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Projection methods for computation of spectral characteristics of weakly guiding optical waveguides

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The original problem on surface and leaky eigenmodes of a weakly guiding step-index optical waveguide is considered. The original problem is reduced to a nonlinear spectral problem for the set of weakly singular boundary integral equations. We approximate the integral operator by collocation and Galerkin methods. Their convergence and quality are proved by numerical experiments.

1 Introduction

Many different numerical techniques are applied for computing eigenmodes of dielectric waveguides [1, 2]; namely, Finite-element, Finite-difference, beam propagation, and spline collocation methods, as well as multidomain spectral approach. Often the authors concentrate on algorithm's features and physical interpretation of the numerical results rather than on fundamental mathematical aspects including the existence, properties, and distribution of the spectra on the complex plane. This study develops a new approach to mathematical and numerical analysis of dielectric waveguides based on the methods of spectral theory of operator-valued functions and integral equations [3]–[5].

We consider the spectral problem on surface and leaky eigenmodes of a weakly guiding step-index optical waveguide. The statement of the problem and its reduction to nonlinear spectral problem with Fredholm integral operator are given in [6]. The convergence of the Galerkin method for numerical solution of this problem was proved theoretically in [7]. For numerical solution of this nonlinear spectral problem in this study we propose the collocation method. We realized this method and the Galerkin method practically, computed some examples, and compared these two methods. The collocation method demonstrated better speed of convergence.

2 Statement of the problem: geometry and relations

Let the three-dimensional space be occupied by an isotropic source-free medium, and let the refractive index be prescribed as a positive real-valued function $n = n(x)$ independent of the longitudinal coordinate and equal to a constant $n_\infty > 0$ outside a cylinder. We consider the generalized natural (surface and leaky) modes of a step-index optical fiber and suppose that the refractive index is equal to a constant $n_+ > n_\infty$ inside the cylinder. The axis of the cylinder is parallel to the longitudinal coordinate and its cross section is a bounded domain Ω with a twice continuously differentiable boundary Γ (see Fig. 1). The domain Ω is a subset of a circle with radius R_0. Denote by Ω_∞ the unbounded domain $\Omega_\infty = \mathbb{R}^2 \setminus \overline{\Omega}$.

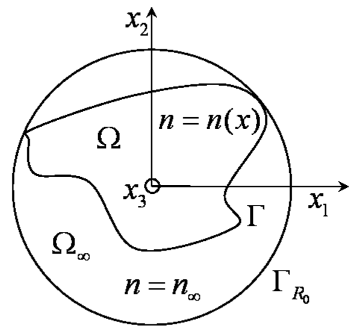

Figure 1: Cross section of the waveguide

We consider a weakly guiding optical waveguide. In this case function n also satisfies the following condition: $n_+ \approx n_\infty$. The modal problem can be formulated [6] as a scalar eigenvalue problem for the Helmholtz equation:

$$\Delta u + \chi_+^2(\beta)u = 0, \quad x \in \Omega, \tag{1}$$

$$\Delta u + \chi_\infty^2(\beta)u = 0, \quad x \in \Omega_\infty, \tag{2}$$

where

$$\chi_+ = \sqrt{k^2 n_+^2 - \beta^2}, \quad \chi_\infty = \sqrt{k^2 n_\infty^2 - \beta^2},$$

$k^2 = \omega^2 \varepsilon_0 \mu_0$; ω is a given radian frequency and ε_0, μ_0 are the free-space dielectric and magnetic constants, respectively; β is an unknown complex propagation constant (eigenvalue). Eigenfunction u also has to satisfy the conjugation conditions

$$u^+ = u^-, \quad \frac{\partial u^+}{\partial \nu} = \frac{\partial u^-}{\partial \nu}, \quad x \in \Gamma, \tag{3}$$

and the partial condition at infinity

$$u(r, \varphi) = \sum_{l=-\infty}^{\infty} a_l H_l^{(1)}(\chi_\infty r) \exp(il\varphi), \quad |x| \geq R_0. \tag{4}$$

Here ν is the normal derivative; $H_l^{(1)}$ is the Hankel function of the first kind and index l; (r, φ) are the polar coordinates of the point x. The series in (4) should converge uniformly and absolutely.

3 Nonlinear spectral problem

If function u is an eigenfunction of problem (1)–(4) corresponding to a eigenvalue β, then the following representations are valid:

$$u(x) = \int_\Gamma \Phi_+(\beta; x, y) f_+(y) dl(y), \quad x \in \Omega, \tag{5}$$

$$u(x) = \int_\Gamma \Phi_\infty(\beta; x, y) f_\infty(y) dl(y), \quad x \in \Omega_\infty, \tag{6}$$

where $\Phi(\beta; x, y) = i/4 H_0^{(1)}(\chi(\beta)|x-y|)$. The right sides of these representations are the simple layer potentials.

