Vacuum polarization of a quantized scalar field in the thermal state in a long throat

Arkady A. Popov[*]

Kazan Federal University, 18 Kremlyovskaya St., Kazan 420008, Russia
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Vacuum polarization of scalar fields in the background of a long throat is investigated. The field is assumed to be both massive or massless, with arbitrary coupling to the scalar curvature, and in a thermal state at an arbitrary temperature. Analytical approximation for $\langle \varphi^2 \rangle_{\text{ren}}$ is obtained.

I. INTRODUCTION

The study of vacuum polarization effects in strong gravitational fields is a pertinent issue since such effects may play a role in the cosmological scenario and in the construction of a self-consistent model of black hole evaporation. These effects may be taken into account by solving the semiclassical backreaction equations,

$$G_\nu^\mu = 8\pi \langle T_\nu^\mu \rangle_{\text{ren}}, \tag{1}$$

where $\langle T_\nu^\mu \rangle_{\text{ren}}$ is the expectation value of the stress-energy tensor operator for the quantized fields. Possible vacuum fluctuations of quantized fields can create wormholes [1–4].

The main difficulty in the theory of semiclassical gravity is that the effects of the quantized gravitational field are ignored. The popular solution to this problem is to justify ignoring the gravitational contribution by working in the limit of a large number of fields, in which the gravitational contribution is negligible. Another problem is that the vacuum polarization effects are determined by the topological and geometrical properties of spacetime as a whole or by the choice of quantum state in which the expectation values are taken. It means that calculation of the functional dependence of $\langle T_\nu^\mu \rangle_{\text{ren}}$ on the metric tensor in an arbitrary spacetime presents formidable difficulty. Only in some spacetimes with high degrees of symmetry for the conformally invariant fields $\langle T_{\mu\nu} \rangle_{\text{ren}}$ can be computed and Eqs. (1) can be solved exactly [5–9].

Numerical computations of $\langle T_\nu^\mu \rangle_{\text{ren}}$ are usually extremely intensive [10–17]. In some cases $\langle T_{\mu\nu} \rangle_{\text{ren}}$ is determined by the local properties of a spacetime, and it is possible to calculate the functional dependence of the renormalized expression for the vacuum expectation value of the stress-energy tensor operator of the quantized fields on the metric tensor approximately (too long). One of the most widely known examples of such a situation is the case of a very massive field. In this case $\langle T_\nu^\mu \rangle_{\text{ren}}$ can be expanded in terms of powers of the small parameter

$$\frac{1}{ml} \ll 1, \tag{2}$$

where m is the mass of the quantized field and l is the characteristic scale of the spacetime curvature [18–24].

Approximate calculations for conformally coupled massless fields have also been made. For $\langle T_\nu^\mu \rangle$ in static Einstein spacetimes ($R_{\mu\nu} = \Lambda g_{\mu\nu}$) these include the approximations of Page, Brown, and Ottewill [25–27]. These results have been generalized to arbitrary static spacetimes by Zannias [28]. A different approach to the derivation of approximate expressions for $\langle \varphi^2 \rangle$ and $\langle T_\nu^\mu \rangle$ for conformally coupled massless fields in static spacetimes has been proposed by Frolov and Zel'nikov [29]. Their calculations were based primarily on geometric arguments and the common properties of the stress-energy tensor rather than on a field theory. Using the methods of quantum field theory the expressions for $\langle \varphi^2 \rangle$ and $\langle T_\nu^\mu \rangle$ of a scalar field in static spherically symmetric asymptotically flat spacetimes have been obtained by Anderson, Hiscock, and Samuel [10]. They assumed that the field is massive or massless with an arbitrary coupling ξ to the scalar curvature and in a zero temperature quantum state or a nonzero temperature thermal state. The result was presented as a sum of two parts, numerical and analytical:

$$\langle T_\nu^\mu \rangle_{\text{ren}} = \langle T_\nu^\mu \rangle_{\text{numeric}} + \langle T_\nu^\mu \rangle_{\text{analytic}}. \tag{3}$$

The analytical part of their expression is conserved. This has a trace equal to the trace anomaly for the conformally invariant field. For these reasons they proposed to use $\langle T_\nu^\mu \rangle_{\text{analytic}}$ directly as an approximation for $\langle T_\nu^\mu \rangle_{\text{ren}}$. An analogous result has been obtained by Groves, Anderson, and Carlson [30] in the case of a massless spin $\frac{1}{2}$ field in a general static spherically symmetric spacetime.

