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Inequality for a Trace on a Unital C^*-Algebra

A. M. Bikchentaev*

Kazan (Volga Region) Federal University, Kazan, Russia

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Abstract—A new inequality for a trace on a unital C^*-algebra is established. It is shown that the inequality obtained characterizes the traces in the class of all positive functionals on a unital C^*-algebra. A new criterion for the commutativity of unital C^*-algebras is proved.

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1. INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$, and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $X \in \mathcal{B}(\mathcal{H})$ is said to be a *projection* if $X = X^2 = X^*$. By the *commutant* of the set $\mathcal{X} \subset \mathcal{B}(\mathcal{H})$ one means the set

$$\mathcal{X}' = \{Y \in \mathcal{B}(\mathcal{H}) : XY = YX, X \in \mathcal{X}\}.$$

A $*$-subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$ is called a *von Neumann algebra* acting on the Hilbert space $\mathcal{H}$ if $\mathcal{M} = \mathcal{M}''$. By a *$C^*$-algebra* one means a complex Banach $*$-algebra $\mathcal{A}$ such that $\|A^*A\| = \|A\|^2$ for any $A \in \mathcal{A}$. Every C^*-algebra can be realized as a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (Gel'fand–Naimark; see [1, Theorem 3.4.1]). For a C^*-algebra $\mathcal{A}$, denote by $\mathcal{A}^{\mathrm{sa}}$, $\mathcal{A}^+$, and $\mathcal{A}^{\mathrm{pr}}$ the subsets of Hermitian elements, positive elements, and projections of $\mathcal{A}$, respectively. For a unital $\mathcal{A}$, let I be the unit of $\mathcal{A}$, and let $P^{\perp} = I - P$ for $P \in \mathcal{A}^{\mathrm{pr}}$. A positive linear functional φ on a C^*-algebra $\mathcal{A}$ is called a *state* if $\|\varphi\| = 1$, and it is said to be *tracial* if $\varphi(X^*X) = \varphi(XX^*)$ for all $X \in \mathcal{A}$.

A positive linear functional φ on a von Neumann algebra $\mathcal{M}$ is said to be *normal* if

$$X_i \nearrow X, \quad X_i, X \in \mathcal{M}^+, \quad \Longrightarrow \quad \varphi(X) = \sup \varphi(X_i).$$

For $P, Q \in \mathcal{M}^{\mathrm{pr}}$ we write $P \sim Q$ (the *Murray–von Neumann* equivalence) if $P = U^*U$ and $Q = UU^*$ for some $U \in \mathcal{M}$.

By a *universal representation* of a C^*-algebra $\mathcal{A}$ one means the pair

$$\{\pi, \mathfrak{H}\} = \sum_{\varphi \in \mathcal{S}(\mathcal{A})}^{\oplus} \{\pi_\varphi, \mathfrak{H}_\varphi\},$$

where $\mathcal{S}(\mathcal{A})$ is the set of all states on $\mathcal{A}$ and $(\pi_\varphi, \mathfrak{H}_\varphi)$ is the Gel'fand–Naimark–Segal representation of the C^*-algebra $\mathcal{A}$ associated with φ. In this case, the von Neumann algebra $\mathcal{M} = \pi(\mathcal{A})''$ generated by $\pi(\mathcal{A})$ is referred to as the *universal enveloping von Neumann algebra* of the C^*-algebra $\mathcal{A}$ [2, Chap. III, Definition 2.3].

Let φ be a positive linear functional on a C^*-algebra $\mathcal{A}$, and let π be the universal representation of $\mathcal{A}$. It follows from the construction of π that an arbitrary state on $\mathcal{A}$ becomes a vector state on $\pi(\mathcal{A})$ and, therefore, can be extended to a normal state on the universal enveloping algebra $\mathcal{M} = \pi(\mathcal{A})''$. Therefore, for φ, there is a positive normal functional $\widehat{\varphi}$ on the universal enveloping von Neumann algebra such that

$$\widehat{\varphi}(\pi(\mathcal{A})) = \varphi(\mathcal{A}), \qquad A \in \mathcal{A}^+.$$

*E-mail: Airat.Bikchentaev@kpfu.ru