We use the properties of the simple layer potentials and reduce the original problem to nonlinear spectral problem:

$$\int_\Gamma \Phi_+(\beta; x, y) f_+(y) dl(y) - \int_\Gamma \Phi_\infty(\beta; x, y) f_\infty(y) dl(y) = 0, \ x \in \Gamma, \tag{7}$$

$$\frac{1}{2} f_+(x) + \int_\Gamma \frac{\partial \Phi_+(\beta; x, y)}{\partial \nu(x)} f_+(y) dl(y) + \frac{1}{2} f_\infty(x) - \int_\Gamma \frac{\partial \Phi_\infty(\beta; x, y)}{\partial \nu(x)} f_\infty(y) dl(y), \ x \in \Gamma. \tag{8}$$

It was proved [6] that for each β the integral operator of the problem (7), (8) is Fredholm, and also that the original problem (1)–(4) is spectrally equivalent to the problem (7), (8).

4 Galerkin method

We approximate the eigenfunctions by truncated Fourier series. The set of linear algebraic equations (7), (8) of the Galerkin method looks as follows:

$$\sum_{k=-N}^{N} \alpha_k^{(1)} \lambda_{k,l} + \frac{1}{4\pi^2} \sum_{k=-N}^{N} \alpha_k^{(1)} \int_0^{2\pi}\int_0^{2\pi} h^{(1,1)} e^{ik\tau} e^{-ilt} d\tau dt + \frac{1}{4\pi^2} \sum_{k=-N}^{N} \alpha_k^{(2)} \int_0^{2\pi}\int_0^{2\pi} h^{(1,2)} e^{ik\tau} e^{-ilt} d\tau dt = 0,$$
$$k, l = -N, \dots, 0, \dots, N, \tag{9}$$

$$\sum_{k=-N}^{N} \alpha_k^{(1)} I_{k,l} + \frac{1}{4\pi^2} \sum_{k=-N}^{N} \alpha_k^{(1)} \int_0^{2\pi}\int_0^{2\pi} h^{(2,1)} e^{ik\tau} e^{-ilt} d\tau dt + \frac{1}{4\pi^2} \sum_{k=-N}^{N} \alpha_k^{(2)} \int_0^{2\pi}\int_0^{2\pi} h^{(2,2)} e^{ik\tau} e^{-ilt} d\tau dt = 0,$$
$$k, l = -N, \dots, 0, \dots, N, \tag{10}$$

where

$$h^{(1,1)}(\beta; t, \tau) = \pi\big[\Phi_+(\beta; x(t), y(\tau)) + \Phi_\infty(\beta; x(t), y(\tau))\big] + \ln|\sin\frac{t-\tau}{2}|,$$

$$h^{(1,2)}(\beta; t, \tau) = \pi\big[\Phi_+(\beta; x(t), y(\tau)) + \Phi_\infty(\beta; x(t), y(\tau))\big]|r'(\tau)|,$$

$$h^{(2,1)}(\beta; t, \tau) = 2\pi\left[\frac{\partial \Phi_+(\beta; x(t), y(\tau))}{\partial \nu(x(t))} + \frac{\partial \Phi_\infty(\beta; x(t), y(\tau))}{\partial \nu(x(t))}\right],$$

$$h^{(2,2)}(\beta; t, \tau) = 2\pi\left[\frac{\partial \Phi_+(\beta; x(t), y(\tau))}{\partial \nu(x(t))} - \frac{\partial \Phi_\infty(\beta; x(t), y(\tau))}{\partial \nu(x(t))}\right],$$

$$\lambda_{k,l} = \begin{cases} 0, & \text{for } k \neq l, \\ \ln 2, & \text{for } k = l = 0, \\ \frac{1}{2|k|}, & \text{for } k = l \neq 0, \end{cases} \quad I_{k,l} = \begin{cases} 0, & \text{for } k \neq l, \\ 1, & \text{for } k = l. \end{cases}$$

5 Collocation method

Consider the Collocation method for numerical approximation of integral equations (7)–(8). We divide (see Fig. 2) the boundary Γ into n sub elements s_i, $i = 1, \ldots, n$. Denote $|s_i|$ by τ. Denote the collocation points by ξ_i. We approximate the integral operator by the following representation:

$$(A_j f_j)(\beta)(\beta;t,\tau) = \sum_{i=1}^{n} \alpha_i^{(1)} \int_{s_i} K(\beta;\xi_j,y)dl(y), \quad j = 1,\ldots,n. \quad (11)$$

