ISSN 0001-4346, Mathematical Notes, 2020, Vol. 107, No. 3, pp. 383–391. © Pleiades Publishing, Ltd., 2020.
Russian Text © The Author(s), 2020, published in Matematicheskie Zametki, 2020, Vol. 107, No. 3, pp. 341–350.

Seminorms Associated with Subadditive Weights on C^*-Algebras

A. M. Bikchentaev[1*]

[1] *Kazan (Volga Region) Federal University, Kazan, 420008 Russia*

Received March 20, 2019; in final form, March 20, 2019; accepted May 22, 2019

Abstract—Let φ be a subadditive weight on a C^*-algebra $\mathscr{A}$, and let $\mathfrak{M}_\varphi^+$ be the set of all elements x in $\mathscr{A}^+$ with $\varphi(x) < +\infty$. A seminorm $\|\cdot\|_\varphi$ is introduced on the lineal $\mathfrak{M}_\varphi^{\mathrm{sa}} = \mathrm{lin}_{\mathbb{R}}\,\mathfrak{M}_\varphi^+$, and a sufficient condition for the seminorm to be a norm is given. Let I be the unit of the algebra $\mathscr{A}$, and let $\varphi(I) = 1$. Then, for every element x of $\mathscr{A}^{\mathrm{sa}}$, the limit $\rho_\varphi(x) = \lim_{t\to 0+}(\varphi(I + tx) - 1)/t$ exists and is finite. Properties of ρ_φ are investigated, and examples of subadditive weights on C^*-algebras are considered. On the basis of Lozinskii's 1958 results, specific subadditive weights on $\mathbb{M}_n(\mathbb{C})$ are considered. An estimate for the difference of Cayley transforms of Hermitian elements of a von Neumann algebra is obtained.

DOI: 10.1134/S0001434620030025

Keywords: *Hilbert space, bounded linear operator, Cayley transform, projection, von Neumann algebra, C^*-algebra, subadditive weight, seminorm, matrix norm.*

1. INTRODUCTION

In connection with the progress in the theory of von Neumann algebras and the extension of the scope of its applications, the problem of extending Segal integration theory [1] to normal weights in arbitrary (not necessarily semifinite) von Neumann algebras has become urgent. This problem is also meaningful for noncentral normal states. The solution of the problem of constructing a noncommutative space $L_1(\mathscr{A}, \varphi)$ associated with a faithful normal semifinite weight φ on a von Neumann algebra $\mathscr{A}$ uses the completion of the lineal $\mathfrak{M}_\varphi^{\mathrm{sa}}$ with respect to the norm

$$\|x\|_\varphi = \inf\{\varphi(a + b) : x = a - b,\ a, b \in \mathfrak{M}_\varphi^+\};$$

see [2, Chap. 3, Sec. 15]. Here $\mathfrak{M}_\varphi^+ = \{x \in \mathscr{A}^+ : \varphi(x) < +\infty\}$ and $\mathfrak{M}_\varphi^{\mathrm{sa}} = \mathrm{lin}_{\mathbb{R}}\,\mathfrak{M}_\varphi^+$, and the key role is played by the weight lineal [2, Chap. 3, Sec. 14]

$$D_\varphi = \{\xi \in \mathscr{H} : \exists\,\lambda > 0\ \forall\, x \in \mathscr{A}^+\ \ (\langle x\xi, \xi\rangle \le \lambda\varphi(x))\}. \tag{1}$$

In the present paper, a similar (semi)norm is introduced with respect to a subadditive weight φ on a C^*-algebra $\mathscr{A}$. These weights were studied in [3]–[8]; ideal seminorms were treated in [9] and [10]. Let us list our results. Let φ and ψ be subadditive weights on a hereditary C^*-subalgebra $\mathscr{A}$ of some von Neumann algebra $\mathscr{M}$, and let every projection in $\mathfrak{M}_\varphi^+$ belong to $\mathfrak{M}_\psi^+$. Then, for every $x \in \mathfrak{M}_\varphi^+$, there is an increasing sequence $\{x_n\}_{n=1}^\infty \subset \mathfrak{M}_\psi^+$ such that $\|x_n - x\| \to 0$ as $n \to \infty$ (Theorem 1). Let φ be a subadditive trace on the von Neumann algebra $\mathscr{A}$. Then

$$\varphi(|K(x) - K(y)|) \le 2\varphi(|x - y|) \qquad \text{for all} \quad x, y \in \mathscr{A}^{\mathrm{sa}},$$

where $K(z)$ is the Cayley transform of the operator $z \in \mathscr{A}^{\mathrm{sa}}$ (Theorem 4).

*E-mail: Airat.Bikchentaev@kpfu.ru

Let φ be a subadditive weight on a unital C^*-algebra $\mathscr{A}$ such that $\varphi(I) = 1$. Then, for every $x \in \mathscr{A}^{\rm sa}$, the following limit exists and is finite:

$$\rho_\varphi(x) = \lim_{t \to 0+} \frac{\varphi(I + tx) - 1}{t} \qquad \text{(Lemma)}.$$

We have

- $\rho_\varphi(x + y) \le \rho_\varphi(x) + \rho_\varphi(y)$ for all $x, y \in \mathscr{A}^{\rm sa}$;
- $\rho_\varphi(x) \le \rho_\varphi(y)$ for all $x, y \in \mathscr{A}^{\rm sa}$ with $x \le y$;
- $\rho_\varphi(\lambda x) = |\lambda| \rho_\varphi(\mathrm{sgn} \lambda x)$ for all $\lambda \in \mathbb{R}$ and $x \in \mathscr{A}^{\rm sa}$;
- $|\rho_\varphi(x) - \rho_\varphi(y)| \le \|x - y\|_\varphi$ for all $x, y \in \mathscr{A}^{\rm sa}$ (Theorem 5).

