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Inhomogeneous compact extra dimensions and de Sitter cosmology

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Abstract In the framework of multidimensional $f(R)$ gravity, we study the possible metrics of compact extra dimensions assuming that our 4D space has the de Sitter metric. Manifolds described by such metrics could be formed at the inflationary and even higher energy scales. It is shown that in the presence of a scalar field, it is possible to obtain a variety of inhomogeneous metrics in the extra factor space $\mathbb{M}_2$. Each of these metrics leads to a certain value of the 4D cosmological constant Λ_4, and in particular, it is possible to obtain $\Lambda_4 = 0$, as is confirmed by numerically obtained solutions. A nontrivial scalar field distribution in the extra dimensions is an important feature of this family of metrics. The obtained solutions are shown to be stable under extra-dimensional perturbations.

1 Introduction

The idea of extra dimensions is widely used for explanation of a variety of phenomena, such as the physics beyond the Standard Model, cosmological scenarios including inflationary models and the origin of the dark components of the Universe, etc. [1–4]. Sometimes extra dimensions are endowed with scalar fields and antisymmetric form fields to stabilize their metric. There are models where the Casimir effect is taken into account [5,6]. Thus inclusion of extra dimensions is a promising background for the physics below ~ 10 TeV.

At the same time, it is usually assumed that the Universe has been nucleated due to quantum processes at very high energies [7–10]. The metric of our Universe and the fields inside the horizon experience strong quantum fluctuations that could affect their dynamics and their final states at low energies [11,12], including the shape of extra dimensions.

In this paper, we study the possible influence of a matter field on the metric of extra dimensions. Previous results concerning multidimensional gravity with 4D Minkowski factor space are published in [13–17], where the importance of inhomogeneous extra dimensions was discussed. In particular, the scalar field localization on deformed extra space. Here we extend the same idea to a more general case, the 4D de Sitter metric with an arbitrary Hubble parameter. Such a metric is approximately realized at the inflationary stage and can be valid up to Planck energies.

The models under consideration contain two extra dimensions that form a compact surface of rotation, which in the general case possesses a conical singularity at a particular point ("the second center"). The scalar field is to a large extent concentrated in a neighborhood of this point, showing a behavior similar to what is observed in many brane-world models. However, unlike such models, we assume that the size of extra dimensions is small enough to be invisible in modern accelerator experiments, i.e., we actually adhere to the Kaluza–Klein concept of extra dimensions. A narrower class of models are completely regular, however, it should be noted that their basic physical properties, including their 4D appearance, are almost indistinguishable from those of models with conical singularities.

Our study is based on multidimensional $f(R)$ gravity. The interest in $f(R)$ theories is motivated by inflationary scenarios starting with the work of Starobinsky [18]. Having been developed 40 years ago on the basis of 4D R^2 gravity, this model remains most promising up to now.

The Einstein gravity has been tested and appears to be valid at small energies. The energy between zero and 10 TeV has been studied by various experiments, including those at the LHC collider. As a result, a deviation from the standard theory of gravity was not found. At the same time, we do not know whether it is true at higher energies where the quantum corrections contribute significantly to a Lagrangian. More-

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over, the choice of the Einstein-Hilbert gravity is doubtful at sub-planckian energies – the smallness of quantum correction should be proved in this case.

Any combination of quantities invariant under the general coordinate transformations may be used in the theory if one keeps in mind two issues. Firstly, a theory must restore the Einstein-Hilbert action at low energies. Secondly, any gravitational action including the Einstein-Hilbert one is non-renormalizable and should be considered as an effective theory. The total uncertainty in the choice of the gravitational Lagrangian is a flaw which is party compensated by new opportunities provided.

The simplest extension of general relativity is the one containing a function of the Ricci scalar $f(R)$. Some viable $f(R)$ models in 4D space that satisfy the observational constraints have been proposed in [19–21]. Such modified gravity can be responsible for dark energy [22].

Stabilization of the extra space as a pure gravitational effect in $f(R)$ and more general multidimensional theories with maximally symmetric extra spaces has been studied in [23–25], as well as their ability to describe both early and late inflationary expansion [26–28]. In [15], it was shown that an $f(R)$ model with inhomogeneous extra space is compatible with 4D Minkowski or very weakly curved space-times.

The structure of this paper is as follows. In Sect. 2 we choose the metric and dimensionality of our manifold, the Lagrangian containing gravity with higher derivatives and a scalar field and derive the set of classical equations. In Sect. 3 we discuss the boundary conditions that are necessary in order to obtain a set of solutions and present a number of numerical solutions obtained under these conditions. Section 4 is devoted to a stability study for the obtained solutions. In Sect. 5 we discuss the 4D properties of these solutions. Conclusion are made in Sect. 6.

2 Basic equations

We will consider 6D metrics of the general form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu - e^{2\alpha(u)}du^2 - e^{2\beta(u)}d\varphi^2, \tag{1}$$

where u and φ are extra spatial coordinates, and $\varphi \in [0, 2\pi)$ is assumed to be a polar angle, while the 4D metric tensor $g_{\mu\nu}$ may depend on both 4D coordinates (making it possible to consider, for example, cosmological or static models) and the "radial" fifth coordinate u. The extra factor space $\mathbb{M}_2$ parametrized by (u, φ) is thus a surface of rotation, which can be compact if the circular radius $r(u) \equiv e^{\beta(u)}$ tends to zero at two boundary values of u. The 6-dimensional metric is chosen as the simplest nontrivial metric suitable for our purposes.