The set of linear algebraic equations of the collocation method looks as follows ($j = 1,\ldots,n$):

$$\sum_{i=1}^{n} \alpha_i^{(1)} \int_{s_i} \Phi_+(\beta;\xi_j,y)dl(y) - \sum_{i=1}^{n} \alpha_i^{(2)} \int_{s_i} \Phi_\infty(\beta;\xi_j,y)dl(y) = 0, \quad (12)$$

$$\frac{1}{2}\alpha_i^{(1)} + \sum_{i=1}^{n} \alpha_i^{(1)} \int_{s_i} \frac{\partial \Phi_+(\beta;\xi_j,y)}{\partial \nu(\xi_j)} dl(y) + \frac{1}{2}\alpha_i^{(2)} - \sum_{i=1}^{n} \alpha_i^{(2)} \int_{s_i} \frac{\partial \Phi_\infty(\beta;\xi_j,y)}{\partial \nu(\xi_j)} dl(y) = 0. \quad (13)$$

So by the collocation method and by Galerkin method we reduce the original problem to two different algebraic nonlinear eigenvalue problems:

$$\begin{pmatrix} A^{(1,1)}(\beta) & A^{(1,2)}(\beta) \\ A^{(2,1)}(\beta) & A^{(2,2)}(\beta) \end{pmatrix} \begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

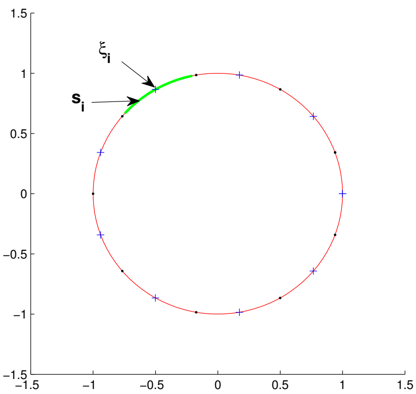

Figure 2: Collocation points

For computations we use the residual inverse iteration process [8].

The singular collocation elements are calculated analytically:

$$a_{i,i}^{(1,1)} = \tau\frac{i}{4}\Big(1 + \frac{2i}{\pi}\Big[\ln\chi_+(\beta) - \ln 2 - \psi(1) + \ln\frac{\tau}{2} - 1\Big]\Big), \quad i = 1,\ldots,n,$$

$$a_{i,i}^{(1,2)} = -\tau\frac{i}{4}\Big(1 + \frac{2i}{\pi}\Big[\ln\chi_\infty(\beta) - \ln 2 - \psi(1) + \ln\frac{\tau}{2} - 1\Big]\Big), \quad i = 1,\ldots,n,$$

$$a_{i,i}^{(2,1)} = \frac{1}{2} + \frac{\tau}{2\pi}, \quad i = 1,\ldots,n,$$

$$a_{i,i}^{(2,2)} = \frac{1}{2} - \frac{\tau}{2\pi}, \quad i = 1,\ldots,n,$$

where ψ is the psi-function.

The other elements are calculated by midpoint method:

$$a_{i,j}^{(1,1)} = \tau\Phi_+(\beta;\xi_i,\xi_j), \quad i,j = 1,\ldots,n, \quad i \neq j,$$

$$a_{i,j}^{(1,2)} = \tau\Phi_\infty(\beta;\xi_i,\xi_j), \quad i,j = 1,\ldots,n, \quad i \neq j,$$

$$a_{i,j}^{(2,1)} = \frac{1}{2} - \frac{i}{4}\tau\chi_+(\beta)\Big(\frac{(\xi_j^{(1)} - \xi_i^{(1)})\nu_1}{|\xi_j - \xi_i|} + \frac{(\xi_j^{(2)} - \xi_i^{(2)})\nu_2}{|\xi_j - \xi_i|}\Big) \cdot H_1^{(1)}(\chi_+(\beta)|\xi_j - \xi_i|), \quad i,j = 1,\ldots,n, \quad i \neq j,$$

$$a_{i,j}^{(2,2)} = \frac{1}{2} - \frac{i}{4}\tau\chi_\infty(\beta)\Big(\frac{(\xi_j^{(1)} - \xi_i^{(1)})\nu_1}{|\xi_j - \xi_i|} + \frac{(\xi_j^{(2)} - \xi_i^{(2)})\nu_2}{|\xi_j - \xi_i|}\Big) \cdot H_1^{(1)}(\chi_\infty(\beta)|\xi_j - \xi_i|), \quad i,j = 1,\ldots,n, \quad i \neq j.$$

Now we describe numerical results based on the collocation method. We present a table for circular waveguide that evaluates dependence for relative error $\varepsilon = |h_6 - \overline{h}_6|/|h_6|$ and $e = \varepsilon/\tau$ of the number of collocation points n. Here $h_6 = 1.02561149$ is the exact value, $\overline{h}_6$ is the approximate value, $h_i = \beta_i/k$.