A sufficient condition for $\|\cdot\|_\varphi$ to be a norm on the lineal $\mathfrak{M}_\varphi^{\rm sa}$ is given (Corollary 3). Particular subadditive weights on $\mathbb{M}_n(\mathbb{C})$ are considered (Example 3).

2. DEFINITIONS AND NOTATION

By a *C^*-algebra* we mean a complex Banach $*$-algebra $\mathscr{A}$ such that $\|x^* x\| = \|x\|^2$ for all $x \in \mathscr{A}$. An element $x \in \mathscr{A}$ is called a *projection* if $x = x^2 = x^*$.

For a C^*-algebra $\mathscr{A}$, by $\mathscr{A}^{\rm pr}$, $\mathscr{A}^{\rm sa}$, and $\mathscr{A}^+$ we denote the subsets of projections, Hermitian elements, and positive elements of $\mathscr{A}$, respectively. Let $\mathscr{A}_1 = \{x \in \mathscr{A} : \|x\| \le 1\}$.

If $x \in \mathscr{A}$, then $|x| = \sqrt{x^* x} \in \mathscr{A}^+$.

If $x \in \mathscr{A}^{\rm sa}$, then $x_+ = (|x| + x)/2$ and $x_- = (|x| - x)/2$ belong to $\mathscr{A}^+$, $x = x_+ - x_-$, and $x_+ x_- = 0$.

We denote the unit of a unital algebra $\mathscr{A}$ by I. A C^*-subalgebra $\mathscr{B}$ of a given C^*-algebra $\mathscr{A}$ is said to be *hereditary* if, for $a \in \mathscr{A}^+$ and $b \in \mathscr{B}^+$, the inequality $a \le b$ implies $a \in \mathscr{B}$.

Let $\mathscr{H}$ be a Hilbert space over the field $\mathbb{C}$, and let $\mathscr{B}(\mathscr{H})$ be the $*$-algebra of all bounded linear operators on $\mathscr{H}$. By the *commutant* of a set $\mathscr{X} \subset \mathscr{B}(\mathscr{H})$ we mean the set

$$\mathscr{X}' = \{y \in \mathscr{B}(\mathscr{H}) : xy = yx \text{ for all } x \in \mathscr{X}\}.$$

By a *von Neumann algebra* acting on a Hilbert space $\mathscr{H}$ we mean a $*$-subalgebra $\mathscr{A}$ of the algebra $\mathscr{B}(\mathscr{H})$ for which $\mathscr{A} = \mathscr{A}''$. Every C^*-algebra can be realized as a C^*-subalgebra of $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$ (Gel'fand–Naimark; see [11, Theorem 3.4.1]). The following definition was given in [3] (see also [4, Remark 1.6]).

Definition 1. By a *subadditive weight* on a C^*-algebra $\mathscr{A}$ we mean a mapping $\varphi\colon \mathscr{A}^+ \to [0, +\infty]$ satisfying the conditions

(i) $x \le y \Rightarrow \varphi(x) \le \varphi(y)$ for all $x, y \in \mathscr{A}^+$,

(ii) $\varphi(x + y) \le \varphi(x) + \varphi(y)$ for all $x, y \in \mathscr{A}^+$,

(iii) $\varphi(\lambda x) = \lambda \varphi(x)$ for all $x \in \mathscr{A}^+$, $\lambda \ge 0$ (here $0 \cdot (+\infty) \equiv 0$).

For a subadditive weight φ, we set

$$\mathfrak{M}_\varphi^+ = \{x \in \mathscr{A}^+ : \varphi(x) < +\infty\}, \qquad \mathfrak{M}_\varphi^{\rm sa} = \mathrm{lin}_{\mathbb{R}}\, \mathfrak{M}_\varphi^+, \qquad \mathfrak{M}_\varphi = \mathrm{lin}_{\mathbb{C}}\, \mathfrak{M}_\varphi^+.$$

A subadditive weight φ is said to be *faithful* if $\varphi(x) = 0 \Rightarrow x = 0$ for all $x \in \mathscr{A}^+$; it is called a *subadditive trace* if $\varphi(x^* x) = \varphi(x x^*)$ for all $x \in \mathscr{A}$. A subadditive weight φ on a von Neumann algebra $\mathscr{A}$ is said to be *normal* if $x_i \nearrow x$ and $(x_i, x \in \mathscr{A}^+) \Rightarrow \varphi(x) = \sup_i \varphi(x_i)$.

