ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2019, Vol. 40, No. 10, pp. 1450–1454. © Pleiades Publishing, Ltd., 2019.

# Rearrangements of Tripotents and Differences of Isometries in Semifinite von Neumann Algebras

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(Submitted by S. A. Grigoryan)

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Received April 20, 2019; revised April 29, 2019; accepted May 5, 2019

**Abstract**—Let $\tau$ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$, and $\mathcal{M}^{\mathrm{u}}$ be a unitary part of $\mathcal{M}$. We prove a new property of rearrangements of some tripotents in $\mathcal{M}$. If $V \in \mathcal{M}$ is an isometry (or a coisometry) and $U - V$ is $\tau$-compact for some $U \in \mathcal{M}^{\mathrm{u}}$ then $V \in \mathcal{M}^{\mathrm{u}}$. Let $\mathcal{M}$ be a factor with a faithful normal trace $\tau$ on it. If $V \in \mathcal{M}$ is an isometry (or a coisometry) and $U - V$ is compact relative to $\mathcal{M}$ for some $U \in \mathcal{M}^{\mathrm{u}}$ then $V \in \mathcal{M}^{\mathrm{u}}$. We also obtain some corollaries.

**DOI:** 10.1134/S1995080219100068

Keywords and phrases: *Hilbert space, linear operator, isometry, unitary operator, idempotent, tripotent, projection, compact operator, von Neumann algebra, trace, rearrangement.*

## 1. INTRODUCTION

A bounded linear operator $A$ on a Hilbert space $\mathcal{H}$ is called a *tripotent* if $A = A^3$, an *idempotent* if $A = A^2$, and a *projection* if $A = A^2 = A^*$. Let $P$ and $Q$ be idempotents on $\mathcal{H}$. Various properties of the difference $P - Q$ (invertibility, Fredholm property, trace-class property, positivity, etc.) were studied in [1–11]. Every tripotent is the difference $P - Q$ of some idempotents $P$ and $Q$ with $PQ = QP = 0$ [7, Proposition 1]. Hence tripotents inherit some properties of idempotents [8].

The results obtained in this paper are as follows. Let $\tau$ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$, $I$ be the unit of $\mathcal{M}$. Denote by $\mu_t(X)$ a *rearrangement* of an operator $X \in \mathcal{M}$, and by $\mathcal{M}^{\mathrm{pr}}$, $\mathcal{M}^{\mathrm{id}}$ and $\mathcal{M}^{\mathrm{u}}$ the subsets of projections, idempotents, and unitary operators ($A^*A = AA^* = I$) in $\mathcal{M}$, respectively.

For every $P \in \mathcal{M}^{\mathrm{id}}$ there exists a unique decomposition $P = \widetilde{P} + Z$, with $\widetilde{P} \in \mathcal{M}^{\mathrm{pr}}$ and nilpotent $Z \in \mathcal{M}$, $Z^2 = 0$, moreover, $Z\widetilde{P} = 0$, $\widetilde{P}Z = Z$, see [9, Theorem 1.3]. Let a tripotent $A \in \mathcal{M}$ be such that $A = P - Q$ with $P \in \mathcal{M}^{\mathrm{id}}$, $Q \in \mathcal{M}^{\mathrm{pr}}$ and $PQ = QP = 0$. Let $P = \widetilde{P} + Z$ be the decomposition described above. Then $\widetilde{P}Q = 0$ and for $R = \widetilde{P} + Q \in \mathcal{A}^{\mathrm{pr}}$, and for all $t > 0$ we have $\mu_t(A) = \mu_t(A)\chi_{[0,\tau(R))}(t) \geq \mu_t(R) = \chi_{[0,\tau(R))}(t)$ (Theorem 1); here $\chi_B$ is the indicator function of a set $B \subset \mathbb{R}$. The condition $Q = Q^*$ is essential in Theorem 1. Corollary 1 gives an application to $F$-normed symmetic spaces on $(\mathcal{M}, \tau)$.

If $V \in \mathcal{M}$ is an isometry (or a coisometry) and $U - V$ is $\tau$-compact for some $U \in \mathcal{M}^{\mathrm{u}}$ then $V \in \mathcal{M}^{\mathrm{u}}$ (Theorem 3). Let a number $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and a $\tau$-compact operator $K \in \mathcal{M}$ be such that an operator $\lambda I + K$ is an isometry. Then the operator $K$ is normal (Corollary 3). Let $\mathcal{M}$ be a factor with a faithful normal trace $\tau$ on it. If $V \in \mathcal{M}$ is an isometry (or a coisometry) and $U - V$ is compact relative to $\mathcal{M}$ for some $U \in \mathcal{M}^{\mathrm{u}}$ then $V \in \mathcal{M}^{\mathrm{u}}$ (Theorem 4).

