DOI 10.1007/s10958-018-3946-2
Journal of Mathematical Sciences, Vol. 233, No. 5, September, 2018

FORMAL MATRICES AND RINGS CLOSE TO REGULAR

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Abstract. This paper contains new and known results on formal matrix rings close to regular. The main results are given with proofs.

1. Preliminaries

All rings are assumed to be associative and with nonzero identity element; all modules are assumed to be unitary. Let $R_1, R_2, \dots, R_n$ be rings and let M_{ij} be (R_i, R_j)-bimodules such that $M_{ii} = R_i$ for all $1 \le i, j \le n$. In addition, let $\varphi_{ijk}\colon M_{ij} \otimes_{R_j} M_{jk} \to M_{ik}$ be (R_i, R_k)-bimodule homomorphisms such that φ_{iij} and φ_{ijj} are canonical isomorphisms for all $1 \le i, j \le n$. We set $a \circ b = \varphi_{ijk}(a \otimes b)$ for $a \in M_{ij}$ and $b \in M_{jk}$. We denote by K the set of all $n \times n$ matrices (m_{ij}) with elements $m_{ij} \in M_{ij}$ for all $1 \le i, j \le n$. It is easy to verify that K is a ring with respect to ordinary operations of addition and of multiplication if and only if $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a \in M_{ik}$, $b \in M_{kl}$, and $c \in M_{lj}$, $1 \le i, k, l, j \le n$. The obtained ring K is called a *formal matrix ring* of order n; it is denoted by $K(\{M_{ij}\}\colon \{\varphi_{ikj}\})$. If

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

are formal matrix rings of order 2, then the ordered family $(R, S, M, N, \varphi, \psi)$ is called a *Morita context* or a *pre-equivalence situation*.

The formal matrix ring $K(\{M_{ij}\}\colon \{\varphi_{ikj}\})$ of order n, in which $M_{ij} = R$ for all $1 \le i, j \le n$, is called a *formal matrix ring over R of order n*; it is denoted by $K_n(R)$ or $K_n(R\colon \{\varphi_{ikj}\})$. For a formal matrix ring $K_n(R\colon \{\varphi_{ijk}\})$ over R of order n, we set $\eta_{ijk} = \varphi_{ijk}(1 \otimes 1)$ for all $1 \le i, j, k \le n$. Then $a \circ b = \varphi_{ijk}(a \otimes b) = \eta_{ijk} ab$ for all $a, b \in R$. For every $a \in R$, we have $a\eta_{ijk} = \varphi_{ijk}(a \otimes 1) = \varphi_{ijk}(1 \otimes a) = \eta_{ijk} a$. Therefore, $\eta_{ijk} \in C(R)$, and the following conditions hold:

(1) $\eta_{iij} = \eta_{ijj} = 1$, $1 \le i, j \le n$;
(2) $\eta_{ijk}\eta_{ikl} = \eta_{ijl}\eta_{jkl}$, $1 \le i, j, k, l \le n$.

The first condition holds, since φ_{iij} and φ_{ijj} are canonical isomorphisms. Since the operation $\circ$ is associative, we have $\eta_{ijk}\, \eta_{ikl}\, abc = \eta_{ijl}\, \eta_{jkl}\, abc$ for all $a, b, c \in R$. By setting $a = b = c = 1$, we obtain the second condition. For every family $\{\eta_{ijk} \mid 1 \le i, j, k \le n\}$ of central elements of R satisfying the first condition and the second condition, we can set $\varphi_{ijk}(a \otimes b) = \eta_{ijk} ab$ for all $a, b \in R$. It is directly verified that $K_n(R\colon \{\varphi_{ikj}\})$ is a formal matrix ring over R of order n. Therefore, the formal matrix ring $K_n(R\colon \{\varphi_{ikj}\})$ is uniquely defined by the family $\{\eta_{ijk} \mid 1 \le i, j, k \le n\}$ of central elements. In this case, the formal matrix ring $K_n(R\colon \{\varphi_{ikj}\})$ is denoted by $K_n(R\colon \{\eta_{ikj}\})$.