3. SUBADDITIVE WEIGHTS AND SEMINORMS ON C^*-ALGEBRAS

Theorem 1. *Let φ and ψ be subadditive weights on a hereditary C^*-subalgebra $\mathscr{A}$ of some von Neumann algebra $\mathscr{M}$, and let every projection in $\mathfrak{M}_\varphi^+$ belong to $\mathfrak{M}_\psi^+$. Then, for every $x \in \mathfrak{M}_\varphi^+$, there is an increasing sequence $\{x_n\}_{n=1}^\infty \subset \mathfrak{M}_\psi^+$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.*

Proof. Without loss of generality, we assume that $\|x\| \le 1$. Then

$$x = \sum_{k=1}^\infty \frac{1}{2^k} p_k$$

for some sequence $\{p_k\}_{k=1}^\infty \subset \mathscr{A}^{\mathrm{pr}}$ [11, Theorem 4.1.13]. We set

$$x_n = \sum_{k=1}^n \frac{1}{2^k} p_k, \qquad n \in \mathbb{N}.$$

Then $x_n \nearrow x$ and $\|x_n - x\| \to 0$ as $n \to \infty$. Since $p_k \le 2^k x$, it follows that $p_k \in \mathfrak{M}_\varphi^+ \cap \mathscr{A}^{\mathrm{pr}} \subset \mathfrak{M}_\psi^+$ for every $k \in \mathbb{N}$. Therefore, $x_n \in \mathfrak{M}_\psi^+$ for every $n \in \mathbb{N}$. $\square$

Corollary 1. *Under the conditions of Theorem 1, we have $\mathfrak{M}_\varphi^{\mathrm{sa}} \subset \overline{\mathfrak{M}_\psi^{\mathrm{sa}}}^{\|\cdot\|}$ and $\mathfrak{M}_\varphi \subset \overline{\mathfrak{M}_\psi}^{\|\cdot\|}$.*

Theorem 2. *Let φ be a subadditive weight on a C^*-algebra $\mathscr{A}$. Consider the following functions on the real vector space $\mathfrak{M}_\varphi^{\mathrm{sa}}$:*

$$\|x\|_\varphi = \inf\{\varphi(a+b) : x = a - b,\ a, b \in \mathfrak{M}_\varphi^+\},$$
$$\|x\|_{\varphi,1} = \inf\{\varphi(a) + \varphi(b) : x = a - b,\ a, b \in \mathfrak{M}_\varphi^+\}.$$

Then $\|\cdot\|_\varphi$ and $\|\cdot\|_{\varphi,1}$ are seminorms. Moreover,

$$\|x\|_\varphi \le \|x\|_{\varphi,1} \le 2\|x\|_\varphi, \qquad \|x\|_\varphi \le \big\||x|\big\|_\varphi = \varphi(|x|)$$

for all $x \in \mathfrak{M}_\varphi^{\mathrm{sa}}$.

Proof. We prove only the triangle inequality for $\|\cdot\|_\varphi$. Let $x, y \in \mathfrak{M}_\varphi^{\mathrm{sa}}$; we set $\alpha = \|x\|_\varphi$ and $\beta = \|y\|_\varphi$. Then, for $a, b, c, d \in \mathfrak{M}_\varphi^+$ with $a - b = x$ and $c - d = y$, we have $\varphi(a+b) \ge \alpha$, $\varphi(c+d) \ge \beta$, and

$$\forall\, \varepsilon > 0\ \exists\, a_\varepsilon, b_\varepsilon \in \mathfrak{M}_\varphi^+ \qquad (a_\varepsilon - b_\varepsilon = x,\ \varphi(a_\varepsilon + b_\varepsilon) \le \alpha + \varepsilon),$$
$$\forall\, \varepsilon > 0\ \exists\, c_\varepsilon, d_\varepsilon \in \mathfrak{M}_\varphi^+ \qquad (c_\varepsilon - d_\varepsilon = y,\ \varphi(c_\varepsilon + d_\varepsilon) \le \beta + \varepsilon).$$

Thus, for every number $\varepsilon > 0$, there are elements $a_\varepsilon, b_\varepsilon, c_\varepsilon, d_\varepsilon \in \mathfrak{M}_\varphi^+$ such that

$$a_\varepsilon + c_\varepsilon - b_\varepsilon - d_\varepsilon = x + y,$$
$$\varphi(a_\varepsilon + c_\varepsilon + b_\varepsilon + d_\varepsilon) \le \varphi(a_\varepsilon + b_\varepsilon) + \varphi(c_\varepsilon + d_\varepsilon) \le \alpha + \beta + 2\varepsilon.$$

We set $u_\varepsilon = a_\varepsilon + c_\varepsilon$ and $v_\varepsilon = b_\varepsilon + d_\varepsilon$ for all $\varepsilon > 0$. We have

$$\forall\, \varepsilon > 0\ \exists\, u_\varepsilon, v_\varepsilon \in \mathfrak{M}_\varphi^+ \qquad (u_\varepsilon - v_\varepsilon = x + y,\ \varphi(u_\varepsilon + v_\varepsilon) \le \alpha + \beta + 2\varepsilon);$$

since $\varepsilon > 0$ is arbitrary, we obtain $\|x + y\|_\varphi \le \|x\|_\varphi + \|y\|_\varphi$ for $x, y \in \mathfrak{M}_\varphi^{\mathrm{sa}}$. This concludes the proof of the theorem. $\square$

Corollary 2. *If $x \in \mathfrak{M}_\varphi^{\mathrm{sa}}$ and $\|x\| \le \varepsilon$, then $\|x\|_\varphi \le \varepsilon\varphi(I)$.*

Proof. For $x \in \mathfrak{M}_\varphi^{\mathrm{sa}}$, we have the implications

$$\||x|\| = \|x\| \le \varepsilon \iff -\varepsilon I \le x \le \varepsilon I \iff |x| \le \varepsilon I$$
$$\implies \|x\|_\varphi \le \big\||x|\big\|_\varphi = \varphi(|x|) \le \varphi(\varepsilon I) = \varepsilon\varphi(I). \quad \square$$

The proof of the following proposition is similar to that of Proposition 16.2 in [2].

