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# SSP rings and modules

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We study $\mathcal{A}$-Ci modules ($i = 2, 3$), first introduced in [K. Oshiro, Continuous modules and quasi-continuous modules, *Osaka J. Math.* **20** (1983) 681–694], and $\mathcal{A}$-SSP modules. We consider the cases when these classes of modules coincide. As a consequence, we obtain some results related to simple-direct-injective modules. We also investigate some properties of SSP formal matrix rings and describe semiartinian SSP rings.

*Keywords*: SSP modules; semiartinian rings; formal matrix rings.

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## 1. Introduction

Throughout this paper, $R$ denotes an associative ring with identity, and modules will be unitary right $R$-modules.

A module $M$ is called an *SSP module* (respectively, *SIP module*) if the sum (respectively, the intersection) of any two direct summands of $M$ is also a direct summand of $M$. A ring $R$ is called a *right SSP ring* (respectively, *right SIP ring*) if

$R_R$ is an SSP module (respectively, SIP-module). Because right SSP rings are left SSP rings, we will not use the terms right SSP and left SSP, and call these rings SSP rings, simply. From [3, Lemma 1.9], it implies that all SSP rings are SIP rings. SSP and SIP modules have also been studied [7, 9, 14]. An important generalization of SSP module (respectively, SIP module) is the concept of $C3$ module (respectively, $D3$ module). These modules have recently been studied in [4, 5].

Let $\mathcal{A}$ be a set of submodules of $M$. We say that a module $M$ is an $\mathcal{A}$-SSP module, if for any submodules $A \in \mathcal{A}$ and $X \leq M$, $A \leq_{\oplus} M, X \leq_{\oplus} M$ then $A + X \leq_{\oplus} M$. In this paper we study the relationship between $\mathcal{A}$-Ci modules ($i = 2, 3$), first introduced in [12], and $\mathcal{A}$-SSP modules. We also studied semiartinian SSP rings and SSP formal matrix rings.

Throughout this paper, the notations $N \leq M$, $N \leq_e M$ and $N \ll M$ mean that $N$ is a submodule, an essential submodule, and a small submodule of $M$, respectively. The Jacobson radical and the maximal regular ideal in $R$ are denoted by $J(R)$ and $\mathrm{Reg}(R)$, respectively. The Jacobson radical of a right $R$-module $M$ is denoted by $J(M)$.

The paper uses standard concepts and notations of the theory of rings and modules (see, eg., [13]).



## 2. $\mathcal{A}$-Ci Modules

Let $M$ be a right $R$-module and $\mathcal{A}$ be a set of submodules of $M$. Following [12, 10], we recall the following conditions:

$\mathcal{A}$-$(C1)$: For all $A \in \mathcal{A}$, there exists $A^* \leq_{\oplus} M$ such that $A \leq_e A^*$.
$\mathcal{A}$-$(C2)$: For all $A \in \mathcal{A}$, if $X \leq_{\oplus} M$ is such that $A \cong X$, then $A \leq_{\oplus} M$.
$\mathcal{A}$-$(C3)$: For all $A \in \mathcal{A}$ and $X \leq_{\oplus} M$, if $A \leq_{\oplus} M$ and $A \cap X = 0$ then $A \oplus X \leq_{\oplus} M$.

**Lemma 2.1.** *Let $M$ be a right $R$-module and $\mathcal{A}$ be a set of submodules of $M$ which is closed under isomorphic images. If $M$ is an $\mathcal{A}$-$C2$ module then $M$ is an $\mathcal{A}$-$C3$ module.*

**Lemma 2.2.** *Let $M$ be a right $R$-module and $\mathcal{A}$ be a set of submodules of $M$ which is closed under isomorphic images. If $M$ is an $\mathcal{A}$-$C2$ module ($\mathcal{A}$-$C3$ module) then so are all direct summands of $M$.*