Let R be a ring and let $\beta_1, \dots, \beta_n \in C(R)$ with $n \ge 2$. We define η_{ijk} for all $1 \le i, j, k \le n$ by the relation

$$\eta_{ijk} = \begin{cases} 1 & \text{if } i = j \text{ or } j = k, \\ \beta_j & \text{if } i,\ j,\ k \text{ are distinct}, \\ \beta_i\beta_j & \text{if } i = k \ne j. \end{cases}$$

Translated from Fundamentalnaya i Prikladnaya Matematika, Vol. 21, No. 1, pp. 5–21, 2016.

 1072–3374/18/2335–0604 © 2018 Springer Science+Business Media, LLC

It is directly verified that the family $\{\eta_{ijk} \mid 1 \le i, j, k \le n\}$ satisfies conditions (1) and (2) and, consequently, defines a formal matrix ring over R of order n. We denote by $\mathbb{M}_{\beta_1,\ldots,\beta_n}(R)$ the formal matrix ring $K_n(R\colon \{\varphi_{ijk}\})$ defined by the set $\{\eta_{ijk}\}$. Therefore, $\mathbb{M}_{\beta_1,\ldots,\beta_n}(R)$ coincides with the set of all matrices of order n over R with ordinary operation of addition and operation of multiplication defined as follows. For two matrices of order n over R, (a_{ij}) and (b_{ij}),

$$(a_{ij})(b_{ij}) = (c_{ij}), \quad \text{where} \quad c_{ij} = \sum_{k=1}^{n} \beta_i^{\delta_{ij}-\delta_{ik}} \beta_k^{1-\delta_{jk}} a_{ik} b_{kj}.$$

Formal matrix rings and their modules have been intensively studied lately. Modules over formal matrix rings are considered in [10–13, 16]. Various ring properties of formal matrix rings are studied in [12, 18, 19, 21]. Grothendieck groups and Whitehead groups of formal matrix rings are studied in [18]. The isomorphism problem for formal matrix rings is studied in [15, 21]. The ideal lattice of such rings is studied in [7].

Let

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix},$$

X be a right R-module, Y be a right S-module, and let us have an R-module homomorphism $f\colon Y \otimes_S N \to X$ and an S-module homomorphism $g\colon X \otimes_R M \to Y$. We set $yn := f(y \otimes n)$, $xm := g(x \otimes m)$ and require that the relations $(yn)m = y(nm)$ and $(xm)n = x(mn)$ hold for all $x \in X$, $y \in Y$, $m \in M$, and $n \in N$. In this case, the group of vector rows (X, Y) is naturally provided by the structure of a right K-module. It is easy to show that any right K-module can be considered as a module of vector rows. Homomorphisms of K-modules can be considered as pairs consisting of an R-homomorphism and an S-homomorphism. Namely, if $\Gamma\colon (X, Y) \to (X', Y')$ is a homomorphism, then there exist an R-homomorphism $\alpha\colon X \to X'$ and an S-homomorphism $\beta\colon Y \to Y'$ such that $\Gamma(x, y) = \big(\alpha(x), \beta(y)\big)$. In addition, the relations $\alpha(yn) = \beta(y)n$ and $\beta(xm) = \alpha(x)m$ hold for all $x \in X$, $y \in Y$, $m \in M$, and $n \in N$.

We recall some constructions from [16]. Let A be a nonzero right R-module. We denote by $H(A)$ the right K-module $\big(A, \mathrm{Hom}_R(N, A)\big)$ such that homomorphisms of module multiplication are the mapping $A \otimes M \to \mathrm{Hom}_R(N, A)$, $a \otimes m \mapsto \big(n \mapsto a(mn)\big)$ and the mapping $\mathrm{Hom}_R(N, A) \otimes N \to A$, $f \otimes n \mapsto f(n)$. We denote by $T(A)$ the right K-module $(A, A \otimes M)$ such that homomorphisms of the module multiplication are the identity automorphism $A \otimes M \to A \otimes M$ and the mapping $(A \otimes M) \otimes N \to A$, $(a \otimes m)n = amn$.

