ISSN 1066-369X, Russian Mathematics , 2017, Vol. 61, No. 1, pp. 76–80. © Allerton Press, Inc., 2017.
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# Two Classes of $\tau$-Measurable Operators Affiliated with a von Neumann Algebra

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Received May 23, 2016

**Abstract**—Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$, $\tau$ be a faithful normal semifinite trace on $\mathcal{M}$. We define two (closed in the topology of convergence in measure $\tau$) classes $\mathcal{P}_1$ and $\mathcal{P}_2$ of $\tau$-measurable operators and investigate their properties. The class $\mathcal{P}_2$ contains $\mathcal{P}_1$. If a $\tau$-measurable operator $T$ is hyponormal, then $T$ lies in $\mathcal{P}_1$; if an operator $T$ lies in $\mathcal{P}_k$, then $UTU^*$ belongs to $\mathcal{P}_k$ for all isometries $U$ from $\mathcal{M}$ and $k = 1, 2$; if an operator $T$ from $\mathcal{P}_1$ admits the bounded inverse $T^{-1}$, then $T^{-1}$ lies in $\mathcal{P}_1$. We establish some new inequalities for rearrangements of operators from $\mathcal{P}_1$. If a $\tau$-measurable operator $T$ is hyponormal and $T^n$ is $\tau$-compact for some natural number $n$, then $T$ is both normal and $\tau$-compact. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \mathrm{tr}$, then the class $\mathcal{P}_1$ coincides with the set of all paranormal operators on $\mathcal{H}$.

**DOI:** 10.3103/S1066369X17010091

Keywords: *Hilbert space, von Neumann algebra, normal trace, $\tau$-measurable operator, rearrangement, topology of convergence in measure, $\tau$-compact operator, integrable operator, hyponormal operator, quasinormal operator, paranormal operator, projection.*

**Introduction.** Let $\mathcal{M}$ be a von Neumann operator algebra on a Hilbert space $\mathcal{H}$, $\tau$ be a faithful normal semifinite trace on $\mathcal{M}$, $\widetilde{\mathcal{M}}$ be a $*$-algebra of all $\tau$-measurable operators, a number $0 < p < +\infty$ and $L_p(\mathcal{M}, \tau)$ be the space of integrable (with respect to $\tau$) with degree $p$ operators. In this paper we introduce two (closed in the topology of convergence in measure $\tau$) classes $\mathcal{P}_1$ and $\mathcal{P}_2$ with $\mathcal{P}_1 \subset \mathcal{P}_2$ elements of $\widetilde{\mathcal{M}}$ and investigate their properties. We show that if an operator $T \in \widetilde{\mathcal{M}}$ is hyponormal, then $T \in \mathcal{P}_1$; if an operator $T \in \mathcal{P}_1$, then $UTU^* \in \mathcal{P}_1$ for all isometries $U \in \mathcal{M}$; if an operator $T \in \mathcal{P}_1$ has the inverse $T^{-1} \in \mathcal{M}$, then $T^{-1} \in \mathcal{P}_1$. For $T \in \widetilde{\mathcal{M}}$ we have $T \in \mathcal{P}_2 \Leftrightarrow T^* \in \mathcal{P}_2$. If $T \in \widetilde{\mathcal{M}}$ and the operators $U, V \in \mathcal{M}$ are isometries, then the rearrangement relation $\mu_t(UTV^*) = \mu_t(T)$ holds for all $t > 0$; if $T \in \mathcal{P}_2$, then $UTU^* \in \mathcal{P}_2$. Let $T \in \widetilde{\mathcal{M}}$ and a unitary operator $S \in \mathcal{M}^{\mathrm{sa}}$ be so that $ST = TS$. Then $T \in \mathcal{P}_k \Leftrightarrow ST \in \mathcal{P}_k$, $k = 1, 2$.

If an operator $T \in \mathcal{P}_1$, then $T^2 \in \mathcal{P}_1$ and $\mu_t(T^2) \geq \mu_t(T)^2$ for all $t > 0$. If an operator $T \in \mathcal{P}_1 \cap \mathcal{M}$, then $T^n \in \mathcal{P}_1$ for all $n \in \mathbb{N}$. The set $\mathcal{P}_1 \cap \mathcal{M}$ is $\|\cdot\|$-closed in $\mathcal{M}$. Consider an operator $T \in \mathcal{P}_1 \cap \mathcal{M}$ and $n \in \mathbb{N}$. Then $\mu_t(T^n) \geq \mu_t(T)^n$ for all $t > 0$ and we have the equivalences $T \in \mathcal{F}(\mathcal{M}) \Leftrightarrow T^n \in \mathcal{F}(\mathcal{M})$; $T \in \widetilde{\mathcal{M}}_0 \Leftrightarrow T^n \in \widetilde{\mathcal{M}}_0$; $T \in L_{pn}(\mathcal{M}, \tau) \Leftrightarrow T^n \in L_p(\mathcal{M}, \tau)$, $0 < p < +\infty$. Every operator $T \in \mathcal{P}_1 \cap \mathcal{M}$ is normaloid. If an operator $(0 \neq)T \in \mathcal{M}$ is quasi-nilpotent, then $T \notin \mathcal{P}_1$. The class $\mathcal{P}_2$ does not contain either non-selfadjoint symmetries ($T^2 = I$) or non-selfadjoint idempotents ($T^2 = T$). If an operator $T \in \widetilde{\mathcal{M}}$ is hyponormal and $T^n \in \widetilde{\mathcal{M}}_0$ for some $n \in \mathbb{N}$, then $T$ belongs to $\widetilde{\mathcal{M}}_0$ and is normal.

Let $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{M}$. We have that $T \in \mathcal{P}_1 \Leftrightarrow T$ is paranormal. The class $\mathcal{P}_1$ is sequentially closed in the strong operator topology and contains a non-hyponormal operator. If $\mathcal{H}$ is separable and infinite dimensional, then $\mathcal{P}_1 \neq \mathcal{P}_2$.

**1. Definition and notation.** Let $\mathcal{M}$ be a von Neumann operator algebra on a Hilbert space $\mathcal{H}$, $\mathcal{M}^{\mathrm{pr}}$ be the lattice of projections on $\mathcal{M}$, $\mathcal{M}^+$ be the positive element cone in $\mathcal{M}$. Let $I$ be the identity of $\mathcal{M}$ and $P^{\perp} = I - P$ for $P \in \mathcal{M}^{\mathrm{pr}}$.

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