In such space-times, we consider a theory with the action

$$S = \int d^6x\sqrt{|g_6|}\left[\frac{m_D^4}{2} f(R) + \frac{1}{2} g^{AB}\phi_{,A}\phi_{,B} - V(\phi)\right], \quad A, B = 0, \dots, 5. \tag{2}$$

where $g_6 = \det(g_{AB})$, $f(R)$ and $V(\phi)$ are some functions (to be chosen later) of the 6D scalar curvature R and the scalar field ϕ, respectively. Variation of (2) with respect to ϕ and g^{AB} leads to the field equations

$$\Box\phi + V_\phi = 0, \quad \text{where } \Box\phi = \nabla_A\nabla^A\phi, \quad V_\phi = dV/d\phi, \tag{3}$$

$$-\frac{1}{2}\delta_A^B f(R) + \left[R_A^B + \nabla_A\nabla^B - \delta_A^B\Box\right] f_R = -\frac{1}{m_D^2} T_A^B, \quad f_R = df/dR, \tag{4}$$

and the stress-energy tensor of the scalar field $\phi = \phi(y)$ reads

$$T_A^B[\phi] = \phi_{,A}\phi^{,B} - \tfrac{1}{2}\delta_A^B\phi_{,C}\phi^{,C} + \delta_A^B V. \tag{5}$$

Before writing the particular equations to be solved, it is helpful to present brief expressions for the Ricci tensor components R_B^A assuming a general diagonal metric in arbitrary dimensions,

$$ds^2 = \sum_A \eta_A e^{2b_A(X)}(dx^A)^2, \quad A = 0, \dots, D-1, \tag{6}$$

where $b_A(X)$ are arbitrary functions of x^A, and $\eta_A = \pm 1$. Then for the diagonal components of R_B^A we have

$$R_{\underline{M}}^{\underline{M}} = \sum_{A\neq M}\left(\Box_A b_M + \Box_M b_A - \eta_M e^{-2b_M} b_{A,M}\sum_{B\neq A,M} b_{B,M}\right), \tag{7}$$

where no summing is assumed over an underlined index,

$$b_{A,B} = \partial_B b_A, \qquad \Box_M f(X) = \frac{1}{\sqrt{g}}\partial_M\left(\sqrt{g}g^{\underline{MM}}\partial_M f(X)\right)$$

for an arbitrary function $f(X)$, and $g = |\det(g_{MN})|$. The off-diagonal components of the Ricci tensor are more conveniently written with lower indices, namely,

$$R_{MN} = \sum_{A\neq M,N}\left(b_{A,MN} + b_{A,M}b_{A,N} - b_{M,N}b_{A,M} - b_{N,M}b_{A,N}\right). \tag{8}$$

In the present study, we consider a cosmological (de Sitter) metric in 4D space-time and the extra dimensions using the Gaussian u coordinate (length along the coordinate axis of $x^4 = u$), so that the metric (1) takes the form

$$ds^2 = dt^2 - e^{2Ht}\delta_{ij}dx^i dx^j - du^2 - r(u)^2 d\varphi^2, \quad i, j = 1, 2, 3, \tag{9}$$

where $H = \text{const}$ is the Hubble constant. Accordingly, in terms of (6) we now have

$$b_0 = 0, \quad b_i = Ht, \quad b_4 = 0, \quad b_5 = \ln r(u), \tag{10}$$

and the expressions for R^A_B are greatly simplified: the only nonzero components of R^A_B and the scalar R are (the prime stands for d/du)

$$R^t_t = R^i_i = 3H^2, \qquad R^u_u = R^\varphi_\varphi = -\frac{r''}{r}, \tag{11}$$

$$R = 12H^2 - \frac{2r''}{r}, \tag{12}$$

Assuming $\phi = \phi(u)$, Eq. (3) and noncoinciding equations from (4) may be written as

$$\phi'' + \phi'\frac{r'}{r} = V_\phi, \tag{13}$$

$$-\frac{1}{2}f(R) + 3H^2 f_R + f''_R + \frac{r'}{r}f'_R = m_D^{-2}\left(-\frac{\phi'^2}{2} - V\right), \tag{14}$$

$$-\frac{1}{2}f(R) - \frac{r''}{r}f_R + \frac{r'}{r}f'_R = m_D^{-2}\left(\frac{\phi'^2}{2} - V\right), \tag{15}$$

$$-\frac{1}{2}f(R) - \frac{r''}{r}f_R + f''_R = m_D^{-2}\left(-\frac{\phi'^2}{2} - V\right), \tag{16}$$

where $f'_R = df_R/du$, etc.