The Jacobson radical and the largest regular ideal of the ring R is denoted by $J(R)$ and $\mathrm{Reg}(R)$, respectively. For a right R-module M, we denote by $J(M)$ the Jacobson radical of M.

Let

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

be a formal matrix ring. A bimodule M is said to be *N-regular* (*right N-fully idempotent*) if $m \in mNm$ (respectively, $m \in mNmS$) for every $m \in M$. M-regular bimodules and right M-fully idempotent bimodules N are similarly defined.

2. Formal Matrix Rings Close to Regular

For a formal matrix ring $K = K(\{M_{ij}\}\colon \{\varphi_{ikj}\})$ of order n and for each $1 \le i, j \le n$, we denote by $\mathrm{Reg}(M_{ij})$ the set of the form $\{m \in M_{ij} \mid m \in mM_{ji}m\}$.

Theorem 2.1 ([25]). *For a formal matrix ring $K = K(\{M_{ij}\}\colon \{\varphi_{ikj}\})$ of order n, we have*

$$\mathrm{Reg}(K) = \{r \in K \mid M_{ti} r_{ij} M_{js} \subset \mathrm{Reg}(M_{ts})\}.$$

Corollary 2.2. *For a formal matrix ring $K = K(\{M_{ij}\}\colon \{\varphi_{ikj}\})$ of order n, the following conditions are equivalent*:

(1) *K is a regular ring*;

(2) *for each subscript pair $1 \le i, j \le n$ and each $m \in M_{ij}$, we have that $m \in mM_{ji}m$.*

A ring R is called an I_0-ring if for every arbitrary $r \in R \setminus J(R)$, there exists an element $s \in R \setminus \{0\}$ with $s = srs$.

Theorem 2.3 ([25]). *For a formal matrix ring $K = K(\{M_{ij}\}\colon \{\varphi_{ikj}\})$ of order n, the following conditions are equivalent*:

(1) *K is an I_0-ring*;
(2) *for every $1 \le i \le n$, the ring R_i is an I_0-ring.*

A ring R is said to be *right* (*left*) *fully idempotent* if $I^2 = I$ for every right (respectively, left) ideal I of the ring R. If the relation $I^2 = I$ holds for every ideal I of the ring R, then the ring R is said to be *fully idempotent*. An element r of the ring R is said to be *right fully idempotent* if $rR = rRrR$. For a ring R, an ideal I of R is said to be *right fully idempotent* if every element of I is right fully idempotent. By [23, 12.17], every ring R has the largest fully idempotent right ideal which is denoted by $I(R)$.

For a formal matrix ring $K = K(\{M_{ij}\}\colon \{\varphi_{ikj}\})$ of order n and for each $1 \le i, j \le n$, we denote by $I(M_{ij})$ the set of the form $\{m \in M_{ij} \mid m \in mM_{ji}mR_j\}$. We denote by re_{ij} the element of the formal matrix ring $K = K(\{M_{ij}\}\colon \{\varphi_{ikj}\})$ such that the component located at the intersection of the ith row and the jth column is equal to r and the remaining components are equal to zero.

Theorem 2.4. *For an arbitrary formal matrix ring $K = K(\{M_{ij}\}\colon \{\varphi_{ikj}\})$ of order n, we have*

$$I(K) = \{r \in K \mid M_{ti}r_{ij}M_{js} \subset I(M_{ts})\}.$$

Proof. We denote by I' the set $\{r \in K \mid M_{ti}r_{ij}M_{js} \subset I(M_{ts})\}$. It follows from the proof of Theorem 5.3 in [4] that I' is an ideal of the ring K. We show that $I(K) \subset I'$. Since $e_{ii}Ke_{jj}I(K)e_{ss}Ke_{tt} \subset I(K)$, it is sufficient to show that all components of an arbitrary ideal $I(K)$ are right fully idempotent. Let $a \in I(R)$. Therefore, for each pair of subscripts i, j, we have

$$e_{ii}ae_{jj} = e_{ii}ae_{jj}\biggl(\sum_{k=1}^{m} b_k e_{ii}ae_{jj}c_k\biggr) = e_{ii}ae_{jj}\biggl(\sum_{k=1}^{m} b_k e_{ii}ae_{jj}c_k e_{jj}\biggr), \quad a_{ij} = a_{ij}\biggl(\sum_{k=1}^{m}(b_k)_{ji}a_{ij}(c_k)_{jj}\biggr).$$