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## 2. DEFINITIONS AND NOTATION

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$, and let $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all bounded linear operators on $\mathcal{H}$. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be an *isometry*, if $A^*A = I$; a *coisometry*, if $A^*$ is an isometry; a *semiorthogonal projection*, if $A^*A = (A + A^*)/2$ [10, 11]. The *commutant* of a set $\mathcal{X} \subset \mathcal{B}(\mathcal{H})$ is defined as the set

$$\mathcal{X}' = \{Y \in \mathcal{B}(\mathcal{H}) \colon \ \ XY = YX \ \text{ for all } \ X \in \mathcal{X}\}.$$

By a *von Neumann algebra* acting on a Hilbert space $\mathcal{H}$ we mean a $*$-subalgebra $\mathcal{M}$ of the algebra $\mathcal{B}(\mathcal{H})$, for which $\mathcal{M} = \mathcal{M}''$. Let $I$ be the unit of an algebra $\mathcal{M}$.

For a von Neumann algebra $\mathcal{M}$, by $\mathcal{M}^{\mathrm{pr}}$, $\mathcal{M}^{\mathrm{id}}$, $\mathcal{M}^{\mathrm{tri}}$, $\mathcal{M}^{\mathrm{u}}$ and $\mathcal{M}^{+}$ we denote the subsets of projections ($A = A^2 = A^*$), idempotents ($A = A^2$), tripotents ($A = A^3$), unitary elements ($A^*A = AA^* = I$) and positive elements of $\mathcal{M}$, respectively. If $A \in \mathcal{M}$, then $|A| = \sqrt{A^*A} \in \mathcal{M}^+$. A formula $A = 2T - I$ defines a bijection between the set $\mathcal{M}^{\mathrm{iso}}$ of all isometries and the set $\mathcal{M}^{\mathrm{sp}}$ of all semiorthogonal projections.

By a *trace* on a von Neumann algebra $\mathcal{M}$ we mean a mapping $\varphi\colon \ \mathcal{M}^+ \to [0, +\infty]$ such that

$$\varphi(X + Y) = \varphi(X) + \varphi(Y), \quad \varphi(\lambda X) = \lambda\varphi(X) \quad \text{for all} \quad X, Y \in \mathcal{M}^+, \quad \lambda \geq 0$$

(here $0 \cdot (+\infty) \equiv 0$), and

$$\varphi(Z^*Z) = \varphi(ZZ^*) \quad \text{for all} \quad Z \in \mathcal{M}.$$

A trace $\varphi$ is said to be *faithful*, if $\varphi(X) = 0 \Rightarrow X = 0$ for $X \in \mathcal{M}^+$; *semifinite*, if $\varphi(X) = \sup\{\varphi(Y) : Y \in \mathcal{M}^+, \ Y \leq X, \ \varphi(Y) < +\infty\}$ for every $X \in \mathcal{M}^+$; *normal*, if $X_i \nearrow X$ ($X_i, X \in \mathcal{M}^+$) $\Rightarrow \varphi(X) = \sup\limits_i \varphi(X_i)$.

An operator $A \in \mathcal{M}$ is *hyponormal*, if $A^*A \geq AA^*$; *normal*, if $A^*A = AA^*$. An operator $A \in \mathcal{M}$ is said to be *compact relative to a semifinite von Neumann algebra* $\mathcal{M}$, if it belongs to the two-sided closed ideal generated by the finite projections of $\mathcal{M}$.

A von Neumann algebra $\mathcal{M}$ is said to be a *factor*, if $\mathcal{M} \cap \mathcal{M}' = \{\lambda I : \ \lambda \in \mathbb{C}\}$.

Let $\tau$ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$. Denote by $\mu_t(X)$ a *rearrangement* of an operator $X \in \mathcal{M}$, i.e. nonincreasing right continuous function $\mu(X)\colon (0, +\infty) \to [0, +\infty)$, given by the formula

$$\mu_t(X) = \inf\{||XP|| : \ \ P \in \mathcal{M}^{\mathrm{pr}}, \quad \tau(I - P) \leq t\}, \quad t > 0.$$

Define $\mu_\infty(X) = \lim\limits_{t\to+\infty} \mu_t(X)$ for $X \in \mathcal{M}$. The set $\mathcal{M}_0 = \{X \in \mathcal{M} : \ \mu_\infty(X) = 0\}$ is an ideal of $\tau$-compact operators in $\mathcal{M}$. Every operator $X \in \mathcal{M}_0$ is compact relative to the algebra $\mathcal{M}$ [12, p. 31].