Let us stress that the single parameter of length dimensionality in problem (1) is the Planck length l_{Pl}. This implies that the characteristic scale l of the spacetime curvature [which corresponds to the solution of Eqs. (1)] can differ from l_{Pl} only if there is a large dimensionless parameter. As an example of such a parameter one can

*apopov@ksu.ru

The constant m_{DS} is equal to the mass m of the field for a massive scalar field. For a massless field m_{DS} is an arbitrary parameter due to the infrared cutoff in $\langle \varphi^2 \rangle_{DS}$. A particular choice of the value of m_{DS} corresponds to a finite renormalization of the coefficients of terms in the gravitational Lagrangian and must be fixed by experiment or observation,

$$\langle \phi^2 \rangle_{DS} = \frac{1}{8\pi^2} \left\{ \frac{2}{f(\tau - \tilde{\tau})^2} + \frac{f'^2}{f^2} + \left[m^2 + \left(\xi - \frac{1}{6} \right) \left(\frac{2}{r^2} - \frac{f''}{f} - \frac{2r^{2''}}{r^2} + \frac{f'^2}{2f^2} - \frac{f'r^{2'}}{fr^2} + \frac{r^{2'2}}{2r^4} \right) \right] \right.$$
$$\left. \times \left[C + \frac{1}{2} \ln \left(\frac{m_{DS}^2 f(\tau - \tilde{\tau})^2}{4} \right) \right] - \frac{m^2}{2} - \frac{f'r^{2'}}{12fr^2} - \frac{f''}{12f} \right\}. \tag{36}$$

The procedure described above gives a renormalized expression for $\langle \varphi^2 \rangle_{ren}$ in the framework of the second-order WKB approximation:

$$4\pi^2 \langle \varphi^2 \rangle_{ren} = 4\pi^2 (\langle \varphi^2 \rangle_{ren}^{(0)} + \langle \varphi^2 \rangle_{ren}^{(2)})$$
$$= \frac{4\mu^2}{r^2} J(a/\mu) - \frac{1}{4r^2} \left(2\xi - \frac{1}{4} \right) + \frac{1}{4r^2} \left[m^2 r^2 + 2 \left(\xi - \frac{1}{6} \right) \right] \ln \left| \frac{m^2 r^2 + 2\xi - 1/4}{m_{DS}^2 r^2} \right|$$
$$- \frac{\mu^2}{r^2} I_1(\mu) + \frac{r^{2''}}{24r^2} - \frac{r^{2'2}}{48r^4} + \frac{f'r^{2'}}{48fr^2} + \left[\frac{f''}{12f} + \frac{r^{2''}}{6r^2} - \frac{f'^2}{24f^2} - \frac{r^{2'2}}{24r^4} + \frac{f'r^{2'}}{12fr^2} + \xi \left(-\frac{f''}{2f} - \frac{r^{2''}}{r^2} + \frac{f'^2}{4f^2} + \frac{r^{2'2}}{4r^4} - \frac{f'r^{2'}}{2fr^2} \right) \right]$$
$$\times \left(\frac{1}{2} \ln \left| \frac{m^2 r^2 + 2\xi - 1/4}{m_{DS}^2 r^2} \right| - \mu \frac{dI_0(\mu)}{d\mu} \right), \tag{37}$$

where

$$a = \frac{2\pi Tr}{\sqrt{f}}, \qquad \mu^2 = m^2 r^2 + 2\xi - \frac{1}{4}. \tag{38}$$

The terms of a fourth WKB order are given in Appendix B. Note that $\lim_{T \to 0} J(a/\mu) = 0$.

IV. CONCLUSION

We have obtained an analytical approximation for $\langle \varphi^2 \rangle_{ren}$ of quantized scalar fields in the background of a long throat (15)–(17). The field is assumed to be both massive or massless, with arbitrary coupling to the scalar curvature, and in a thermal state at an arbitrary temperature.