We show that $I' \subset I(K)$. We assume that the ideal I' contains an element which is not right fully idempotent. In I', we choose an element r such that r is not fully idempotent and the row $r_{11}, \dots, r_{1n}, \dots, r_{n1}, \dots, r_{nn}$ has the largest number of the first zeros. Let $r_{i_0j_0}$ be the first nonzero element in this row. We have

$$r_{i_0j_0} = r_{i_0j_0}\sum_k a_k r_{i_0j_0} b_k,$$

where $a_k \in M_{j_0i_0}$, $b_k \in R_{j_0j_0}$ for every k. Then

$$r - r\sum_k a_k e_{j_0i_0} r b_k e_{j_0j_0} = \sum_{i,j} r_{ij}e_{ij} - \biggl(\sum_{i,j} r_{ij}e_{ij}\biggr)\biggl(\sum_k a_k e_{j_0i_0}\biggl(\sum_{i,j} r_{ij}e_{ij}\biggr) b_k e_{j_0j_0}\biggr) = \sum_{i,j} g_{ij}e_{ij},$$

where $g_{ij} = r_{ij}$, provided $j \ne j_0$, and

$$g_{ij_0} = r_{ij_0} - \sum_k r_{ij_0} a_k r_{i_0j_0} b_k.$$

It is clear that $g_{ij} = 0$ if either $i < i_0$, or $i = i_0$, $j < j_0$, or $i = i_0$, $j = j_0$. Consequently, it follows from the choice of arbitrary r that the element

$$r - r\sum_k a_k e_{j_0i_0} r b_k e_{j_0j_0}$$

is right fully idempotent. Then it follows from [4, Lemma 5.2] that any r is fully idempotent; this contradicts our original assumptions. $\square$

Corollary 2.5. *For a formal matrix ring $K = K(\{M_{ij}\}\colon \{\varphi_{ikj}\})$ of order n, the following conditions are equivalent*:

(1) *K is a right fully idempotent ring*;

(2) *for each subscript pair $1 \le i, j \le n$ and any $m \in M_{ij}$, we have $m \in mM_{ji}mR_j$.*

The following assertion follows from Corollaries 2.2 and 2.5.

Corollary 2.6. *For a formal matrix ring $K = K(\{M_{ij}\}\colon \{\varphi_{ikj}\})$ of order n such that the ring R_i is commutative for each $1 \le i \le n$, the following conditions are equivalent*:

(1) *K is a right fully idempotent ring*;

(2) *K is a left fully idempotent ring*;

(3) *K is a regular ring.*

Corollary 2.7. *Let $K = K_n(R\colon \{\eta_{ikj}\})$ be a formal matrix ring over R of order n.*

(1) *If every element of the set $\{\eta_{ikj}\}$ is not a zero-divisor, then we have*

(a) $I(K) = \begin{pmatrix} I(R) & I(R) & \dots & I(R) \\ I(R) & I(R) & \dots & I(R) \\ \dots & \dots & \dots & \dots \\ I(R) & I(R) & \dots & I(R) \end{pmatrix}$;

(b) $\mathrm{Reg}(K) = \begin{pmatrix} \mathrm{Reg}(R) & \mathrm{Reg}(R) & \dots & \mathrm{Reg}(R) \\ \mathrm{Reg}(R) & \mathrm{Reg}(R) & \dots & \mathrm{Reg}(R) \\ \dots & \dots & \dots & \dots \\ \mathrm{Reg}(R) & \mathrm{Reg}(R) & \dots & \mathrm{Reg}(R) \end{pmatrix}$.