3 Models with inhomogeneous extra space

3.1 Equations and boundary conditions

In our calculations, in order to avoid dealing with third- and fourth-order derivatives, it will be convenient, within the same set of equations, to use the Ricci scalar $R(u)$ as one more unknown function. in addition to $r(u)$ and $\phi(u)$. As three independent equations for this system, we can take, for example, (13), (12) and a combination of (16) and (12):

$$\phi'' + \phi'\frac{r'}{r} = V_\phi, \tag{17}$$

$$R = 12H^2 - 2\frac{r''}{r}, \tag{18}$$

$$-\frac{1}{2}f(R) + f''_R + \left(\frac{R}{2} - 6H^2\right)f_R = m_D^{-2}\left(-\frac{\phi'^2}{2} - V\right), \tag{19}$$

resolved with respect to the higher derivatives ϕ'', r'', R''. We will also use the combination (15) + (16) − (14) − $f_R\cdot$(12), which leads to

$$-\frac{f(R)}{2} + (R - 15H^2)f_R = m_D^{-2}\left(\frac{\phi'^2}{2} - V\right), \tag{20}$$

and contains lower-order derivatives, as a restriction on the solutions of the coupled second order differential equations.

As boundary conditions, we use the requirement of $u = 0$ being a regular center on the (u, ϕ) surface and the corresponding requirements for ϕ and R:

$$r(0) = 0, \qquad r'(0) = 1 \tag{21}$$

$$\phi(0) = \phi_0, \quad \phi'(0) = 0, \quad R(0) = R_0, \tag{22}$$

where all quantities with the index 0 are constants. These initial parameters are related by the condition following from Eq. (20),

$$-\frac{f(R_0)}{2} + (R_0 - 15H^2)f_R(R_0) = -m_D^{-2}V(\phi_0). \tag{23}$$

This means that for given $f(R)$ the quantity R_0 is related to H and ϕ_0, so that any two of these three parameters are free.

We also have from (19) and (21) for $u \to 0$

$$R'(0) = 0, \quad \lim_{u\to 0}\frac{\phi'}{r} = \phi''_0, \quad r''(0) = 0. \tag{24}$$

We will seek solutions for $u > 0$ in which the circular radius $r \to 0$ at some $u = u_{\max}$, which provides compactness of the extra space parametrized by u and φ.

The total energy on the (u, φ) surface for a specific solution is

$$\rho(\phi_0) = 2\pi\int_0^{u_{\max}} du\, r(u)\left[\frac{\phi'^2}{2} + V\right], \tag{25}$$

it may be interpreted as the energy density of the scalar field stored in the extra dimensions. This energy density depends on the parameter ϕ_0 expressing the boundary scalar field value in $\mathbb{M}_2$.

3.2 Pure gravity

Let us first consider the case $\phi = \phi_0$, in which the scalar field is distributed uniformly in space and does not depend on time, and the equations can be solved analytically. In this case the scalar field potential is constant, $V = V_0 = V(\phi_0)$.

Equation (20) in this case leads to

$$\frac{1}{2}f(R) + (15H^2 - R)f_R = 0, \tag{26}$$

which means that also $R = R_0 = \text{const}$, hence the 2D extra space is maximally symmetric for any given $f(R)$, and from (12) it follows

$$\frac{r''}{r} = 6H^2 - \frac{R_0}{2}. \tag{27}$$

Now, the difference of Eqs. (14)–(16) reduces to

$$3H^2 f_R + \frac{r'}{r}f'_R + \frac{r''}{r}f_R = 0. \tag{28}$$

If we assume that $f_R(R_0) \neq 0$, and also notice that $f'_R = 0$ due to $R = \text{const}$, Eq. (28) reduces to

$$\frac{r''}{r} = -3H^2, \tag{29}$$

and we also have

$$R = R_0 = 18H^2, \qquad \frac{f}{f_R} = 6H^2. \tag{30}$$

Under our conditions at $u = 0$, the solution of (29) reads

$$r = \frac{1}{\sqrt{3}H} \sin\left(\sqrt{3}H\,u\right), \tag{31}$$

and the metric has the form

$$ds^2 = dt^2 - \mathrm{e}^{2Ht}\delta_{ij}\,dx^i dx^j - \frac{1}{3H^2}(d\theta^2 + \sin^2\theta d\varphi^2),$$
$$\theta = \sqrt{3}H\,u, \tag{32}$$

the extra space being a 2-sphere.

We have shown that with any choice of the initial function $f(R)$ the only solution with the metric (9) for pure gravity (or with a constant scalar field) corresponds to a spherical extra space.

This result deserves attention at high energies where the Hubble parameter is large enough. A common starting point is to fix the properties of extra dimensions, their size in particular. These properties depend on the Lagrangian parameters, including the topology of extra space, but do not depend on our 4D metric. According to (30), (32), the state of affairs is different at least for the class of models containing all sorts of $f(R)$. The extra space is inevitably maximally symmetric, and its radius is stiffly related to the Hubble constant, $r = \sqrt{3}H^{-1}$.

In particular, if we choose $f(R) = aR^2 + R + (c - V)$ (see Eq. (34) further on), Eq. (26) gives the following relation between the parameters:

$$108aH^4 + 12H^2 + c - V = 0 \quad \text{or}$$
$$H^2 = \frac{-b \pm \sqrt{b^2 - 3a(c - V)}}{18a} = \frac{R_0}{18}. \tag{33}$$

The possibility of complex roots in this expression shows that not any choice of the parameters leads to a valid solution, since obviously H^2 must be real.

At the inflationary stage of the Universe evolution, r is close to 10^{-27} cm, and it is about 10^{-33} cm at the Planck scale. However, if we consider very small H, for example, corresponding to the present epoch, the Ricci scalar of extra dimensions will be close to zero, meaning their huge size, incompatible with observations.