Let $f : A \to B$ be a homomorphism. We denote by $\langle f \rangle$ the submodule of $A \oplus B$ as follows:

$$\langle f \rangle = \{a + f(a) \mid a \in A\}.$$

**Theorem 2.1.** *Let $M$ be a right $R$-module and $\mathcal{A}$ be a set of submodules of $M$ which is closed under isomorphic images and summands. If every submodule of $M$ is $\mathcal{A}$-projective, then the following conditions are equivalent:*

(1) *If whenever two direct summands $A, B$ of $M$ with $A \in \mathcal{A}$, then $A+B$ is a direct summand of $M$.*

(2) *$M$ is an $\mathcal{A}$-$C3$ module.*
(3) *For any decomposition $M = A_1 \oplus A_2$ with $A_1 \in \mathcal{A}$, then every homomorphism $f : A_1 \to A_2$ has the image, a direct summand of $A_2$.*

**Proof.** (1) $\Rightarrow$ (2) The implication is obvious.

(2) $\Rightarrow$ (3) Let $f : A_1 \to A_2$ be an $R$-homomorphism with $A_1 \in \mathcal{A}$. By the hypothesis, there exists a decomposition $A_1 = \mathrm{Ker}(f) \oplus B$ for a submodule $B$ of $A_1$. Then $B \oplus A_2$ is a direct summand of $M$. Note that if a module satisfies (2), so are its direct summands of $M$. Hence $B \oplus A_2$ satisfies (2). Let $g := f|_B : B \to A_2$. Then $g$ is a monomorphism and $\mathrm{Im}(g) = \mathrm{Im}(f)$. It is easy to see that $B \oplus A_2 = \langle g \rangle \oplus A_2$, $\langle g \rangle \cap B = 0$ and $\langle g \rangle \simeq B$. Note that $B, \langle g \rangle \in \mathcal{A}$. As $B \oplus A_2$ satisfies (2), $B \oplus \langle g \rangle$ is a direct summand of $B \oplus A_2$. Thus $B \oplus \langle g \rangle = B \oplus \mathrm{Im}(g)$, which implies that $\mathrm{Im}(g)$ or $\mathrm{Im}(f)$ is a direct summand of $A_2$.

(3) $\Rightarrow$ (1) Let $N$ and $K$ be summands of $M$ such that $N \in \mathcal{A}$. Write $M = N \oplus N'$ and $M = K \oplus K'$ for some $N', K' \leq M$. Consider the canonical projections $\pi_K : M \to K$ and $\pi_{N'} : M \to N'$. Let $A := \pi_{N'}(\pi_K(N))$. Then $A = (N + K) \cap (N + K') \cap N'$, and so is a direct summand of $M$ by (3). Write $M = A \oplus L$ for a submodule $L \leq M$. Clearly,

$$(N + K) \cap [(N + K') \cap (N' \cap L)] = 0.$$

Hence, $N' = A \oplus (N' \cap L)$ and $M = (N \oplus A) \oplus (N' \cap L)$. Since $A \leq N + K$ and $A \leq N + K'$, we get

$$N + K = (N \oplus A) \cap [(N + K) \cap (N' \cap L)]$$

and

$$N + K' = (N \oplus A) \cap [(N + K') \cap (N' \cap L)].$$

They imply

$$\begin{aligned} M &= N + K' + K \\ &= (N \oplus A) + [(N + K) \cap (N' \cap L)] + [(N + K') \cap (N' \cap L)] \\ &\leq (N + K) + [(N + K') \cap (N' \cap L)]. \end{aligned}$$

Thus $M = (N + K) \oplus [(N + K') \cap (N' \cap L)$. $\square$

**Theorem 2.2.** *Let $M$ be a right $R$-module and $\mathcal{A}$ be a set of submodules of $M$ which is closed under isomorphic images and summands. If every factor module of $M$ is $\mathcal{A}$-projective or every submodule of $M$ is $\mathcal{A}$-injective, then the following conditions are equivalent*:

(1) *If whenever two direct summands $A, B$ of $M$ with $A \in \mathcal{A}$, then $A + B$ is a direct summand of $M$.*
(2) *$M$ is an $\mathcal{A}$-$C3$ module.*

(3) *For any decomposition $M = A_1 \oplus A_2$ with $A_1 \in \mathcal{A}$, then every homomorphism $f : A_1 \to A_2$ has the image, a direct summand of $A_2$.*
(4) *$M$ is an $\mathcal{A}$-$C2$ module.*

**Proof.** (1) $\Rightarrow$ (2) The implication is obvious.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) The implication is similar to the argument in the proof of Theorem 2.1.

(4) $\Rightarrow$ (2) It follows from Lemma 2.2(1).

(3) $\Rightarrow$ (4) Let $\sigma : A \to B$ be an isomorphism with $A \in \mathcal{A}$ a summand of $M$ and $B \leq M$. We show that $B$ is a direct summand of $M$. Write $M = A \oplus T$ for a submodule $T$ of $M$. We have that $A/A \cap B$ is an image of $M$ and obtain that $A \cap B$ is a direct summand of $A$. Take $A = (A \cap B) \oplus C$ for a submodule $C$ of $A$. Now $M = (A \cap B) \oplus (C \oplus T)$. Clearly, $A \cap [(C \oplus T) \cap B] = 0$ and $B = (A \cap B) \oplus [(C \oplus T) \cap B]$. Let $H := \sigma^{-1}((C \oplus T) \cap B)$. Then $H$ is a submodule of $A$, $H \cap [(C \oplus T) \cap B] = 0$ and there exists a submodule $H'$ of $H$ such that $A = H \oplus H'$. Note that $M = H \oplus (H' \oplus T)$. Consider the projection $\pi : M \to H' \oplus T$. Then

$$H \oplus [(C \oplus T) \cap B] = H \oplus \pi((C \oplus T) \cap B).$$

By (3), the image of the homomorphism $\pi|_{(C \oplus T) \cap B} \circ \sigma|_H : H \to H' \oplus T$ is a direct summand of $H' \oplus T$ because $H$ is contained in $\mathcal{A}$. Write $H' \oplus T = \pi|_{(C \oplus T) \cap B}\sigma(H) \oplus K$ for a submodule $K$ of $H' \oplus T$. Then $H' \oplus T = \pi((C \oplus T) \cap B) \oplus K$. It follows that

$$M = H \oplus \pi((C \oplus T) \cap B) \oplus K = H \oplus [(C \oplus T) \cap B] \oplus K.$$

By the modular law, $C \oplus T = [(C \oplus T) \cap B] \oplus [(H \oplus K) \cap (C \oplus T)]$. Thus

$$\begin{aligned} M &= (A \cap B) \oplus [(C \oplus T) \cap B] \oplus [(H \oplus K) \cap (C \oplus T)] \\ &= B \oplus [(H \oplus K) \cap (C \oplus T)]. \end{aligned}$$

□

**Corollary 2.1.** *Let $N$ be a right $R$-module. The following conditions are equivalent*:

(1) *$N$ is semisimple injective.*
(2) *For any right $R$-module $M$, $M$ is an $\mathcal{A}$-$C3$ module and every factor module of $M$ is $\mathcal{A}$-projective where*

$$\mathcal{A} = \{A \leq M \mid \exists\, X \leq N, f : X \to M, f(X) \leq^e A\}.$$

**Proof.** (1) $\Rightarrow$ (2) Assume that $N$ is a semisimple injective module. For any right $R$-module $M$,

$$\begin{aligned} \mathcal{A} &= \{A \leq M \mid \exists\, X \leq N, f : X \to M, f(X) \leq^e A\} \\ &= \{A \leq M \mid A \text{ is embeddable in } N\}. \end{aligned}$$

Thus every factor module of $M$ is $\mathcal{A}$-projective and $M$ is an $\mathcal{A}$-$C3$ module by Theorem 2.2.