**Lemma 1** (see [13]). *Let $X, Y \in \mathcal{M}$. Then*

1) $\mu_t(X) = \mu_t(|X|) = \mu_t(X^*)$ *for all* $t > 0$;
2) *if* $|X| \leq |Y|$ *then* $\mu_t(X) \leq \mu_t(Y)$ *for all* $t > 0$;
3) $\mu_{s+t}(XY) \leq \mu_s(X)\mu_t(Y)$ *for all* $s, t > 0$;
4) $\mu_t(f(|X|)) = f(\mu_t(X))$ *for all continuous increasing functions* $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $t > 0$;
5) $\mu_{0+}(X) = \lim\limits_{t\to 0+} \mu_t(X) = \sup\limits_{t>0} \mu_t(X) = ||X||$.

One can define a rearrangement for every $\tau$-measurable operator $X$, i.e. for every $X \in \widetilde{\mathcal{M}}$, see [13]. An $F$-normed subspace $\mathcal{E} \subset \widetilde{\mathcal{M}}$ is said to be a symmetric $F$-normed space on $(\mathcal{M}, \tau)$, if

$$Y \in \mathcal{E}, X \in \widetilde{\mathcal{M}} \quad \text{and} \quad \mu(X) \leq \mu(Y) \Rightarrow X \in \mathcal{E} \quad \text{and} \quad ||X||_{\mathcal{E}} \leq ||X||_{\mathcal{E}}.$$

Let $m$ be a linear Lebesgue measure on $\mathbb{R}$. A noncommutative $L_p$-Lebesgue space ($0 < p < +\infty$) affiliated with $(\mathcal{M}, \tau)$ can be defined as

$$L_p(\mathcal{M}, \tau) = \{X \in \widetilde{\mathcal{M}} : \ \ \mu(X) \in L_p(\mathbb{R}^+, m)\}$$

with the $F$-norm (the norm for $1 \leq p < +\infty$) $||X||_p = ||\mu(X)||_p$, $X \in L_p(\mathcal{M}, \tau)$.

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \mathrm{tr}$ is the canonical trace then $\mathcal{M}_0$ coincides with the ideal $\mathfrak{S}(\mathcal{H})$ of all compact operators on $\mathcal{H}$, and

$$\mu_t(X) = \sum_{n=1}^{\infty} s_n(X)\chi_{[n-1,n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^{\infty}$ is a sequence of the operator $X$ $s$-numbers [14, Ch. 1]; here $\chi_A$ is the indicator function of a set $A \subset \mathbb{R}$. Then the space $L_p(\mathcal{M}, \tau)$ is a Shatten–von Neumann ideal $\mathfrak{S}_p(\mathcal{H})$, $0 < p < \infty$.

## 3. ON GENERALIZED SINGULAR NUMBERS OF TRIPOTENTS

For every $P \in \mathcal{M}^{\mathrm{id}}$ there exists a unique decomposition $P = \widetilde{P} + Z$, with $\widetilde{P} \in \mathcal{M}^{\mathrm{pr}}$ and nilpotent $Z \in \mathcal{M}$, $Z^2 = 0$, moreover, $Z\widetilde{P} = 0$, $\widetilde{P}Z = Z$, see [9, Theorem 1.3]. For every $A \in \mathcal{M}^{\mathrm{tri}}$ there exists a unique pair $P, Q \in \mathcal{M}^{\mathrm{id}}$ such that $A = P - Q$ and $PQ = QP = 0$ [7, Proposition 1].

Let $\tau$ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$. If $A \in \mathcal{M}^{\mathrm{tri}}$ and $A = A^*$, then $A = P - Q$ with $P, Q \in \mathcal{M}^{\mathrm{pr}}$ and $PQ = 0$ [7, Corollary 3]. We have $A^2 = |A| = P + Q \in \mathcal{M}^{\mathrm{pr}}$ and item 1) of Lemma 1 yields

$$\mu_t(A) = \mu_t(|A|) = \mu_t(P + Q) = \chi_{(0,\tau(P+Q))}(t) \quad \text{for all} \quad t > 0.$$