To avoid such a strong constraint, one can add matter fields (a scalar one in our case) or/and widen the Lagrangian by adding other invariants like the Ricci tensor squared, making it possible to obtain inhomogeneous extra dimensions. A detailed discussion on the basis of other extra space metrics can be found in [29,30].

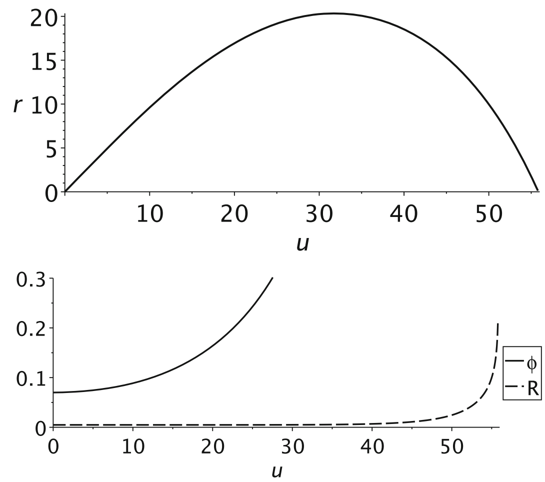

Fig. 1 The extra space metric function $r(u)$, the Ricci scalar $R(u)$ of 6D space and the scalar field $\phi(u)$ for $f(R) = aR^2 + bR + c$ and $V(\phi) = (m^2/2)\phi^2$ (units $m_D = 1$). The parameter values are $m = 0.1,\ b = 1,\ a = -100,\ c = -0.0021$. Additional conditions are: $\phi_0 = 0.07,\ H = 0$. The value of $R(0)$ follows from Eq. (23). Here $R(0) \simeq 0.00485$

In the next section, we show that smallness of the Hubble parameter does not mean smallness of the extra-dimensional Ricci scalar. The size of extra space could be small enough not to be in conflict with observations.

3.3 Numerical solutions: conical singularities

To obtain examples of numerical solutions of interest, let us choose the following functions in the action (2):

$$f(R) = aR^2 + bR + c, \qquad a, b, c = \text{const},$$
$$V(\phi) = \tfrac{m^2}{2}\,\phi^2, \qquad m = \text{const}. \tag{34}$$

The figures below present solutions for different values of the parameters.

The numerical results are presented in Figs. 1, 2, 3, 4, 5 and 6. A variation of the parameter values can lead to qualitatively different metrics in $\mathbb{M}_2$. For example, the curvature may change its sign, as is seen from Fig. 5.

The point $u = 0$ is a regular center by our boundary conditions. One can notice that in general our solutions contain conical singularities at $u = u_{\max}$. Indeed, as is seen from Fig. 4, left panel, where the first derivative $r'(u_{\max}) \neq -1$, the curvature R is infinite at the point $u = u_{\max}$. A similar thing happens with a usual cone as a 2D surface in 3D flat space: in that case, the metric on the conical surface is everywhere flat, hence the curvature tensor and all its invariants are zero, while at the top the curvature is infinite. Our case is quite similar, the only difference is that the curvature around the top ($u = u_{\max}$) is not zero but finite. Indeed, the limiting