(2) $\Rightarrow$ (1) Let $B$ be a submodule of $N$. Then $M = B \oplus E(B)$ is an $\mathcal{A}$-$C3$ module where $\mathcal{A} = \{A \leq M \,|\, \exists\, X \leq N, f : X \to M, f(X) \leq^e A\}$. As $B \in \mathcal{A}$, then by Theorem 2.2 the inclusion map $\iota : B \to E(B)$ splits. It means that $B = E(B)$ is injective. So $B$ is a direct summand of $N$. It shown that $N$ is a semisimple injective module. $\square$

**Theorem 2.3.** *Let $M$ be a right $R$-module and $\mathcal{A}$ be a set of Artinian submodules of $M$ which is closed under isomorphic images and summands. If every submodule of $M$ is $\mathcal{A}$-projective, then the following conditions are equivalent*:

(1) *$M$ is an $\mathcal{A}$-$C3$ module.*
(2) *$M$ is an $\mathcal{A}$-$C2$ module.*



**Proof.** (1) $\Rightarrow$ (2) Let $M_1$ be submodule of $M$, which is isomorphic to a direct summand $M_2$ of $M$ and $M_1 \in \mathcal{A}$. Then $M = M_2 \oplus M_2'$. If $M_1 \subset M_2$, then by $M_2$ that is Artinian and $M_1 \cong M_2$, implies that $M_1 = M_2$. Let $M_1 \nsubseteq M_2$ and $\pi : M_2 \oplus M_2' \to M_2'$ be projection. According to the hypothesis, $\mathrm{Ker}(\pi_{|M_1})$ is a direct summand of $M_1$, then $M_1 = M_1 \cap M_2 \oplus N_1$. Since $N_1 \cong \pi(M_1)$, $M_1 \cong M_2$, then there is an isomorphism $\phi : N' \to \pi(M_1)$, where $N'$ is a direct summand of $M_1$. Since $\langle\phi\rangle \in \mathcal{A}$ and $\langle\phi\rangle \cap M_2 = 0$, $M_2 + \langle\phi\rangle = M_2 \oplus N_1$ is a direct summand of $M$. Therefore, $N_1$ is a non-zero direct summand of $M$. It is clear that $M_1 \cap M_2 \in \mathcal{A}$ and $M_1 \cap M_2$ is isomorphic to a direct summand of $M$. If $M_1 \cap M_2$ is not a direct summand of $M$, by using an argument that is similar to the argument presented above, we can show that $M_1 \cap M_2 = N_2 \oplus N_2'$, where $N_2$ is a non-zero direct summand of $M$, $N_2'$ is a submodule of $M$, which is isomorphic to a direct summand of $M$ and $N_2, N_2' \in \mathcal{A}$. Since each module of the class $\mathcal{A}$ is Artinian, by conducting similar constructions that continue for some $k$, we obtain a decomposition $M_1 = N_1 \oplus \cdots \oplus N_k$, where $N_i$ is a direct summand of $M$ and $N_i \in \mathcal{A}$ for each $i$. Since $M$ is an $\mathcal{A}$-$C3$ module, $N_1 \oplus \cdots \oplus N_k$ is a direct summand of $M$.

(2) $\Rightarrow$ (1) It follows from Lemma 2.2(1). $\square$

**Corollary 2.2 ([6, Proposition 2.1]).** *The following conditions are equivalent for a module $M$*:

(1) *For any simple submodules $A$, $B$ of $M$ with $A \cong B \leq_\oplus M$, $A \leq_\oplus M$.*
(2) *For any simple summands $A, B$ of $M$, $A \oplus B \leq_\oplus M$.*
(3) *For any finitely generated semisimple submodules $A$, $B$ of $M$ with $A \cong B \leq_\oplus M$, $A \leq_\oplus M$.*
(4) *For any finitely generated semisimple summands $A, B$ of $M$, $A \oplus B \leq_\oplus M$.*

**Proof.** The equivalences (1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (3) follow from Theorem 2.2. The implication (4) $\Rightarrow$ (1) is obvious.