**Theorem 1.** *Let $\tau$ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$. Let $A \in \mathcal{M}^{tri}$ be such that $A = P - Q$ with $P \in \mathcal{M}^{\mathrm{id}}$, $Q \in \mathcal{M}^{\mathrm{pr}}$ and $PQ = QP = 0$. Let $P = \widetilde{P} + Z$ be the decomposition described above. Then $\widetilde{P}Q = 0$ and for $R = \widetilde{P} + Q \in \mathcal{M}^{\mathrm{pr}}$, and for all $t > 0$ we have*

$$\mu_t(A) = \mu_t(A)\chi_{[0,\tau(R))}(t) \geq \mu_t(R) = \chi_{[0,\tau(R))}(t). \tag{1}$$

*Proof.* We have

$$PQ = \widetilde{P}Q + ZQ = 0, \tag{2}$$

and passing to adjoint operators, we conclude that $Q\widetilde{P} + QZ^* = 0$. Now from the equality $QP = Q\widetilde{P} + QZ = 0$ we have $QZ = QZ^*$. Multiplying both sides of the last equality from the left by the operator $ZQ$, we obtain $0 = QZ^*ZQ = |ZQ|^2$. Hence $ZQ = 0$ and by (2) we obtain $\widetilde{P}Q = 0$. Therefore, $R = \widetilde{P} + Q \in \mathcal{M}^{\mathrm{pr}}$. Since $A^2 \in \mathcal{M}^{\mathrm{id}}$ for every $A \in \mathcal{M}^{\mathrm{tri}}$, we infer that $A^2 = R + Z$ is the decomposition described above by [9, Theorem 1.3]. It is easy to see that $\widetilde{P}Z^* = (Z\widetilde{P})^* = 0$ and

$$AA^* = (\widetilde{P} + Z - Q)(\widetilde{P} + Z^* - Q) = \widetilde{P} + Q + ZZ^* = R + ZZ^*.$$

For all $t > 0$ we have $\mu_t(R) = \chi_{[0,\tau(R))}(t) \in \{0, 1\}$ and

$$\mu_t(A) = \mu_t(|A^*|) = \sqrt{\mu_t(|A^*|^2)} = \sqrt{\mu_t(R + ZZ^*)} \geq \sqrt{\mu_t(R)} = \mu_t(R)$$

by items 1), 2) and 4) of Lemma 1 and monotonocity of the real function $f(\lambda) = \sqrt{\lambda}$ on $\mathbb{R}^+$. Note that $RA = A$. If $\tau(R) = +\infty$, then (1) holds. If $a = \tau(R) < +\infty$, then $b = \tau(\widetilde{P}) = \tau(P) = a - \tau(Q)$ [15, Theorem 4.6] and $\mu_t(R) = \chi_{[0,a)}(t)$ for all $t > 0$. Let $t > a$ be arbitrary and $s \in [0, 1]$ be such that $st > a$. Then

$$\mu_t(A) = \mu_t(RA) \leq \mu_{st}(R)\mu_{(1-s)t}(A) = 0$$

by item 3) of Lemma 1. Therefore, (1) holds and Theorem 1 is proved. □

**Remark 1.** If $\tau(R) < \infty$ then by (1) we have $A \in \mathcal{M}_0$. If $Q = 0$ then by Theorem 1 for $P \in \mathcal{M}^{\mathrm{id}}$ we obtain $\mu_t(P) \subset \{0\} \cup [1, ||P||]$, cf. with [16, Lemma 3.8].

**Remark 2.** The condition $Q = Q^*$ is essential in Theorem 1. Consider the idempotents

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$

in $(\mathbb{M}_2(\mathbb{C}), \mathrm{tr})$. Then $PQ = QP = 0$ and for the tripotent $A = P - Q$ we have $\mu_t(A) = \sqrt{3-2\sqrt{2}} \in (0,1)$ for $1 < t < 2$. See also [17].

**Corollary 1.** *Let $(\mathcal{E}, ||\cdot||_{\mathcal{E}})$ be a $F$-normed symmetic space on $(\mathcal{M}, \tau)$. If $A \in \mathcal{A}^{tri}$ as in Theorem 1 lies in $\mathcal{E}$, then $R \in \mathcal{E}$ and $||R||_{\mathcal{E}} \le ||A||_{\mathcal{E}}$.*

**Theorem 2.** *Let $\tau$ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$, $A \in \mathcal{M}^{tri}$ and $Z$, $R$ be as in Theorem 1. If $Z \neq 0$ and $\tau(R) < +\infty$, then there exists a number $t > 0$ such that $\mu_t(A) > \mu_t(R)$.*

*Proof.* If $X, Y \in \mathcal{M}^+$, $Y \neq 0$ and $X \geq \mu_\infty(X) \cdot I$, then there exists a number $t > 0$ such that $\mu_t(X) < \mu_t(X+Y)$ [18, Proposition 2.2]. It remains to put $X = R$, $Y = ZZ^*$ and note that $\mu_\infty(X) = 0$. Theorem is proved. □

**Corollary 2.** *In conditions of Theorem 2 we have $||R||_p \le ||A||_p$ for all $0 < p < \infty$.*

## 4. WHEN AN ISOMETRY OPERATOR IS UNITARY?

Let $\tau$ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$.

