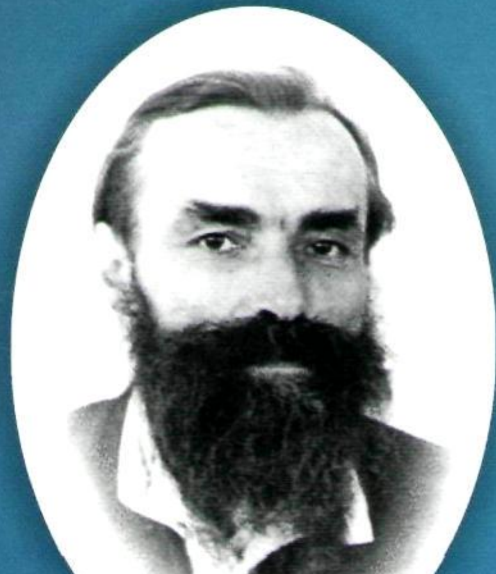
АЛГЕБРА
И
МАТЕМАТИЧЕСКАЯ ЛОГИКА



Материалы международной конференции, посвященной 100-летию со дня рождения профессора В.В. Морозова, (Казань, 25 – 30 сентября 2011 г.) и молодежной школы-конференции "Современные проблемы алгебры и математической логики" (Казань, 22 сентября – 3 октября 2011 г.)

конечных подгрупп.

Свойство 6. *Если a — элементарная точка группы G, то в нормализаторе $N_G(K)$ конечной нильпотентной группы K, нормализуемой элементом a, точка a содержится в конечном числе конечных подгрупп.*

Свойство 7. *Нормализатор элементарной точки обладает конечной периодической частью.*

Свойство 8. *Бесконечная черниковская группа не обладает элементарными точками.*

ON REPRESENTATION OF TRIPOTENTS AND REPRESENTATIONS VIA TRIPOTENTS

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Let $\mathcal{A}, \mathcal{D}$ be algebras. An element $A \in \mathcal{A}$ is called *idempotent* if $A^2 = A$; and *tripotent* if $A^3 = A$. Let

$$\mathcal{A}^{\mathrm{id}} = \{A \in \mathcal{A}:\ A^2 = A\}, \quad \mathcal{A}^{\mathrm{tr}} = \{A \in \mathcal{A}:\ A^3 = A\}.$$

In [1] we study the following questions: if both A and B are tripotents, then: under what conditions are $A + B$ and AB tripotent? Under what conditions do A and B commute? We decompose any tripotent into the difference of two mutually orthogonal idempotents. This representation is unique. We extend the partial order from the Hilbert space idempotents quantum logic (see [2]) to the set of all tripotents and show that every normal tripotent is self-adjoint.

We say that a linear map $\phi : \mathcal{A} \to \mathcal{D}$ *preserves idempotents* if $\phi(A) \in \mathcal{D}^{\mathrm{id}}$ whenever $A \in \mathcal{A}^{\mathrm{id}}$. Similarly, ϕ *preserves tripotents*

if $\phi(A) \in \mathcal{D}^{\mathrm{tr}}$ whenever $A \in \mathcal{A}^{\mathrm{tr}}$. We prove that any linear map that preserves idempotents also preserves tripotents. The converse holds for unital algebras and maps which preserve units. The large class of maps preserving idempotents was considered in [3].

We say that an element $A \in \mathcal{A}$ is *a rational convex combination of tripotents* if $A = \lambda_1 A_1 + \ldots + \lambda_n A_n$ with λ_i non-negative rational numbers and $A_i \in \mathcal{A}^{\mathrm{tr}}$, and $\lambda_1 + \ldots + \lambda_n = 1$. We say that A is *average* of tripotents if $A = (A_1 + \ldots + A_n)/n$ where $A_i \in \mathcal{A}^{\mathrm{tr}}$, $i = 1, 2, \ldots, n$. We denote the set of all averages of tripotents by $\mathrm{aver}\mathcal{A}^{\mathrm{tr}}$. Let $\mathcal{A}$ be *-algebra and let $\mathcal{A}^{\mathrm{sa}} = \{A \in \mathcal{A}:\ A = A^*\}$, $\mathcal{A}^{\mathrm{pr}} = \mathcal{A}^{\mathrm{id}} \cap \mathcal{A}^{\mathrm{sa}}$. For $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$ we describe the following sets: the set of all finite sums of tripotents, $\mathrm{conv}\mathcal{A}^{\mathrm{tr}}$ and $\mathrm{aver}\mathcal{A}^{\mathrm{tr}}$. In contrast with the set $\mathrm{conv}\mathcal{A}^{\mathrm{pr}}$, the sets $\mathrm{conv}\mathcal{A}^{\mathrm{id}}$ and $\mathrm{conv}\mathcal{A}^{\mathrm{tr}}$ are not closed. We show that every matrix $A \in \mathcal{A}$ with $\mathrm{Tr}A \in \mathbb{Z}$ is a finite sum of elements from $\mathcal{A}^{\mathrm{tr}}$. This result is similar to one of [4] which states that $A \in \mathcal{A}$ is a sum of idempotents if and only if $\mathrm{Tr}A \in \mathbb{Z}$ and $\mathrm{Tr}A \geqslant \mathrm{rank}A$.

Recall that finite sums of elements of $\mathcal{A}^{\mathrm{pr}}$, $\mathcal{A}^{\mathrm{id}}$ and $\mathcal{A}^{\mathrm{tr}} \cap \mathcal{A}^{\mathrm{sa}}$ were considered in [5], [4] and [6], respectively. Survey [7] contains results on linear combinations, sums, convex combinations and/or averages of operators from the classes of diagonal operators, unitary operators, isometries, projections, symmetries, idempotents and some other classes but does not contain results on tripotents. Thus our report supplements the results given in the survey.

We also give the new proof of rational trace matrix representations presented in Theorem 3.6 of [8]. Finally, we pose three open problems.

Problem 1. *Find the sufficient and necessary condition on $A, B \in \mathcal{A}^{\mathrm{tr}}$, such that $AB \in \mathcal{A}^{\mathrm{tr}}$.*