(2) $\Rightarrow$ (4) It is enough to show that, if $M_1, \ldots, M_n$ are simple summands of $M$, then $M_1 + \cdots + M_n$ is a summand of $M$. That is easy to prove by induction. $\square$

Note that a module $M$ satisfying the condition of Corollary 2.2 is called simple-direct-injective (see [6]).

## 3. SSP-Rings

The following statement follows from [2, 3, 7].

**Theorem 3.1.** *The following conditions are equivalent for a quasi-projective module $M$:*

(1) *If $f, g \in \mathrm{End}_R(M)$ are regular homomorphisms, then $fg$ is a regular homomorphism.*
(2) *If $e, f \in \mathrm{End}_R(M)$ are idempotent homomorphisms, then $fe$ is a regular homomorphism.*
(3) $\mathrm{End}_R(M)$ *is a right SSP ring.*
(4) $\mathrm{End}_R(M)$ *is a left SSP ring.*
(5) *For any decomposition $M = A \oplus B$ and any homomorphism $f \in \mathrm{Hom}_R(A, B)$, the image of the homomorphism $f$ is a direct summand of $M$.*

**Lemma 3.1.** *The following conditions are equivalent for a ring $R$:*

(1) *If $a, b \in R$ are regular, then $ab$ is also regular.*
(2) *If $e, f \in R$ are idempotent elements, then $ef$ is regular.*
(3) ${}_RR$ *is an SSP module.*
(4) $R_R$ *is an SSP module.*
(5) *For any idempotent $e \in R$, every element of the set $eR(1-e)$ and every element of the set $(1-e)Re$ are regular.*

The previous lemma gives the equivalent definition of an SSP ring. For example, every regular ring and every normal ring are SSP rings.

**Lemma 3.2.** *If $R$ is an SSP ring, then $eRe$ is an SSP ring for any idempotent $e \in R$.*

**Lemma 3.3.** *Let $K = \left(\begin{smallmatrix} R & M \\ N & S \end{smallmatrix}\right)$ be a Morita context. If $K$ is an SSP ring, then $M$ is an $N$-regular module and $N$ is an $M$-regular module.*

**Lemma 3.4.** *The following conditions are equivalent for a ring $R$:*

(1) *$R$ is an SSP ring.*
(2) *$R/\mathrm{Reg}(R)$ is an SSP ring.*

**Proof.** (1) $\Rightarrow$ (2) Using Lemma 3.1, we only need to show that the product of two idempotents of the ring $R/\mathrm{Reg}(R)$ is a regular element. Let $e_1, e_2$ are idempotents of the ring $R/\mathrm{Reg}(R)$. Then by [11, Lemma 3], there exist idempotents $f_1, f_2 \in R$ such that $e_1 = f_1 + \mathrm{Reg}(R), e_2 = f_2 + \mathrm{Reg}(R)$. Since $R$ is an SSP ring, $f_1 f_2$ is a regular element of the ring $R$. Therefore, $e_1 e_2$ is a regular element of the ring $R/\mathrm{Reg}(R)$.

(2) $\Rightarrow$ (1) We will show that the product of two regular elements of the ring $R$ is also a regular element. Let $a, b$ are regular elements of the ring $R$. Since $R/\mathrm{Reg}(R)$ is an SSP ring, there exists $c \in R$ such that $abcab - ab \in \mathrm{Reg}(R)$ by Lemma 3.1. Then we have $abcab - ab = (abcab - ab)d(abcab - ab)$ for some $d \in R$. Therefore, $ab \in abRab$. $\square$

A module $M$ is called a *retractable module* if $\mathrm{Hom}(M, N) \neq 0$, $\forall N(\neq 0) \leq M$.