Proposition 1. *Let φ be a subadditive weight on a von Neumann algebra $\mathscr{A}$ acting on a Hilbert space $\mathscr{H}$, let D_φ be the lineal defined by formula (1), and let $\xi \in \mathscr{H}$. A vector functional $\omega_\xi(x) \equiv \langle x\xi, \xi\rangle$ is $\|\cdot\|_\varphi$-continuous on the real space $\mathfrak{M}_\varphi^{\mathrm{sa}}$ if and only if $\xi \in D_\varphi$.*

Theorem 3. *Let φ be a subadditive trace on a von Neumann algebra $\mathscr{A}$. Then*

i) *the mapping $x \mapsto \|x\|_{\varphi,2} = \varphi(|x|)$ defines a seminorm on $\mathfrak{M}_\varphi^{\mathrm{sa}}$;*

ii) *$\|x\|_\varphi \le \varphi(|x|) \le \|x\|_{\varphi,1}$ for all $x \in \mathfrak{M}_\varphi^{\mathrm{sa}}$.*

Proof. (i) By Corollary 1 of [12], for every $x, y \in \mathscr{A}^{\mathrm{sa}}$, there is a unitary operator $u \in \mathscr{A}^{\mathrm{sa}}$ such that

$$|x+y| \le 2^{-1}(|x| + |y| + u(|x| + |y|)u).$$

Therefore,

$$\varphi(|x+y|) \le \varphi(|x| + |y|) \le \varphi(|x|) + \varphi(|y|).$$

(ii) The first inequality was established in Theorem 2, and the proof of the second one is similar to that of Proposition 15.7 in [2], where it was shown that, for $x \in \mathfrak{M}_\varphi^{\mathrm{sa}}$ and $a, b \in \mathfrak{M}_\varphi^{+}$ with $a - b = x$, we have $\varphi(a) \ge \varphi(x_+)$ and $\varphi(b) \ge \varphi(x_-)$. $\square$

Remark 1. If a C^*-algebra $\mathscr{A}$ is unital and $x, y \in \mathscr{A}$, then, for every number $\varepsilon > 0$, there are unitary $u, v \in \mathscr{A}$ such that $|x+y| \le u|x|u^* + v|y|v^* + \varepsilon I$; see Theorem 4.2 of [13]. Therefore, the mapping $x \mapsto \|x\|_{\varphi,2} = \varphi(|x|)$ defines a seminorm on $\mathscr{A}$ for every finite subadditive trace φ on $\mathscr{A}$.

Theorems 2 and 3 imply the following corollary.

Corollary 3. *Let φ be a faithful subadditive trace on a von Neumann algebra $\mathscr{A}$. Then the mappings*

$$x \mapsto \|x\|_{\varphi,2} = \varphi(|x|), \qquad x \mapsto \|x\|_\varphi = \inf\{\varphi(a+b) : x = a - b,\ a, b \in \mathfrak{M}_\varphi^{+}\},$$
$$x \mapsto \|x\|_{\varphi,1} = \inf\{\varphi(a) + \varphi(b) : x = a - b,\ a, b \in \mathfrak{M}_\varphi^{+}\}$$

define equivalent norms on the real vector space $\mathfrak{M}_\varphi^{\mathrm{sa}}$.

Let $\mathscr{A}$ be a unital C^*-algebra. As is well known, the Cayley transform

$$K(x) = \frac{x + iI}{x - iI} = (x - iI)^{-1}(x + iI) = (x + iI)(x - iI)^{-1}$$

of an element $x \in \mathscr{A}^{\mathrm{sa}}$ is a unitary element of $\mathscr{A}$.

Theorem 4. *Let φ be a subadditive trace on a von Neumann algebra $\mathscr{A}$. Then*

$$\varphi(|K(x) - K(y)|) \le 2\varphi(|x - y|) \qquad \textit{for all} \quad x, y \in \mathscr{A}^{\mathrm{sa}}.$$

Proof. *Step* 1. Let us show that $\varphi(|azb|) \le \varphi(|z|)$ for all $a, b \in \mathscr{A}_1$ and $z \in \mathscr{A}$. To this end, we shall prove that $\varphi(|c|) = \varphi(|c^*|)$ for all $c \in \mathscr{A}$. Let $c = u|c|$ be the polar decomposition of the operator $c \in \mathscr{A}$. Then $|c^*| = u|c|u^*$ (see [14]). We have

$$\varphi(|c^*|) = \varphi(u|c|u^*) = \varphi(u\sqrt{|c|} \cdot \sqrt{|c|}u^*) = \varphi(\sqrt{|c|}u^* \cdot u\sqrt{|c|}\,) = \varphi(|c|).$$

Now, for all $a, b \in \mathscr{A}_1$ and $z \in \mathscr{A}$, from the operator monotonicity of the function $\lambda \mapsto \sqrt{\lambda}$ ($\lambda \in \mathbb{R}^+$) we obtain

$$\varphi(|azb|) = \varphi(\sqrt{b^*z^*a^*azb}\,) \le \varphi(\sqrt{b^*z^*zb}\,) = \varphi(|zb|) = \varphi(|(zb)^*|)$$
$$= \varphi(|b^*z^*|) = \varphi(\sqrt{zbb^*z^*}\,) \le \varphi(\sqrt{zz^*}\,) = \varphi(|z^*|) = \varphi(|z|).$$

