalov, V. P. and Y. V. Shestopalov, *Spectral Theory and Excitation of Open Structures*, Peter Peregrinus Ltd., 1996.
4. Chernokozhin, E., Y. Smirnov, and Y. Shestopalov, *Logarithmic Integral Equations in Electromagnetics*, VSP Int. Science Publishers, Utrecht, Boston, Koln, Tokyo, 2000.
5. Kartchevski, E. M., A. I. Nosich, and G. W. Hanson, "Mathematical analysis of the generalized natural modes of an inhomogeneous optical fiber," *SIAM J. Appl. Math.*, Vol. 65, No. 6, 2033–2048, 2005.
6. Dautov, R. Z. and E. M. Karchevskii, *Integral Equations Method and Exact Nonlocal Boundary Conditions in the Theory of Dielectric Waveguides*, Kazan State University, Kazan, 2009.
7. Karchevskii, E. M., "The fundamental wave problem for cylindrical dielectric waveuides," *Differential Equations*, Vol. 36, 1109–1111, 2000.
8. Veselov, G. N. and S. B. Raevskiy, *Layered Metal-dielectric Waveguides*, Radio i Sviaz', Moscow, 1988.

9. Karchevskii, E. M., "On the investigation of the spectrum of natural waves in dielectric waveguides," *Comput. Math. Math. Phys.*, Vol. 39, No. 9, 1493–1498, 1999.
10. Karchevskii, E. M., "Mathematical analysis and numerical modeling of the guided modes of the step-index optical fibers," *SIAM Proc. in Applied Mathematics*, Vol. 102, 414–419, 2000.
11. Goncharenko, A. M. and V. A. Karpenko, *The Foundations of the Theory of Optical Waveguides*, Nauka i Tekhnika, Minsk, 1983.
12. Karchevskii, E. M., "Investigation of a numerical method for solving a spectral problem in the theory of dielectric waveguides," *Russian Math. (Iz. VUZ)*, Vol. 43, No. 1, 8–15, 1999.
13. Kartchevski, E. M. and G. W. Hanson, "Mathematical analysis of the guided modes of integrated optical guides," *Mathematical and Numerical Aspects of Wave Propagation — WAVES 2003*, 445–450, Springer, Berlin, 2003.
14. Karchevskii, E. M. and S. I. Solov'ev, "Investigation of a spectral problem for the Helmholtz operator in the plane," *Differ. Equ.*, Vol. 36, No. 4, 631–634. 2000.
15. Frolov, A. G. and E. M. Kartchevskiy, "Integral equation methods in optical waveguide theory," *Springer Proceedings in Mathematics and Statistics*, Vol. 52, 2013.