**Theorem 3.2.** *Let $P$ be a quasi-projective retractable module. If $J(P)$ is an essential submodule of $P$, then the following conditions are equivalent*:

(1) *$P$ is an SSP module.*
(2) *$S = \mathrm{End}_R(P)$ is a normal ring.*

**Proof.** (1) $\Rightarrow$ (2) Assume that $e \in S$ is not a central idempotent. Without loss of generality, we can assume that $(1-e)Se \neq 0$. Then there exists a non-zero homomorphism $\psi \in \mathrm{Hom}_R(eP, (1-e)P)$. From Theorem 3.1, $\mathrm{Im}(\psi)$ is a direct summand of $P$. Then $\mathrm{Ker}(\psi)$ is a direct summand of $eP$. Therefore, $eP$ and $(1-e)P$ contain non-zero direct summands which are isomorphic. Let $fP$ be a non-zero direct summand of $eP$, which is isomorphic to some direct summand of $(1-e)P$. Then $f$ is an idempotent of the ring $S$. Since $J(P)$ is essential in $P$, the submodule $fP$ contains a non-zero element $m \in J(P)$. Since $P$ is retractable, there exists a non-zero homomorphism modules $\phi : P \to mR$. If $\phi_{|fP} \neq 0$ then there exists a non-zero homomorphism $\varphi : eP \to (1-e)P$ such that $\mathrm{Im}(\varphi) \subset J((1-e)P)$, that is impossible by Theorem 3.1. If $\phi_{|fP} = 0$ then there exists a non-zero homomorphism from $(1-f)P$ to $J(fP)$, which is also impossible. This contradiction shows that $S$ is a normal ring.

The implications (2) $\Rightarrow$ (1) follows from Theorem 3.1. $\square$

**Theorem 3.3.** *Let $R$ be a right semiartinian ring. Then the following conditions are equivalent*:

(1) *$R$ is an SSP ring.*
(2) *$R/\mathrm{Reg}(R)$ is a normal ring.*
(3) *$eR(1-e) \subset \mathrm{Reg}(R)$ for any idempotent $e \in R$.*

**Proof.** (1) $\Rightarrow$ (2) Put $\overline{R} = R/\mathrm{Reg}(R)$. Call $A$ a right ideal of the ring $\overline{R}$ with $\mathrm{Soc}(\overline{R}_{\overline{R}}) = A \oplus J(\overline{R}) \cap \mathrm{Soc}(\overline{R}_{\overline{R}})$. Assume that $S$ is a simple submodule of $A_{\overline{R}}$ and $rS$ is not a submodule of $A_{\overline{R}}$ for some $r \in \overline{R}$. Then $\pi(rS)$ is a simple submodule of $J(\overline{R}) \cap \mathrm{Soc}(\overline{R}_{\overline{R}})$, where $\pi : A \oplus J(\overline{R}) \cap \mathrm{Soc}(\overline{R}_{\overline{R}}) \to J(\overline{R}) \cap \mathrm{Soc}(\overline{R}_{\overline{R}})$ is the natural projection. On the other hand, there exists a submodule $B$ of $\overline{R}_{\overline{R}}$ such that $\overline{R}_{\overline{R}} = S \oplus B$. It follows that $J(\overline{R}) = J(B)$ and $J(B) \cap \mathrm{Soc}(\overline{R}_{\overline{R}})$ contains a submodule $S_0$ which is isomorphic to the module $S$. Then there is a homomorphism $f : S \to B$, such that $\mathrm{Im}(f) = S_0$. By Lemma 3.4, the ring $\overline{R}$ is an SSP ring. Therefore $S_0$ is a direct summand of $B$ by Theorem 3.1, which contradicts $S_0 \subset J(B)$. Thus, $A$ is an ideal of $R$. We will show that $A$ is a regular ideal. Let $a \in A$. Since $a\overline{R}$ is a semisimple module of finite length and $a\overline{R} \cap J(\overline{R}) = 0$, $a\overline{R}$ is a direct summand of $\overline{R}_{\overline{R}}$. Therefore, $a \in a\overline{R}a = aAa$. Since $\mathrm{Reg}(\overline{R}) = 0$, $A = 0$ and therefore $\mathrm{Soc}(\overline{R}_{\overline{R}}) \subset J(\overline{R})$. Then, the implication follows from Theorem 3.2.