Step 2. For $x, y \in \mathscr{A}^{\mathrm{sa}}$, we have

$$\begin{aligned}K(x) - K(y) &= (x - iI)^{-1}(x + iI) - (y + iI)(y - iI)^{-1} \\ &= (x - iI)^{-1}[(x + iI)(y - iI) - (x - iI)(y + iI)](y - iI)^{-1} \\ &= -2i(x - iI)^{-1} \cdot (x - y) \cdot (y - iI)^{-1}.\end{aligned}$$

Since the function $f(\lambda) = 1/(\lambda - i)$, $\lambda \in \mathbb{R}$, satisfies the inequality $|f(\lambda)| \le 1$, $\lambda \in \mathbb{R}$, we see that $f(x) = (x - iI)^{-1}, f(y) = (y - iI)^{-1} \in \mathscr{A}_1$. By Step 1, we have

$$\begin{aligned}\varphi(|K(x) - K(y)|) &= \varphi(|-2i(x - iI)^{-1} \cdot (x - y) \cdot (y - iI)^{-1}|) \\ &= 2\varphi(|(x - iI)^{-1} \cdot (x - y) \cdot (y - iI)^{-1}|) \le 2\varphi(|x - y|).\end{aligned}$$

This concludes the proof of the theorem. □

4. A FUNCTIONAL ASSOCIATED WITH A SUBADDITIVE WEIGHT

Lemma. *Let φ be a subadditive weight on a unital C^*-algebra $\mathscr{A}$, and let $\varphi(I) = 1$. Then, for every $x \in \mathscr{A}^{\mathrm{sa}}$, the limit*

$$\rho_\varphi(x) = \lim_{t \to 0+} \frac{\varphi(I + tx) - 1}{t}$$

exists and is finite.

Proof. For $x = 0$, the assertion is obvious. For $x \ne 0$, we set

$$J_x = \{t \in \mathbb{R} : -\|x\|^{-1} < t < \|x\|^{-1}\}.$$

We have $I + tx \in \mathscr{A}^+$ for all $t \in J_x$. Conider the function f defined by the formula

$$f(t) = \varphi(I + tx), \qquad t \in J_x.$$

Let us show the continuity of the function f on J_x. Take $s, t \in J_x$ such that $|s - t| < \varepsilon \|x\|^{-1}$. Then $\|(s - t)x\| < \varepsilon$ and $\|(s - t)x\|_\varphi < \varepsilon$ by Corollary 2. Since $I + sx = I + tx + (s - t)x$, it follows from the triangle inequality that

$$\|I + sx\|_\varphi < \|I + tx\|_\varphi + \varepsilon.$$

Therefore, $\|I + sx\|_\varphi - \|I + tx\|_\varphi < \varepsilon$. Interchanging the roles of s and t, we obtain the inequality $\|I + tx\|_\varphi - \|I + sx\|_\varphi < \varepsilon$. Thus,

$$\big|\|I + sx\|_\varphi - \|I + tx\|_\varphi\big| < \varepsilon,$$

and the equalities $\|I + ux\|_\varphi = \varphi(I + ux)$ for $u \in \{s, t\}$ imply

$$|\varphi(I + sx) - \varphi(I + tx)| < \varepsilon.$$

Therefore, f is continuous on J_x. For all $s, t \in J_x$, we have

$$f\left(\frac{s + t}{2}\right) = \varphi\left(\frac{1}{2}(I + sx) + \frac{1}{2}(I + tx)\right) \le \frac{1}{2}\varphi(I + sx) + \frac{1}{2}\varphi(I + tx) = \frac{1}{2}f(s) + \frac{1}{2}f(t),$$

i.e., the function f is convex on J_x. By virtue of a well-known fact (see, e.g., Theorem 111 of [15]), the function f has finite right derivative at every point $t \in J_x$. In particular, the limit

$$\lim_{t \to 0+} \frac{f(t) - f(0)}{t}$$

exists and is finite. Since $f(0) = 1$, this limit is equal to $\rho_\varphi(x)$. Note that, by well-known properties of convex functions, the fraction $(f(t) - f(0))/t$ monotonically decreases to $\rho_\varphi(x)$ as $t \downarrow 0$. This concludes the proof of the lemma. □

Theorem 5. *Let φ be a subadditive weight on a unital C^*-algebra $\mathscr{A}$, and let $\varphi(I) = 1$. Then*

i) $\rho_\varphi(x+y) \le \rho_\varphi(x) + \rho_\varphi(y)$ *for all* $x, y \in \mathscr{A}^{\mathrm{sa}}$;

ii) $\rho_\varphi(x) \le \rho_\varphi(y)$ *for all* $x, y \in \mathscr{A}^{\mathrm{sa}}$ *with* $x \le y$;

iii) $\rho_\varphi(\lambda x) = |\lambda| \rho_\varphi(\mathrm{sgn}\lambda x)$ *for all* $\lambda \in \mathbb{R}$ *and* $x \in \mathscr{A}^{\mathrm{sa}}$;

iv) $\rho_\varphi(x + \lambda I) = \rho_\varphi(x) + \lambda$ *for all* $x \in \mathscr{A}^{\mathrm{sa}}$ *and* $\lambda \in \mathbb{R}$;

v) $|\rho_\varphi(x) - \rho_\varphi(y)| \le \|x - y\|_\varphi$ *for all* $x, y \in \mathscr{A}^{\mathrm{sa}}$.