(2) *K is right fully idempotent if and only if R is right fully idempotent and $\{\eta_{ikj}\} \subset U(R)$.*

(3) *K is regular if and only if R is regular and $\{\eta_{ikj}\} \subset U(R)$.*

Proof. For arbitrary elements $r_1, r_2 \in R$, we denote by $r_1 *_{ijk} r_2$ the expression $\phi_{ijk}(r_1 \otimes r_2)$.

(1). We show that the relation from (a) holds. We denote by I' the right part of (a). The inclusion $I(K) \subset I'$ directly follows from Theorem 2.4. We show that the converse inclusion holds. Let $r \in I'$. For arbitrary $1 \le i, j, s, t \le n$ and $a, b \in R$, we have the relations

$$ar_{ij}b\eta_{sjt}\eta_{sij}\eta_{tst} = ar_{ij}b\eta_{sjt}\eta_{sij}\eta_{tst} \sum_{1 \le l \le k} c_l a r_{ij} b \eta_{sjt}\eta_{sij}\eta_{tst} d_l,$$

$$ar_{ij}b\eta_{sjt}\eta_{sij} = ar_{ij}b\eta_{sjt}\eta_{sij} \sum_{1 \le l \le k} c_l a r_{ij} b \eta_{sjt}\eta_{sij}\eta_{tst} d_l,$$

$$(a *_{sij} r_{ij}) *_{sjt} b = \big((a *_{sij} r_{ij}) *_{sjt} b\big) \sum_{1 \le l \le k} \Big(c_l *_{tst} \big((a *_{sij} r_{ij}) *_{sjt} b\big)\Big) d_l.$$

Then $r \in I(K)$ by Theorem 2.4. The relation from (b) is similarly proved.

(2). $\Longrightarrow$. It follows from Corollary 2.5 that for arbitrary $1 \le i, j \le n$, we have

$$1 = \sum_{1 \le t \le k} r_t *_{iji} s_t = \eta_{iji} \sum_{1 \le t \le k} r_t s_t.$$

Therefore, elements of the form η_{iji} are invertible in R. For every $1 \le k \le n$, we have $\eta_{iji} = \eta_{ijk}\eta_{jik}$. Therefore, $\{\eta_{ikj}\} \subset U(R)$.

$\Longleftarrow$. This implication directly follows from the first part of the original theorem.

(3). The proof is similar to the proof of (2). $\square$

3. Formal Matrix Rings Which Are Semi-Artinian or max-Rings

A ring R is said to be *right semi-Artinian* if every nonzero right R-module contains a simple submodule. If every nonzero right R-module contains a maximal submodule, then the ring R is called a *right* max-*ring*.

Theorem 3.1 ([2, Theorem 4.2]). *For a formal matrix ring*

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix},$$

the following conditions are equivalent:

(1) *K is a right semi-Artinian ring*;

(2) *R and S are right semi-Artinian rings.*

Proof. (1) $\Longrightarrow$ (2). Let A be a nonzero right R-module. It follows from the assumption that the right K-module $H(A) = \big(A, \mathrm{Hom}_R(N, A)\big)$ contains a simple submodule (X, Y). If $X = 0$, then $f(N) = fN = 0$ for every $f \in Y$. Consequently, $Y = 0$, which is impossible. Therefore, $X \neq 0$ and it follows from simplicity of the module (X, Y) that X is a simple submodule of the R-module A. It follows from the above argument that every nonzero right R-module contains a simple submodule; consequently, R is a right semi-Artinian ring. With the use of a similar argument, we can show that S is a right semi-Artinian ring.