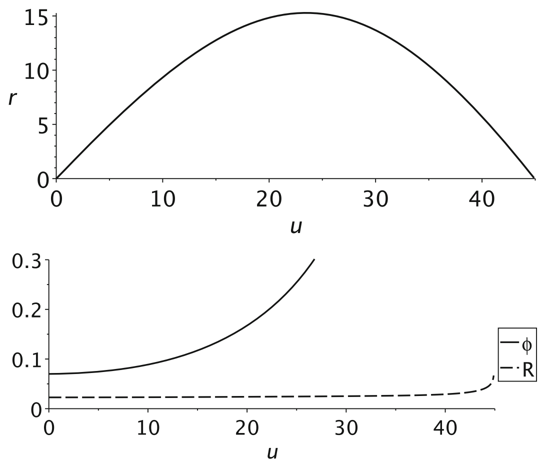

Fig. 2 The same as in Fig. 1 but $H = 0.035$, $R(0) \simeq 0.0227767$

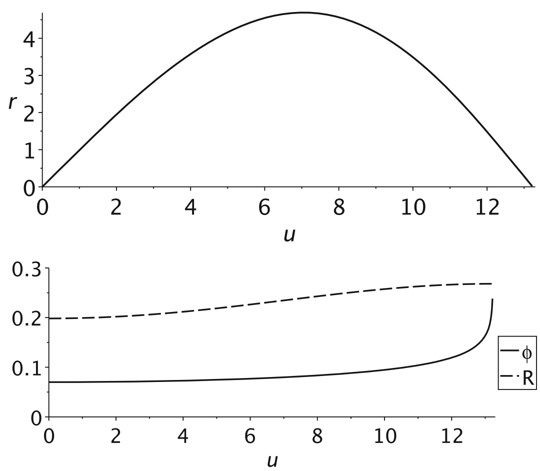

Fig. 3 The same as in Fig. 1 but $H = 0.1$, $R(0) \simeq 0.198327$

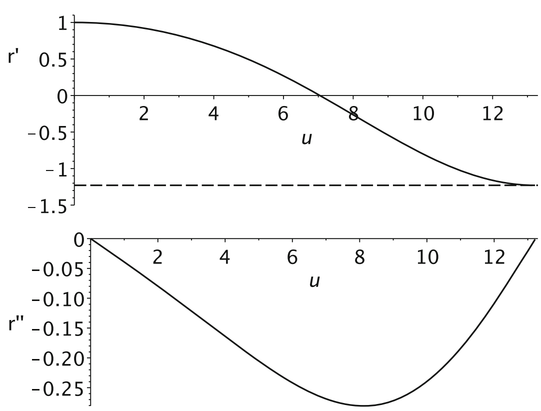

Fig. 4 Additional information about the case discussed in Fig. 3

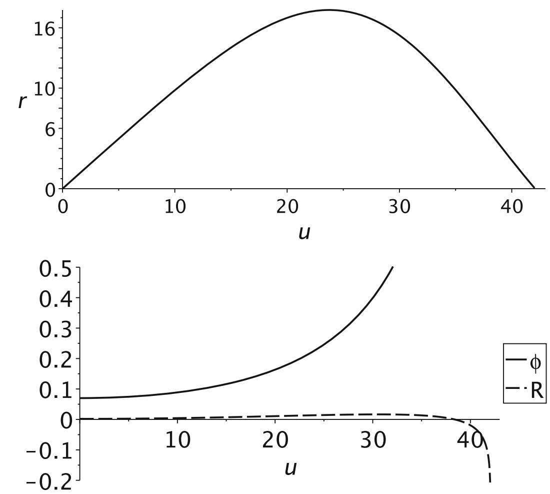

Fig. 5 The parameter values: $m = 0.1$, $b = 1$, $a = 10$, $c = 0.0021$. Additional conditions: $\phi_0 = 0.07$, $H = 0$, $R(0) \simeq 0.00189$ follows from Eq. (23)

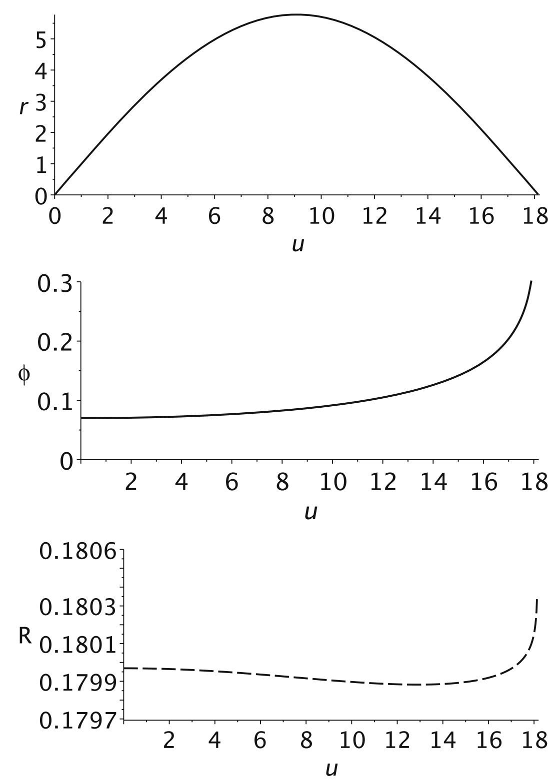

Fig. 6 Solution without a conical singularity, $r'(u_{\max}) = -1$. The extra space metric function $r(u)$, the Ricci scalar $R(u)$ of 6D space and the scalar field $\phi(u)$ for $f(R) = aR^2 + bR + c$ and $V(\phi) = (m^2/2)\phi^2$ (units $m_D = 1$). The parameter values are $m = 0.1$, $b = 1$, $a = -10.9$, $c = -0.0021$. Additional conditions are: $\phi_0 = 0.07$, $H = 0.1$ The value of $R(0)$ follows from Eq. (23). Here $R(0) \simeq 0.179969$, $r'(u_{\max}) = -1$

values of R as $u \to u_{\max}$ are finite, see Fig. 3, right panel. Moreover, the l.h.s. in Eq. (20) is finite in the same limit, so finite is also the r.h.s. characterizing the scalar field. Such metrics are used in the extra-dimensional context, see, e.g., [31] and references therein.

It is also of interest to check whether or not there are solutions free from such a singularity, other than the evident maximally symmetric metric, $R = \text{const}$. The question is: can we find a nontrivial nonsingular solution? The answer is yes, and it is presented in Fig. 6. To be sure that this is not a numerical effect, we shifted the value $\phi_0 = 0.07$ in both directions and accordingly obtained $r'(u_{\max}) < -1$ and $r'(u_{\max}) > -1$.

In this solution with a regular metric, as has been verified by our numerical calculations, the curvature R and the scalar field are also finite, so such models are completely regular. However, models with conical singularities do not differ too substantially from these regular models in their physical properties.

A few general remarks on possible singularities in extra dimensions. Evidently, the classical equations written above are invalid at energy scales larger than the Planck scale. For 4D physics the corresponding length scale is about $l_4 = 1/m_4 \sim 10^{-33}$ cm. At scales near l_4 and smaller, quantum fluctuations are strong, and any solution to the classical equations is invalid. In a D-dimensional world a similar scale is $l_D = 1/m_D = 1$ by our convention. There are two consequences if we intend to work on the classical level: (i) the size of extra dimensions must be much larger than unity; (ii) any peculiarities with the coordinate interval $\delta u \simeq l_D = 1$ are meaningless without thorough analysis of quantum effects. In particular, if a classical solution contains a singularity, as it happens in most of our solutions at $u = u_{\max}$, it is reasonable to suppose that such a singularity is smoothed by quantum effects and should not be taken seriously. The stability of such configurations based on the methods discussed in [32] is considered below.