n	τ	$\overline{h}_6$	ε	e
100	0.0634	1.024 + 9e-03i	0.0011	0.018
250	0.0252	1.025 + 4e-03i	0.0004	0.017
500	0.0125	1.025 + 2e-03i	0.0002	0.016
1000	0.0062	1.025 + 8e-05i	0.0001	0.016
2000	0.0031	1.025 + 4e-05i	5e-05	0.016

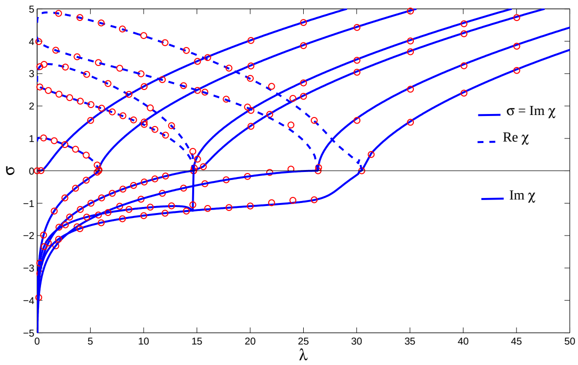

Figure 3: The first ten dispersion curves for leaky and surface waves for circular waveguide calculated by the Collocation method (marked by circles) with comparison to the exact solutions (plotted by solid lines), where $\lambda = k^2(n_+^2 - n_\infty^2)$, $\sigma = \sqrt{\beta^2 - k^2 n_\infty^2}$.

Our numerical calculations show that the collocation method has the first rate of convergence. The dispersion curves for surface and leaky modes of the circular step-index fiber calculated by collocation method in comparation with exact solutions are presented at figure 3. Figures 4, 5 show some isolines of the eigenfunctions for circular and square waveguide.

6 Comparison of methods

The following table describes the behavior of inner convergence for square waveguide. We compare approximations obtained by collocation method and by Galerkin method with h_6 which was calculated for $n = 4000$ by collocation method.

n	$time(s.)$	$\overline{\varepsilon}$	N	$time(s.)$	ε
100	12	1e-3	1	1449	9e-3
500	167	3e-4	3	6484	2e-4
2000	2308	4e-5	4	10450	6e-5
4000	9457	0			

Here n is the number of collocation points, N is the retained terms of the Fourier series for Galerkin method, $\overline{\varepsilon}$ is the relative error of collocation method, ε is the relative error of Galerkin method, $time(sec.)$ is the time of computing. The collocation method demonstrates better speed of convergence in this experiment as well as in other numerical experiments for waveguides with complicated boundaries.

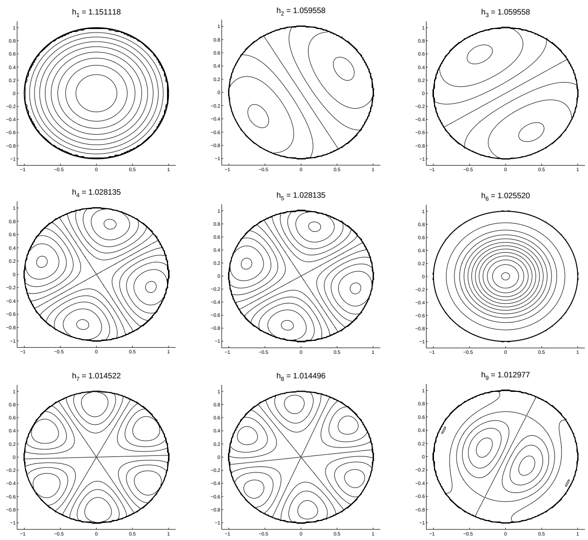

Figure 4: Isolines of some eigenfunctions of circular waveguide

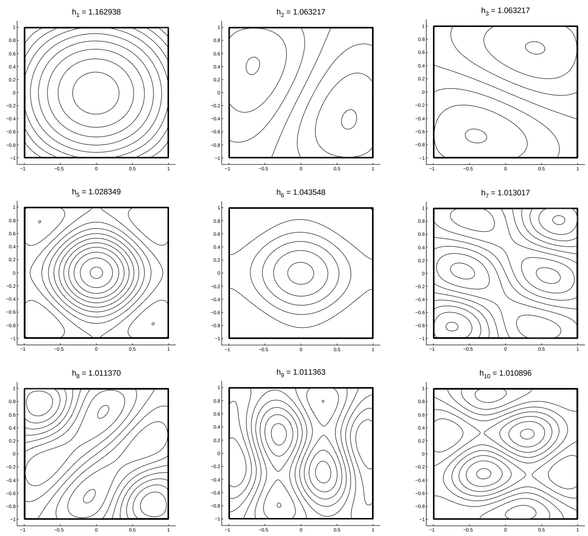

Figure 5: Isolines of some eigenfunctions of square waveguide

Acknowledgements

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