(2) $\Longrightarrow$ (1). Let (A, B) be a right K-module and let (A_0, B_0) be a nonzero submodule in (A, B). Without loss of generality, we can assume that $A_0 \neq 0$. Since R and S are right semi-Artinian rings, $\mathrm{Soc}(A)$ is essential in A and $\mathrm{Soc}(B)$ is essential in B. Then the module A_0 contains a simple submodule aR, where $a \in A_0$. If $aRM = aM = 0$, then $(aR, 0)$ is a simple submodule of the K-module (A_0, B_0). If $aM \neq 0$, then it follows from the essentiality of the submodule $\mathrm{Soc}(B)$ in the module B that the S-module aM contains a simple submodule bS, where $b \in B_0$. It is clear that the element b has the form $b = am$, where $m \in M$. If $bN = 0$, then $(0, bS)$ is a simple submodule of the K-module (A_0, B_0). If $bN \neq 0$, then $bN = amN \subset aMN$ is a nonzero submodule of the simple module aR. Consequently, $bN = aR$. Since $aRM = bNM$ and the module bS is simple, we have $aRM = bS$. Since aR is a simple R-module, bS is a simple S-module and $aRM = bS$, $bSN = aR$, we have that (aR, bS) is a simple submodule in the K-module (A_0, B_0). □

Theorem 3.2 ([2, Theorem 4.3]). *For a formal matrix ring*

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix},$$

the following conditions are equivalent:

(1) *K is a right* max-*ring*;

(2) *R and S are right* max-*rings.*

Proof. (1) $\Longrightarrow$ (2). Let A be a nonzero right R-module. It follows from the assumption of (1) that the right K-module $T(A) = (A, A \otimes M)$ contains a maximal submodule (X, Y). If $X = A$, then it is clear that $Y = A \otimes M$, which is impossible. Therefore, $X \neq A$. It follows from the maximality of the submodule (X, Y) that X is a maximal submodule of the R-module A. It follows from the above argument that every nonzero right R-module contains a maximal submodule; consequently, R is a right max-ring. By the use of a similar argument, we can show that S is a right max-ring.

(2) $\Longrightarrow$ (1). Let (A, B) be a nonzero right K-module and let (X, Y) be a proper submodule of the module (A, B). Since R and S are right max-rings, nonzero factor modules of the modules A and B contain maximal submodules. If $(A/X)M \neq B/Y$, then B has a maximal submodule Y' such that $(A/X)M \subset Y'/Y$. In this case, it is easy to see that the module $(A/X, Y'/Y)$ is a maximal submodule of the module $(A/X, B/Y)$. If $(B/Y)M \neq A/X$, then we can use a similar argument to show that the module $(A/X, B/Y)$ contains a maximal submodule. We assume that $(A/X)M = B/Y$ and $(B/Y)N = A/X$. The module A has a maximal submodule A_0 with $X \subset A_0$. In the module B, we consider a submodule B_0 such that

$$B_0/Y = \{\bar{b} \in B/Y \mid \bar{b}N \subset A_0/X\}.$$

It is clear that $B_0/Y \neq B/Y$ and $(A_0/X)M \subset B_0/Y$. We show that B_0 is a maximal submodule of the S-module B. Let $\bar{b} \notin B_0/Y$. Then $\bar{b}N \not\subseteq A_0/X$; consequently,

$$\bar{b}N + A_0/X = A/X, \quad \bar{b}NM + (A_0/X)M = (A/X)M = B/Y.$$

Therefore, the relation $\bar{b}S + B_0/Y = B/Y$ holds for any arbitrary $\bar{b} \in (B/Y) \setminus (B_0/Y)$; consequently, $(B/Y)/(B_0/Y)$ is a simple S-module. Since

$$(A_0/X)M \subset B_0/Y, \quad (B_0/Y)N \subset A_0/X,$$

we have that $(A_0/X, B_0/Y)$ is a submodule of the T-module $(A/X, B/Y)$. It is easy to see that the right K-module $\big((A/X)/(A_0/X), (B/Y)/(B_0/Y)\big)$ has length at most two. Consequently, the submodule (X, Y) is contained in a maximal submodule of the module (A, B). □

In [16], right perfect formal matrix rings

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

are described in the case where $MN = 0$ and $NM = 0$; in [19], these rings are described in the case where the right modules M_S and N_R are finitely generated.