One can also notice that the size r of the internal space presented in the figures is of the order of $\sim 10 = 10/m_D$. That means that quantum fluctuations are suppressed, and our classical equations are applicable. Another constraint relates to the Newton law of gravity at low energies. To clarify the question, let us follow the paper [33]. It was reasonably mentioned there that two point particles separated by a distance l feel a gravitational potential $V(l) \propto 1/l^{D-3}$ that is dictated by the Gauss theorem in D-dimensional space. The experimental result is $V(l) \propto 1/l$ if $l > 10^{-2}$ cm which is true for our 4-dim space (see, e.g., [34]). We are dealing with $D = 6$ and hence $V(l) \propto 1/l^3$ if $l \ll r$. Now we cite [33]: "... if the masses are placed at distances $l \gg r$,their gravitational flux lines cannot continue to penetrate in the extra dimensions, and the usual $1/l$ potential is obtained." The conclusion is that the inequality

$$r \simeq 10/m_D \ll 10^{-2}\,\text{cm} \tag{35}$$

should hold in order to avoid problems for our model. Our numerical calculations presented in Sect. 5 give $m_D \sim m_4/20$, where m_4 is the Planck mass, see Eq. (46) and above. Hence, $r \simeq 10/m_D \simeq 200/m_4 \sim 10^{-31}$ cm, and the inequality (35) holds with great accuracy.

4 Stability

A general analysis of the stability is a very complicated task. Here we show that our solutions are stable relative to perturbations homogeneous in 4D space, depending only on t and u. More definitely, we study the evolution of metric (9) with the small deviations

$$\begin{aligned}
&\phi(t,u) = \phi_c(u) + \delta\phi(t,u),\\
&r(t,u) = r_c(u) + \delta r(t,u), \quad R(t,u) = R_c(u) + \delta R(t,u),\\
&\delta\phi \ll \phi_c, \quad \delta r \ll r_c, \quad \delta H \ll H_c, \quad \delta R \ll R_c.
\end{aligned} \tag{36}$$

Here and below the index "c" relates to the static solutions. We substitute them into the classical equations and show that there are no growing modes. There are three unknown functions $\delta r(t,u)$, $\delta\phi(t,u)$, $\delta R(t,u)$, so that we need three classical equations linearized with respect to these quantities, which may be written in the form

$$\begin{aligned}
&-\Box_2\delta\phi - 3H_c\delta\dot\phi + \frac{1}{r_c}r_c'\delta\phi' + \frac{1}{r_c}\phi_c'\delta r' - \frac{1}{r_c^2}\phi_c' r_c'\,\delta r\\
&-m^2\delta\phi = 0,
\end{aligned} \tag{37}$$

$$\delta R = \frac{2}{r_c}\Box_2\delta r + 6\frac{H_c}{r_c}\delta\dot r + \frac{2}{r_c^2}r_c''\,\delta r, \tag{38}$$

$$\begin{aligned}
&a\Bigg[\frac{4R_c}{r_c}\Box_2\delta r - 10\Box_2\delta R + 12\frac{H_c R_c}{r_c}\delta\dot r + \frac{10R_c'}{r_c}\delta r'\\
&-30H_c\delta\dot R + \frac{10r_c'}{r_c}\delta r' + \frac{1}{r_c^2}\left(4r_c''R_c - 10r_c'R_c'\right)\delta r\\
&+\frac{1}{r_c^2}\left(4r_c''R_c - 10r_c'R_c'\right)\delta r + \left(24H_c^2 - \frac{4r_c''}{r_c} - 6R_c\right)\delta R\Bigg]\\
&+b\left[\frac{2}{r_c}\Box_2\delta r + \frac{6H_c}{r_c}\delta\dot r + \frac{2r_c''}{r_c^2}\delta r - 3\delta R\right] + 6m^2\phi_c\delta\phi\\
&+4\phi_c'\delta\phi' = 0,
\end{aligned} \tag{39}$$

where the dot denotes $\partial/\partial t$, $\Box_2 = \partial^2_{tt} - \partial^2_{uu}$, a and b are coefficients from (34).

In fact, we arbitrarily choose initial deviations from the static field configuration and perform numerical analysis. We show that the solutions to the classical equations have no growing modes and relax (due to friction) to the static homogeneous configuration. The friction is supplied by the

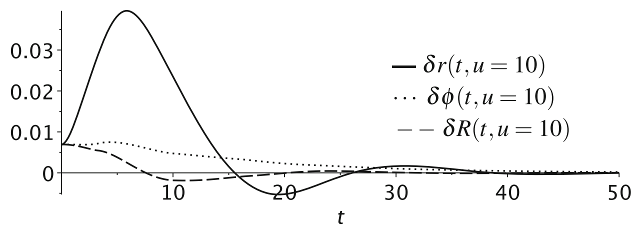

Fig. 7 Time dependence of the perturbations to the solution presented in Fig. 3

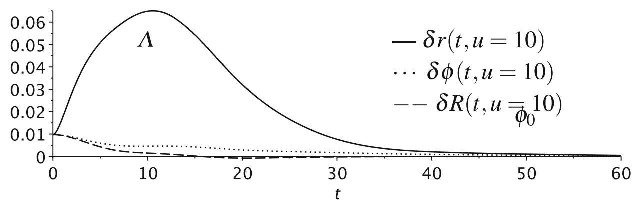

Fig. 8 Time dependence of the perturbations to the solution presented in Fig. 6 (values at $u = 10$)

nonzero Hubble parameter H and (as confirmed by calculations) is absent if $H = 0$.