**Theorem 3.** *If $V \in \mathcal{M}$ is an isometry (or a coisometry) and $U - V \in \mathcal{M}_0$ for some $U \in \mathcal{M}^u$ then $V \in \mathcal{M}^{\mathrm{u}}$.*

*Proof. Step 1.* Let $V \in \mathcal{M}$ be an isometry and let $U = I$. Then $K = I - V \in \mathcal{M}_0$ and $P = VV^* \in \mathcal{M}^{\mathrm{pr}}$. We have

$$K^*K - KK^* = I - P \geq 0, \tag{3}$$

i.e., an operator $K$ is hyponormal. Then an operator $K$ is normal by [19, Theorem 2.2] (or by [20, Corollary 4.3]). Now by (3) we have $P = I$ and $V \in \mathcal{M}^{\mathrm{u}}$.

*Step 2.* Let an isometry $V \in \mathcal{M}$ and an operator $U \in \mathcal{M}^{\mathrm{u}}$ be such that $U - V \in \mathcal{M}_0$. Since $\mathcal{M}_0$ is an ideal in $\mathcal{M}$, we have

$$(U - V)U^* = I - VU^* \in \mathcal{M}_0.$$

Obviously, $(VU^*)^* \cdot VU^* = I$, i.e. an operator $VU^*$ is an isometry. By Step 1 we have $VU^* \in \mathcal{M}^{\mathrm{u}}$. Therefore, $V = VU^* \cdot U \in \mathcal{M}^{\mathrm{u}}$ as a product of unitary operators from $\mathcal{M}$.

*Step 3.* Let a coisometry $V \in \mathcal{M}$ and an operator $U \in \mathcal{M}^{\mathrm{u}}$ be such that $U - V \in \mathcal{M}_0$. Then $V^*$ is an isometry, $U^* \in \mathcal{M}^{\mathrm{u}}$, and $U^* - V^* = (U - V)^* \in \mathcal{M}_0$. By Step 2 we have $V^* \in \mathcal{M}^{\mathrm{u}}$. Hence $V \in \mathcal{M}^{\mathrm{u}}$ and Theorem 3 is proved. □

**Corollary 3.** *Let a number $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and an operator $K \in \mathcal{M}_0$ be such that an operator $\lambda I + K$ is an isometry. Then the operator $K$ is normal.*

**Corollary 4.** *Let $S, T \in \mathcal{M}^{sp}$ and $S - T \in \mathcal{M}_0$. If the operator $T$ is normal then the operator $S$ is also normal.*

*Proof.* The formula $V_A = 2A - I$ ($A \in \mathcal{M}^{\mathrm{sp}}$) determines a bijection between $\mathcal{M}^{\mathrm{sp}}$ and the set of all isometries from $\mathcal{M}$. Moreover, $V_A \in \mathcal{M}^{\mathrm{u}}$ if and only if an operator $A$ is normal. □

**Theorem 4.** *Let $\mathcal{M}$ be a factor with a faithful normal trace $\tau$ on it. If $V \in \mathcal{M}$ is an isometry (or a coisometry) and $U - V$ is compact relative to $\mathcal{M}$ for some $U \in \mathcal{M}^u$ then $V \in \mathcal{M}^{\mathrm{u}}$.*

*Proof.* If an operator $T \in \mathcal{M}$ is hyponormal and compact relative to $\mathcal{M}$ then $T$ is normal [21, Theorem]. Therefore, we can repeat the proof of Theorem 3.

**Corollary 5.** *Let $\mathcal{M}$ be a factor with a faithful normal trace $\tau$ on it. Let a number $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and a compact relative to $\mathcal{M}$ operator $K$ be such that an operator $\lambda I + K$ is an isometry. Then the operator $K$ is normal.*

**Corollary 6.** *Let $\mathcal{M}$ be a factor with a faithful normal trace $\tau$ on it. Let $S, T \in \mathcal{M}^{sp}$ and $S - T$ be compact relative to $\mathcal{M}$. If the operator $T$ is normal then the operator $S$ is also normal.*

## ACKNOWLEDGMENTS

Research was supported by the subsidy allocated to Kazan Federal University for the state assignment in the sphere of scientific activities (1.9773.2017/8.9).

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