Proof. Let us use the lemma.

(i) For sufficiently small numbers $t > 0$, we have

$$\varphi(I + t(x+y)) - 1 = \varphi\biggl(\frac{1}{2}(I + 2tx) + \frac{1}{2}(I + 2ty)\biggr) - 1 \le \frac{1}{2}(\varphi(I + 2tx) - 1) + \frac{1}{2}(\varphi(I + 2ty) - 1).$$

Dividing both sides of this inequality by $t > 0$ and passing to the limit as $t \to 0+$, we obtain the desired assertion.

The proof of (ii) is similar to that of (i).

(iii) For all $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in \mathscr{A}^{\mathrm{sa}}$ and for $t > 0$ with $|t\lambda| < \|x\|^{-1}$, we have

$$\frac{\varphi(I + t(\lambda x)) - 1}{t} = \frac{\varphi(I + t(\lambda x)) - 1}{t\lambda}\lambda.$$

Passing to the limit as $t \to 0+$, we obtain the desired fact for $\lambda \ne 0$. For $\lambda = 0$, the equation $\rho_\varphi(\lambda x) = |\lambda| \rho_\varphi(\mathrm{sgn}\,\lambda x)$ is obvious.

(iv) Let $\lambda \in \mathbb{R}$ and $t > 0$ be such that $\lambda t > -1$. For $s = t(1 + \lambda t)^{-1}$, we have $s \to 0+$ as $t \to 0+$ and, for sufficiently small numbers $t > 0$, we obtain

$$\frac{\varphi(I + t(x + \lambda I)) - 1}{t} = \frac{\varphi((1+\lambda t)I + tx) - 1}{t} = \frac{(1+\lambda t)\varphi(I + sx) - 1}{t} = \frac{\varphi(I + sx) - (1+\lambda t)^{-1}}{s} = \frac{\varphi(I + sx) - 1}{s} + \lambda.$$

(v) *Step* 1. Let us show that $|\rho_\varphi(x)| \le \|x\|_\varphi$ for all $x \in \mathscr{A}^{\mathrm{sa}}$. Let $x \in \mathscr{A}^{\mathrm{sa}}$, and let $x = a - b$ for $a, b \in \mathscr{A}^+$. For sufficiently small numbers $t > 0$, we have

$$\varphi(I + tx) = \varphi(I + t(a - b)) \le \varphi(I + t(a + b)) \le \varphi(I) + t\varphi(a + b) = 1 + t\varphi(a + b),$$
$$1 = \varphi(t(a+b) + I - t(a+b)) \le \varphi(t(a+b)) + \varphi(I - t(a+b)) \le t\varphi(a+b) + \varphi(I - t(b-a)) \le t\varphi(a+b) + \varphi(I + tx).$$

Thus,

$$|\varphi(I + tx) - 1| \le t\varphi(a+b), \qquad |(\varphi(I + tx) - 1)t^{-1}| \le \varphi(a+b).$$

Therefore,

$$|(\varphi(I + tx) - 1)t^{-1}| \le \inf\{\varphi(a+b) : x = a - b,\ a, b \in \mathscr{A}^+\} = \|x\|_\varphi,$$

and

$$|\rho_\varphi(x)| = \lim_{t \to 0+} \left|\frac{\varphi(I + tx) - 1}{t}\right| \le \|x\|_\varphi, \qquad x \in \mathscr{A}^{\mathrm{sa}}.$$

Step 2. It follows from part (i), by virtue of Step 1 for $x, y \in \mathscr{A}^{\mathrm{sa}}$, that

$$\rho_\varphi(y) - \rho_\varphi(x) \le \rho_\varphi(y - x) \le \|y - x\|_\varphi = \|x - y\|_\varphi,$$
$$\rho_\varphi(x) - \rho_\varphi(y) \le \rho_\varphi(x - y) \le \|x - y\|_\varphi,$$

as was to be proved. Now, by Theorem 2, we obtain

$$|\rho_\varphi(x) - \rho_\varphi(y)| \le \|x - y\|_{\varphi,1} \qquad \text{for all} \quad x, y \in \mathscr{A}^{\mathrm{sa}}.$$

This concludes the proof of the theorem. □

Corollary 4. *Let φ be a subadditive weight on a unital C^*-algebra $\mathscr{A}$, and let $\varphi(I) = 1$. Then $\rho_\varphi(I) = 1$, and the restriction $\rho_\varphi|_{\mathscr{A}^+}$ is a subadditive weight on $\mathscr{A}$, denoted by ψ, and $\rho_\varphi = \rho_\psi$. Moreover, $\|x\|_\psi \le \|x\|_\varphi$ and $\|x\|_{\psi,1} \le \|x\|_{\varphi,1}$ for all $x \in \mathscr{A}^{\mathrm{sa}}$.*

Proof. It follows from part (ii) of Theorem 5 that $\psi\colon \mathscr{A}^+ \to [0, +\infty)$. For any $x \in \mathscr{A}^{\mathrm{sa}}$, we have

$$\rho_\psi(x) = \lim_{s\to 0+} \frac{\rho_\varphi(I + sx) - 1}{s} = \lim_{s\to 0+}\biggl(\lim_{t\to 0+} \frac{\varphi(I + t(I + sx)) - 1 - t}{st}\biggr)$$
$$= \lim_{s\to 0+}\biggl(\lim_{t\to 0+} \frac{\varphi(I + st/(1+t)x) - 1}{st/(1+t)}\biggr) = \lim_{s\to 0+} \rho_\varphi(x) = \rho_\varphi(x).$$