For the perturbation equations we have used the boundary conditions

$$\delta r(t,0) = \delta\phi(t,0) = \delta R(t,0) = \delta r(t,u_{\max}) = \delta\phi(t,u_{\max}) = \delta R(t,u_{\max}) = 0 \tag{40}$$

and the initial conditions

$$\delta r(0,u) = \delta\phi(0,u) = \delta R(0,u) = 0.01\sin^2(u\pi/u_{\max})$$
$$\delta\dot r(0,u) = \delta\dot\phi(0,u) = \delta R(0,u) = 0. \tag{41}$$

The nonsingular solutions presented in Figs. 3, 6 are also stable. The time behavior of their fluctuations under the same boundary and initial conditions is shown in Figs. 7 and 8.

5 Reduction to 4 dimensions and low energies

The study made above reveals that static inhomogeneous extra dimensions could exist. For given $f(R)$ and $V(\phi)$, their shape and energy density also depend on the initial (random)l value ϕ_0. Let us briefly discuss the observational manifestations of such solutions. As was shown in [16], there exist such extra-dimensional metrics that lead to the 4D cosmological constant Λ_4 arbitrarily close to zero. This effect is a result of interference between the gravitational and scalar field parts of the Lagrangian. The result obtained in [15] is based on approximate equations. In this section, we use the exact set of equations derived from the metric (9) and the action (2).

The quantity Λ_4 can be found by integrating out the internal coordinates in the action (2). We know that the Hubble parameter is at present almost zero as compared to the possible extra-dimensional scales. Hence let us put $H = 0$. In this case R_0 and ϕ_0 are related by (23), and $\Lambda_4(\phi_0)$ is a function of the unique argument ϕ_0. It remains to find this function and its zero points.

Let us consider static solutions found above and use the smallness of H as compared to the extra space Ricci scalar $R_2 = -2r''/r$, see (12). After the decomposition $f(R) = f(R_4 + R_2) \simeq f(R_2) + f_R(R_2)R_4$ we obtain

$$S = 2\pi\int d^4x\sqrt{g_4}\int_0^{u_{\max}} du\, r(u)\left[\frac{1}{2}f_R(R_2)R_4 + \frac{1}{2}f(R_2) - \frac{1}{2}\phi_{,u}^2 - V\big(\phi(u)\big)\right] \tag{42}$$

Comparing this expression with the standard form of the 4D action

$$S_4 = \int d^4x\sqrt{g_4}\left[\frac{1}{2}m_4^2R_4 - \Lambda(\phi_0)\right] \tag{43}$$

we get the observed Planck mass

$$m_4^2(\phi_0) = 2\pi\int_0^{u_{\max}} du\, r(u) f_R(R_2) \tag{44}$$

in the units $m_D = 1$, and

$$\Lambda_4(\phi_0) = -2\pi\int_0^{u_{\max}} du\, r(u)\left[\frac{1}{2}f(R_2) - \frac{1}{2}\phi_{,u}^2 - V\big(\phi(u)\big)\right]. \tag{45}$$

The above discussion used the units $m_D = 1$. Now we can express all quantities in terms of the usual Planck units in 4 dimensions. One can easily find the value $\phi_0 \simeq 0.00073$ that corresponds to the zero value of Λ_4 from Fig. 9, upper panel. The lower panel gives the 4D Planck mass $m_4 \simeq 21.23$ in units $m_D = 1$ at this value of ϕ_0.

Now everything is prepared for calculating the energy density (25) of the scalar field distributed in the extra dimensions. Numerical integration in u gives

$$\rho(\phi_0) \simeq 1.03\, m_D^4 \simeq 1.03(m_4/21.23)^4 \simeq 0.0000051\, m_4^4. \tag{46}$$

The scalar field density stored in the extra space is neutralized by the gravitational term $f(R_2)$, so that the cosmological constant (45) is arbitrarily small for the specific solution to Eqs. (3.1). Such a solution certainly exists due to the continuity of the set of solutions. The quantity ϕ_0 was used as an additional parameter, see (22), to find a specific distribution $\phi(u)$. We see that the set of the scalar field distributions is parametrized by the boundary value ϕ_0. The same can be said about its energy–momentum tensor $T_{AB}(\phi_0)$.