Corollary 3.3. *The following assertions hold.*

(1) *A formal matrix ring $K = K(\{M_{ij}\} \colon \{\varphi_{ikj}\})$ of order n is right semi-Artinian (right* max*-ring) if and only if R_i is a right semi-Artinian ring (right* max*-ring) for every $1 \le i \le n$.*

(2) *A formal matrix ring $K = K(\{M_{ij}\} \colon \{\varphi_{ikj}\})$ of order n is right perfect if and only if R_i is a right perfect ring for every $1 \le i \le n$.*

Proof. (1). The assertion directly follows from Theorems 3.1 and 3.2.

(2). The assertion follows from [23, 6.48] and Theorem 3.1. □

4. Formal Matrix Rings Which Are V-Rings or SV-Rings

A ring over which every simple right module is injective is called a right V-ring. A right semi-Artinian, right V-ring is called a right SV-ring.

Lemma 4.1. *Let*

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

be a formal matrix ring, the module M be a right N-fully idempotent, and let the module N be right M-fully idempotent.

(1) *If (A_0, B_0) is a submodule of the K-module (A, B), then the following conditions are equivalent:*
   (a) *(A_0, B_0) is an essential submodule in the K-module (A, B);*
   (b) *the submodule A_0 is essential in the R-module A and the submodule B_0 is essential in the S-module B.*

(2) *If A is a simple right R-module, then the right K-module $T(A)$ is simple.*

(3) *If A is a simple right R-module and $T(A)$ is an injective right K-module, then the module A is injective.*

Proof. (1). (a) $\Longrightarrow$ (b). Let a be a nonzero element of the module A. Since (A_0, B_0) is an essential submodule of K-module (A, B), for some arbitrary

$$\begin{pmatrix} r & m \\ n & s \end{pmatrix}$$

of K, we have that

$$(a, 0)\begin{pmatrix} r & m \\ n & s \end{pmatrix} \in (A_0, B_0) \setminus \{(0, 0)\}.$$

Then $ar \in A_0$ and $am \in B_0$. If $ar \neq 0$, then $aR \cap A_0 \neq 0$. We assume that $ar = 0$. Then $am \neq 0$. It follows from the assumption of (1) that $m \in mNmS$. Therefore, $amN \neq 0$. Since $B_0N \subset A_0$, we have that $amN \subset A_0 \cap aR$. It follows from the above argument that $aR \cap A_0 \neq 0$. Therefore, the submodule A_0 is essential in the module A. It is similarly proved that the submodule B_0 is essential in the module B.

(b) $\Longrightarrow$ (a). This implication is directly verified.

(2). Let $A = aR$. We show that $T(aR)$ is a simple right K-module. If $aR \otimes M = 0$, then it is clear that the module $T(aR)$ is simple. We assume that $aR \otimes M \neq 0$. We show that $aR \otimes M = a \otimes M$ is a simple right S-module. Let $a \otimes m$ be an arbitrary nonzero element of $a \otimes M$. If $(a \otimes m)N = 0$, then $amN = 0$. Since M is right N-fully idempotent, we have that

$$m = m \sum_{i=1}^{k} n_i m s_i,$$

where $s_i \in S$, $n_i \in N$ for every $1 \leq i \leq k$. Then

$$a \otimes m = a \otimes \left(m \sum_{i=1}^{k} n_i m s_i \right) = \sum_{i=1}^{k} (amn_i \otimes ms_i) = 0,$$

which contradicts the choice of an arbitrary $a \otimes m$. Thus, $(a \otimes m)N \neq 0$ and it follows from the simplicity of the module aR that

$$aR = (a \otimes m)N = amN, \quad aR \otimes M = amN \otimes M = a \otimes mNM \subset a \otimes mS.$$

For arbitrary nonzero element $a \otimes m$ of right S-module $a \otimes M$, we have that $a \otimes M = a \otimes mS$. Consequently, the module $a \otimes M$ is simple. Since $(a \otimes M)N = aR$, we have that $(aR)M = aR \otimes M$ and $T(aR)$ is a simple right K-module.