As an example, we consider spacetime with metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{39}$$

where $f(r)$ is

$$f(r) = -\frac{(r + 3r_+)}{6r_+{}^2 r^2}(r - r_+)^3. \tag{40}$$

It was shown in [35] the WKB approximation (15)–(17) remains valid near the ultraextremal horizon $r = r_+$. It means that we can obtain an explicit expression for $\langle \varphi^2 \rangle_{ren}$ near the ultraextremal horizon $r = r_+$,

$$4\pi^2 \langle \varphi^2 \rangle_{ren} = \frac{4\mu^2}{r_+^2} J(a_+/\mu_+) - \frac{1}{4r_+^2} \left(2\xi - \frac{1}{4} \right) + \frac{1}{4r_+^2} \left[m^2 r_+^2 + 2 \left(\xi - \frac{1}{6} \right) \right] \ln \left| \frac{\mu_+^2}{m_{DS}^2 r^2} \right| - \frac{\mu_+^2}{r_+^2} I_1(\mu_+)$$
$$+ \left[\frac{a_+}{\mu_+^2} \left(\frac{2\xi - 1/4}{r_+ \mu_+} - \frac{f'(r_+)}{2f(r_+)} \right) \frac{dJ(a_+/\mu_+)}{d(a_+/\mu_+)} + \frac{2\xi - 1/4}{24r_+^3 \mu_+^2} + 2 \frac{2\xi - 1/4}{r_+^3} I_1(\mu_+) \right.$$
$$\left. + \frac{m^2 \mu_+}{r_+} \frac{dI_1(\mu_+)}{d\mu_+} + \frac{2\mu_+}{r_+^3} \left(\xi - \frac{1}{6} \right) \frac{dI_0(\mu_+)}{d\mu_+} \right] (r - r_+) + O((r - r_+)^2), \tag{41}$$

where $a_+ = \frac{2\pi Tr_+}{\sqrt{f(r_+)}}, \mu_+^2 = m^2 r_+^2 + 2\xi - 1/4$.

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[1] M. S. Morris and K. S. Thorne, Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity, Am. J. Phys. **56**, 395 (1988).

[2] S. Sushkov, A selfconsistent semiclassical solution with a throat in the theory of gravity, Phys. Lett. A **164**, 33 (1992).

[3] D. Hochberg, A. Popov, and S. V. Sushkov, Self-Consistent Wormhole Solutions of Semiclassical Gravity, Phys. Rev. Lett. **78**, 2050 (1997).

[4] A. Popov, Long throat of a wormhole created from vacuum fluctuations, Classical Quantum Gravity **22**, 5223 (2005).

[5] A. A. Starobinsky, A new type of isotropic cosmological models without singularity, Phys. Lett. **91B**, 99 (1980).

[6] S. G. Mamayev and V. M. Mostepanenko, Isotropic cosmological models determined by vacuum quantum effects, Zh. Eksp. Teor. Fiz. **78**, 20 (1980) [Sov. Phys. JETP **51**, 9 (1980)].

[7] L. A. Kofman, V. Sakhni, and A. A. Starobinskii, Anisotropic cosmological model created by quantum polarization of vacuum, Zh. Eksp. Teor. Fiz. **85**, 1876 (1983) [Sov. Phys. JETP **58**, 1090 (1983)].

[8] L. A. Kofman and V. Sahni, A new self-consistent solution of the Einstein equations with one-loop quantum-gravitational corrections, Phys. Lett. **127B**, 197 (1983).

[9] L. A. Kofman and V. Sahni, Some self-consistent solutions of the Einstein equations with one-loop quantum gravitational corrections: $G_{ik} = 8\pi G \langle T_{ik} \rangle_{\mathrm{vac}}$, Phys. Lett. A **117**, 275 (1986).

[10] P. R. Anderson, W. A. Hiscock, and D. A. Samuel, Stress-energy tensor of quantized scalar fields in static spherically symmetric spacetimes, Phys. Rev. D **51**, 4337 (1995).

[11] K. W. Howard and P. Candelas, Quantum Stress Tensor in Schwarzschild Space-Time, Phys. Rev. Lett. **53**, 403 (1984).

[12] P. Candelas, Vacuum polarization in Schwarzschild spacetime, Phys. Rev. D **21**, 2185 (1980).

[13] M. S. Fawcett, The energy-momentum tensor near a black hole, Commun. Math. Phys. **89**, 103 (1983).

[14] B. P. Jensen and A. Ottewill, Renormalized electromagnetic stress tensor in Schwarzschild spacetime, Phys. Rev. D **39**, 1130 (1989).

[15] B. P. Jensen, J. G. Mc Laughlin, and A. C. Ottewill, Anisotropy of the quantum thermal state in schwarzschild space-time, Phys. Rev. D **45**, 3002 (1992).

[16] P. R. Anderson, W. A. Hiscock, and D. J. Loranz, Semiclassical Stability of the Extreme Reissner-Nordström Black Hole, Phys. Rev. Lett. **74**, 4365 (1995).