Theorem 2 and part (v) of Theorem 5 imply

$$\psi(x) = \rho_\varphi(x) \le \|x\|_\varphi \le \varphi(x) \qquad \text{for all} \quad x \in \mathscr{A}^+.$$

This concludes the proof of the assertion. □

Remark 2. Let φ and ψ be subadditive weights on a unital C^*-algebra $\mathscr{A}$ with $\varphi(I) = \psi(I) = 1$, and let $\lambda \in (0, 1)$. Then

$$\rho_{\lambda\varphi + (1-\lambda)\psi}(x) = \lambda\rho_\varphi(x) + (1 - \lambda)\rho_\psi(x) \qquad \text{for all} \quad x \in \mathscr{A}^{\mathrm{sa}}.$$

Proposition 2. *Let φ be a subadditive weight on a von Neumann algebra $\mathscr{A}$ such that $\varphi(I) = 1$, and let $\psi = \rho_\varphi|_{\mathscr{A}^+}$. If φ is normal, then ψ is also normal. If $\mathscr{A}$ is finite and φ is a subadditive trace, then ψ is also a subadditive trace.*

Proof. Suppose that the algebra $\mathscr{A}$ is finite (i.e., $a \in \mathscr{A}$ and $a^*a = I$ imply $aa^* = I$). Let φ be a subadditive trace, and let $x = u|x|$ be the polar decomposition of an operator $x \in \mathscr{A}$. Then $xx^* = ux^*xu^*$ and there is a unitary operator $v \in \mathscr{A}$ with the property $xx^* = vx^*xv^*$ (see the proof of Theorem 2 in [16]). We have $x^*x = v^*xx^*v$ and

$$\rho_\varphi(x^*x) = \lim_{t\to 0+} \frac{\varphi(I + tv^*xx^*v) - 1}{t} = \lim_{t\to 0+} \frac{\varphi(v^*(I + txx^*)v) - 1}{t}$$
$$= \lim_{t\to 0+} \frac{\varphi(\sqrt{I + txx^*}vv^*\sqrt{I + txx^*}\,) - 1}{t} = \rho_\varphi(xx^*).$$

This concludes the proof of the proposition. □

Example 1. For a vector state $\varphi = \langle(\cdot)\xi, \xi\rangle$, $\xi \in \mathscr{H}$, $\|\xi\| = 1$, on the algebra $\mathscr{B}(\mathscr{H})$, we have $\|x\|_\varphi = |\langle x\xi, \xi\rangle|$ for all $x \in \mathscr{B}(\mathscr{H})^{\mathrm{sa}}$ (see [2, Example 15.6]). Obviously, $\rho_\varphi(x) = \langle x\xi, \xi\rangle$ for all $x \in \mathscr{B}(\mathscr{H})^{\mathrm{sa}}$. Therefore, the inequality of part (v) of Theorem 5 is an equality.

Example 2. Let ψ be a faithful state on a unital C^*-algebra $\mathscr{A}$. Then $\langle x, y\rangle_\psi = \psi(y^*x)$, $x, y \in \mathscr{A}$, is an inner product on $\mathscr{A}$. Therefore, $\varphi(x) = \sqrt{\psi(x^*x)}$, $x \in \mathscr{A}$, is a norm on $\mathscr{A}$. If the state ψ is tracial, then $\varphi|_{\mathscr{A}^+}$ is a subadditive trace on $\mathscr{A}$ [10, Proposition 1] (the converse assertion also holds; see the theorem in [17]). Since the positive functional ψ is automatically continuous, it follows that, for all $x \in \mathscr{A}^{\mathrm{sa}}$, we have

$$\rho_\varphi(x) = \lim_{t\to 0+} \frac{\sqrt{\psi((I + tx)^2)} - 1}{t} = \lim_{t\to 0+} \frac{\psi((I + tx)^2) - \psi(I)}{t(\sqrt{\psi((I + tx)^2)} + 1)} = \psi(x).$$

Example 3. For every vector $\xi=(\xi_1,\xi_2,\dots,\xi_n)\in\mathbb{C}^n$, we set

$$\|\xi\|_1=\max_{1\le k\le n}|\xi_k|,\qquad \|\xi\|_2=\sum_{k=1}^n|\xi_k|,\qquad \|\xi\|_3=\sqrt{\sum_{k=1}^n|\xi_k|^2}.$$

These norms induce the norms

$$\|x\|_1=\max_{1\le k\le n}\sum_{m=1}^n|x_{km}|,\qquad \|x\|_2=\max_{1\le m\le n}\sum_{k=1}^n|x_{km}|,\qquad \|x\|_3=\sqrt{\lambda_1(x^*x)},$$

respectively, on $\mathbb{M}_n(\mathbb{C})$, where $x=[x_{km}]_{k,m=1}^n$ and $\lambda_1(x^*x)$ stands for the greatest eigenvalue of the matrix x^*x. Consider the mappings from $\mathbb{M}_n(\mathbb{C})^+$ to $\mathbb{R}^+$ defined by the formulas

$$\phi(x)=\|x\|_1,\quad \varphi(x)=\|x\|_2,\quad \psi(x)=\|x\|_3,\qquad x\in\mathbb{M}_n(\mathbb{C})^+.$$