The implication (2) $\Rightarrow$ (1) follows from Lemma 3.4.

The equivalence of (2) $\Leftrightarrow$ (3) follows from [11, Lemma 3]. $\square$

**Theorem 3.4.** *Let $R$, $S$ be normal rings and $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ be a formal matrix ring. Then the following conditions are equivalent*:

(1) *$K$ is an SSP ring.*
(2) *$R, S$ are SSP rings and* $\mathrm{Reg}(K) = \begin{pmatrix} \mathrm{Reg}(R) & M \\ N & \mathrm{Reg}(S) \end{pmatrix}$.

**Proof.** (1) $\Rightarrow$ (2) By Lemma 3.3, all elements of the form $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ are regular in the ring $K$. Since Lemma 3.1, $mn$ are regular in the ring $R$ for any $m \in M, n \in N$. Let $\sum_{i\in I} r_i mnr'_i$ be any element of the ideal $RmnR$. Since $mn$ is regular, $mn = mnrmn$ for some $r \in R$. Then since $\sum_{i\in I} r_i mnr'_i = \sum_{i\in I} r_i mnrmnr'_i = mnr(\sum_{i\in I} r_i mnr'_i)$, every element of the ideal $RmnR$ belongs to the set $\{mn \,|\, m \in M, n \in N\}$, and hence, it is regular. So that we have $MN \subset \mathrm{Reg}(R)$. Similarly, we can show that $NM \subset \mathrm{Reg}(S)$. Then from [15, Theorem 5.3], it follows that $\mathrm{Reg}(K) = \begin{pmatrix} \mathrm{Reg}(R) & M \\ N & \mathrm{Reg}(S) \end{pmatrix}$. We obtain that $R$, $S$ are SSP rings by Lemma 3.2.

(2) $\Rightarrow$ (1) Since $K/\mathrm{Reg}(K) \cong R/\mathrm{Reg}(R) \times S/\mathrm{Reg}(S)$ is an SSP ring, then it follows from Lemma 3.4 that $K$ is an SSP ring. $\square$

**Corollary 3.1.** *Let $R, S$ be rings that satisfy every idempotent is trivial and $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ be a formal matrix ring. Then the following conditions are equivalent*:

(1) *$K$ is an SSP ring.*
(2) *Either $M = 0, N = 0$ or $K \cong M_2(T)$, where $T$ is a skew field.*

**Theorem 3.5.** *Let $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ is a formal matrix ring and $R$, $S$ be right semiartinian rings. Then the following conditions are equivalent*:

(1) *$K$ is an SSP ring.*
(2) *$R$, $S$ are SSP rings and* $\mathrm{Reg}(K) = \begin{pmatrix} \mathrm{Reg}(R) & M \\ N & \mathrm{Reg}(S) \end{pmatrix}$.

**Proof.** (1) $\Rightarrow$ (2) According to [1, Theorem 4.2], the ring $K$ is semiartinian. From Theorem 3.3, it follows that $\begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix} \subset \mathrm{Reg}(K)$. Then from [15, Theorem 5.3], it implies that $\mathrm{Reg}(K) = \begin{pmatrix} \mathrm{Reg}(R) & M \\ N & \mathrm{Reg}(S) \end{pmatrix}$.

(2) $\Rightarrow$ (1) Since by Theorem 3.3, the rings $R/\mathrm{Reg}(R)$ and $S/\mathrm{Reg}(S)$ are normal, then the ring $K/\mathrm{Reg}(K) \cong R/\mathrm{Reg}(R) \times S/\mathrm{Reg}(S)$ is normal. Then from Lemma 3.4, it follows that $K$ is an SSP ring. $\square$

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