We conclude that the scalar field can help to stabilize compact extra space. The size of latter is smaller than 10^{-18}cm in spite of zero value of the Hubble parameter H. Therefore,

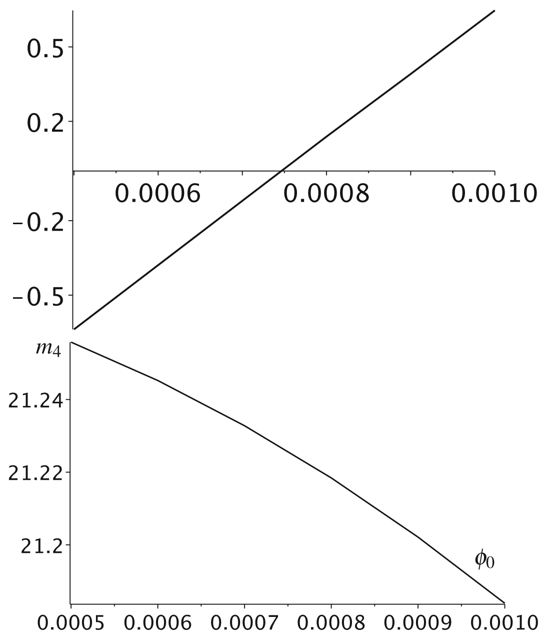

Fig. 9 The dependence of Λ_4 and m_4 on ϕ_0 for $f(R) = aR^2+bR+c$ and $m_D = 1,\ m = 0.1,\ b = 1,\ a = -100,\ c = -0.0021,\ V(\phi) = (m^2/2)\,\phi^2,\ \ H = 0$. $R(0)$ is the positive root of Eq. (23)

there are no contradictions with r^{-2} law of the gravity and the experimental limit found at the LHC collider.

6 Conclusion

In this paper, we have studied the metric of compact extra dimensions at high energy density of the Universe where the 4D space-time is described by the de Sitter metric with an arbitrary value of the Hubble parameter. Numerical solutions to the full set of the classical equations have been analyzed. It is shown that in a theory with given $f(R)$, inclusion of a scalar field leads to a continuous set of static extra space metrics. The properties of such inhomogeneous metrics depend on the scalar field distribution in the extra dimensions.

The extra-dimensional metrics represent a set of the cardinality of continuum even if the Largargian parameters are fixed. These metrics are stable under fluctuations in the extra space, as was shown in Sect. 4.

It has been shown that the form of the stationary extra metric depends also on the value of the Hubble parameter H. The latter slowly changes with time in the early Universe. Therefore, we can approximate it as a constant and apply the obtained results under the assumption $H = \text{const}$. As a result, the extra space metric and the scalar field distribution are changing during the inflationary period.

Our analysis of the classical equations indicates that in the absence of matter fields only a maximally symmetric (spherical) metric in $\mathbb{M}_2$ is possible. This analytic result shows that the Ricci scalar of the extra space is unambiguously related to the Hubble parameter, and hence the extra-dimensional radius is slowly varying with time at the inflationary stage, and a similar picture might be expected for the present epoch. However, at present this radius has to be unacceptably large, this shortcoming being cured by invoking a scalar field, which makes its role very important.

At high energy scales, quantum fluctuations perturb both the metric and the scalar field. It is widely assumed that our manifold was born at sub-Planckian energies, so that the scalar field randomly varies at those times. Let us estimate the conditions at which the fluctuations cannot disturb the extra space metric. Fluctuation are able to produce Kaluza–Klein excitations if the extra-dimensional scale l_e is larger then the fluctuation wavelength $1/k$, where k is magnitude of the its wave vector. For relativistic matter, the energy density $\rho \sim k^4$ while the Hubble parameter is $H \sim \sqrt{\rho/m_4^2}$. These estimates constrain the extra-dimensional scale as $l_e \lesssim 1/\sqrt{Hm_4}$.

This inequality allows us to impose a restriction on the extra space metric which is much stronger than those obtainable in collider experiments. Indeed, the cosmological constant and the gravitational constant do not vary within the present horizon. This means that fluctuations should be damped at the inflationary stage where $H = H_I \simeq 10^{13}$ GeV, so that the scale l_e of the extra dimensions should be smaller than $1/\sqrt{H_I m_4}$. We conclude that the averaged size of the compact extra dimensions should be smaller than $\sim 10^{-28}$ cm. This limit confirms those obtained in [17], where it was shown that the slow roll motion of the inflaton is forbidden if $l_e > H^{-1}$. Recall that the collider limit is $l_e < 10^{-18}$ cm, which is 10 orders of magnitude weaker than our prediction.

In conclusion, we would like to mention one more application of the idea of inhomogeneous extra dimensions. Consider, instead of (9), the 6D metric

$$ds^2 = e^{2\gamma(u)}\eta_{\mu\nu}dx^\mu dx^\nu - du^2 - r(u)^2 d\varphi^2, \qquad (47)$$

where $\eta_{\mu\nu}$ is the 4D Minkowski metric, and the metric in $\mathbb{M}_2$ is the same as in (9), but in terms of the metric (6) we now have

$$b_\mu = \gamma(u), \quad b_4 = 0, \quad b_5 = \ln r(u). \qquad (48)$$

Using the expressions (7) and (8), it is then straightforward to derive the explicit form of field equations for the unknowns $\gamma(u)$, $r(u)$ and $\phi(u)$. A tentative study has shown that there exist solutions with u-dependence of the circular radius r somewhat similar to that shown in Fig. 1 under boundary conditions similar to (21), (22), even if the scalar field is absent. If the size of $\mathbb{M}_2$ is large enough and with proper dependences $r(u)$ and $\phi(u)$, the solutions can admit interpretations in terms of the brane world concept, somewhat similar to [35,36]. Unlike solutions with the metric (9), mostly intended for the early Universe, those with (47) are able to describe the present-day universe with very small 4D curvature, and a study of their possible properties and applications is in progress.

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