Then $\phi(I)=\varphi(I)=\psi(I)=1$ and, by Lemma 4 of [18] (see also [19, pp. 461–465]), we have

$$\rho_\phi(x)=\max_{1\le k\le n}\left\{x_{kk}+\sum_{m=1,\,m\ne k}^n|x_{km}|\right\},\qquad \rho_\varphi(x)=\max_{1\le m\le n}\left\{x_{mm}+\sum_{k=1,\,k\ne m}^n|x_{km}|\right\},$$

and $\rho_\psi(x)$ is equal to the largest eigenvalue of the matrix $x\in\mathbb{M}_n(\mathbb{C})^{\mathrm{sa}}$. Since ψ is a faithful subadditive trace, it follows that $\|\cdot\|_\psi$ and $\|\cdot\|_{\psi,1}$ are norms on $\mathbb{M}_n(\mathbb{C})^{\mathrm{sa}}$ by Corollary 3. Some applications of matrix norms studied by us were given in [20] and [21].

FUNDING

The research was funded by the subsidy allocated to Kazan Federal University for the state assignment in the sphere of scientific activities (project No 1.13556.2019/13.1).

REFERENCES

1. I. Segal, "A non-commutative extension of abstract integration," Ann. of Math. (2) **57** (3), 401–457 (1953).
2. A. N. Sherstnev, *Methods of Bilinear Forms in Noncommutative Theory of Measure and Integral* (Fizmatlit, Moscow, 2008) [in Russian].
3. U. Haagerup, "Normal weights on W^*-algebras," J. Funct. Anal. **19** (3), 302–317 (1975).
4. F. Combes, "Poids sur une C^*-algèbre," J. Math. Pures Appl. (9) **47** (1), 57–100 (1968).
5. A. M. Bikchentaev, "The Haagerup problem on subadditive weights on W^*-algebras," Izv. Vyssh. Uchebn. Zaved. Mat., No. 10, 94–98 (2011) [Russian Math. (Iz. VUZ), **55** (10), 82–85 (2011)].
6. A. M. Bikchentaev, "The Haagerup problem on subadditive weights on W^*-algebras. II," Izv. Vyssh. Uchebn. Zaved. Mat., No. 12, 72–76 (2013) [Russian Math. (Iz. VUZ), **57** (12), 66–69 (2013)].
7. A. A. Lebedev, "On monotone dominated sublinear functionals on the space of measurable functions," Sibirsk. Mat. Zh. **33** (6), 94–105 (1992) [Sib. Math. J. **33** (6), 1028–1038 (1992)].
8. A. A. Lebedev, "Disintegration of dominated monotone sublinear functionals on the space of measurable functions," Sibirsk. Mat. Zh. **34** (6), 130–149 (1993) [Sib. Math. J., **34** (6), 1117–1134 (1993)].
9. A. M. Bikchentaev, "Ideal F-norms on C^*-algebras," Izv. Vyssh. Uchebn. Zaved. Mat., No. 5, 69–74 (2015) [Russian Math. (Iz. VUZ), **59** (5), 58–63 (2015)].
10. A. M. Bikchentaev, "Ideal F-norms on C^*-algebras. II," Izv. Vyssh. Uchebn. Zaved. Mat., No. 3, 90–96 (2019) (in Russian).
11. G. J. Murphy, *C^*-Algebras and Operator Theory* (Academic Press, Inc., Boston, MA, 1990; Faktorial, Moscow, 1997).
12. A. M. Bikchentaev, "Block projection operators in normed solid spaces of measurable operators," Izv. Vyssh. Uchebn. Zaved. Mat., No. 2, 86–91 (2012) [Russian Math. (Iz. VUZ), **56** (2), 75–79 (2012)].
13. C. A. Akemann, J. Anderson, and G. K. Pedersen, "Triangle inequalities in operator algebras," Linear Multilinear Algebra **11** (2), 167–178 (1982).

14. J. von Neumann, "Über Adjungierte Funktionaloperatoren," Ann. of Math. (2) **33** (2), 294–310 (1932).
15. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities* (2nd ed., Cambridge, Cambridge University Press, 1988; Russian transl. of the 1st ed. Inostr. Lit., Moscow, 1948).
16. A. M. Bikchentaev, "Minimality of convergence in measure topologies on finite von Neumann algebras," Mat. Zametki **75** (3), 342–349 (2004) [Math. Notes **75** (3), 315–321 (2004)].
17. A. M. Bikchentaev and O. E. Tikhonov, "Characterization of the trace by monotonicity inequalities," Linear Algebra Appl. **422** (1), 274–278 (2007).
18. S. M. Lozinskii, "Error estimate for numerical integration of ordinary differential equations. I," Izv. Vyssh. Uchebn. Zaved. Mat., No. 5, 52–90 (1958) (in Russian).
19. B. F. Bylov, R. É. Vinograd, D. M. Grobman, and V. V. Nemytskii, *Theory of Ljapunov Exponents and Its Application to Problems of Stability* (Nauka, Moscow, 1966) [in Russian].
20. A. I. Perov, I. D. Kostrub, O. I. Kleshchina and E. E. Dikarev, "Absolute logarithmic norm," Izv. Vyssh. Uchebn. Zaved. Mat., No. 4, 70–85 (2018) [Russian Math. (Iz. VUZ) **62** (4), 60–73 (2018)].
21. A. I. Perov and I. D. Kostrub, "On the spectral abscissa and the logarithmic norm," Mat. Zametki **101** (4), 562–575 (2017) [Math. Notes **101** (4), 677–687 (2017)].