[17] E. R. Bezerra de Mello, V. B. Bezerra, and N. R. Khusnutdinov, Vacuum polarization of a massless spinor field in global monopole spacetime, Phys. Rev. D **60**, 063506 (1999).

[18] V. P. Frolov and A. I. Zel'nikov, Vacuum polarization by a massive scalar field in Schwarzschild spacetime, Phys. Lett. **115B**, 372 (1982).

[19] V. P. Frolov and A. I. Zel'nikov, Vacuum polarization of massive fields in Kerr spacetime, Phys. Lett. **123B**, 197 (1983).

[20] V. P. Frolov and A. I. Zel'nikov, Vacuum polarization of massive fields near rotating black holes. Phys. Rev. D **29**, 1057 (1984).

[21] R. Herman, A method for calculating the imaginary part of the Hadamard Elementary function $G^{(1)}$ in static, spherically symmetric spacetimes, Phys. Rev. D **58**, 084028 (1998).

[22] J. Matyjasek, Stress-energy tensor of neutral massive fields in the Reissner-Nordstrom spacetime, Phys. Rev. D **61**, 124019 (2000).

[23] H. Koyama, Y. Nambu, and A. Tomimatsu, Vacuum polarization of massive scalar fields on the black hole horizon, Mod. Phys. Lett. A **15**, 815 (2000).

[24] J. Matyjasek, Vacuum polarization of massive scalar fields in the spacetime of the electrically charged nonlinear black hole, Phys. Rev. D **63**, 084004 (2001).

[25] D. N. Page, Thermal stress tensors in static Einstein spaces, Phys. Rev. D **25**, 1499 (1982).

[26] M. R. Brown and A. C. Ottewill, Effective actions and conformal transformations, Phys. Rev. D **31**, 2514 (1985).

[27] M. R. Brown, A. C. Ottewill, and D. N. Page, Conformally invariant quantum field theory in static Einstein space-times, Phys. Rev. D **33**, 2840 (1986).

[28] T. Zannias, Renormalized thermal stress tensor for arbitrary static space-times, Phys. Rev. D **30**, 1161 (1984).

[29] V. P. Frolov and A. I. Zel'nikov, Killing approximation for vacuum and thermal stress-energy tensor in static space-times, Phys. Rev. D **35**, 3031 (1987).

[30] P. B. Groves, P. R. Anderson, and E. D. Carlson, Method to compute the stress-energy tensor for the massless spin $1/2$ field in a general static spherically symmetric spacetime, Phys. Rev. D **66**, 124017 (2002).

[31] N. Nakazawa and T. Fukuyama, On the energy-momentum tensor at finite temperature in curved space-time, Nucl. Phys. **B252**, 621 (1985).

[32] P. Candelas and K. W. Howard, Vacuum $\langle \phi^2 \rangle$ in Schwarzschild spacetime, Phys. Rev. D **29**, 1618 (1984).

[33] K. W. Howard, Vacuum $\langle T_\mu^\nu \rangle$ in Schwarzschild spacetime, Phys. Rev. D **30**, 2532 (1984).

[34] P. R. Anderson, A method to compute $\langle \phi^2 \rangle$ in asymptotically flat, static, spherically symmetric spacetimes, Phys. Rev. D **41**, 1152 (1990).

[35] A. Popov and O. Zaslavskii, Quantum-corrected ultraextremal horizons and the validity of the WKB approximation in the massless limit, Phys. Rev. D **75**, 084018 (2007).

[36] A. A. Popov and S. V. Sushkov, Vacuum polarization of a scalar field in wormhole spacetimes, Phys. Rev. D **63**, 044017 (2001).

[37] A. Popov, Local expansion of the bivector of geodesic parallel displacement, Gravitation Cosmol. **13**, 119 (2007).

[38] A. A. Popov, Stress-energy of a quantized scalar field in static wormhole spacetimes, Phys. Rev. D **64**, 104005 (2001).

[39] M. A. Evgrafov, *Analytic Functions* (Nauka, Moskow, 1968).

[40] S. V. Sushkov, Analytical approximation of $\langle \phi^2 \rangle$ for a massive scalar field in static spherically symmetric spacetimes, Phys. Rev. D **62**, 064007 (2000).

[41] N. R. Khusnutdinov, The Abel-Plana formulas, Space, time and fundamental interactions **2**, 71 (2013).