(3). Let B be a right R-module, which is an essential extension of the module A. The embedding $\varepsilon \colon A \to B$ of the module A into the module B induces the homomorphism of K-modules $T(\varepsilon) \colon T(A) \to T(B)$. It follows from (2) that $T(A)$ is a simple module and $T(\varepsilon) \neq 0$. Therefore, $T(\varepsilon)$ is a monomorphism. By the assumption of (3), $T(A)$ is an injective right K-module. Therefore, $T(A)$ is a direct summand of the module $T(B)$. Then A is a direct summand of the module B; consequently, $A = B$. Therefore, A is an injective module. □

Theorem 4.2. *Let*

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

be a formal matrix ring. Then the following conditions are equivalent:

(1) *K is a right V-ring*;

(2) *R and S are right V-rings, the module M is right N-fully idempotent, and the module N is right M-fully idempotent.*

Proof. (1) $\Longrightarrow$ (2). The property that the module M is a right N-fully idempotent and the module N is right M-fully idempotent follows from [4, Corollaries 6.9 and 7.8].

Let A be a simple right R-module. It follows from Lemma 4.1 that the right K-module $T(A)$ is simple. Since K is a right V-ring, $T(A)$ is an injective module. Then it follows from Lemma 4.1 that the R-module A is injective. Therefore, R is a V-ring. It can be similarly proved that S is a V-ring.

(2) $\Longrightarrow$ (1). Let (B_1, B_2) be a K-module and let (A_1, A_2) be a simple essential submodule in (B_1, B_2). Then it follows from Lemma 4.1 that A_i is an essential submodule of the module B_i and A_i is either a simple module or the zero module for every $1 \leq i \leq 2$. Then it follows from the assumption of (2) that $A_1 = B_1$ and $A_2 = B_2$. Therefore, K is a right V-ring. □

Corollary 4.3. *Let*

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

be a formal matrix ring. If R and S are commutative rings, then the following conditions are equivalent:

(1) *K is a right V-ring*;

(2) *K is a regular ring.*

Proof. (1) $\Longrightarrow$ (2). It follows from Theorem 4.2 and [22, 22.4] that R and S are regular rings, the module M is right N-fully idempotent, and the module N is right M-fully idempotent. For an arbitrary element $m \in M$, we have

$$m = m\sum_{i=1}^{k} n_i m s_i = m\sum_{i=1}^{k} s_i n_i m = m\Big(\sum_{i=1}^{k} s_i n_i\Big)m,$$

where $s_i \in S$, $n_i \in N$ for every $1 \le i \le k$. Therefore, the module M is N-regular. We can similarly show that the module N is M-regular. Then it follows from Corollary 2.2 that K is a regular ring.

(2) $\Longrightarrow$ (1). The implication follows from Theorem 4.2 and [22, 22.4]. □

Theorem 4.4. *Let*

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

be a formal matrix ring. Then the following conditions are equivalent:

(1) *K is a right SV-ring*;
(2) *R and S are right SV-rings, the module M is N-regular, and the module N is M-regular*;
(3) *R and S are right SV-rings, the module M is right N-fully idempotent, and the module N is right M-fully idempotent.*

Proof. (1) $\Longrightarrow$ (2). It follows from Theorems 3.1 and 4.2 that R and S are right SV-rings. It follows from [5, Theorem 2.7] that K is a regular ring. Consequently, the module M is N-regular and the module N is M-regular.

(2) $\Longrightarrow$ (3). This assertion is directly verified.

(3) $\Longrightarrow$ (1). The implication follows from Theorems 3.1 and 4.2. □

Remark. Theorems 4.2 and 4.4 were first published in [1].

Corollary 4.5. *Let*

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

be a formal matrix ring. If K is a regular ring, then the following conditions are equivalent:

(1) *K is a right SV-ring*;
(2) *R and S are right SV-rings.*

Corollary 4.6. *Let P be a finitely generated, projective, right R-module and $S = \mathrm{End}_R(P)$. Then the following assertions hold*:

(1) *if R is a right V-ring, then S is a right V-ring*;
(2) *if R is a right* max*-ring, then S is a right* max*-ring*;
(3) *if R is a right SV-ring, then S is a right SV-ring*;
(4) *if R is a right semi-Artinian ring, then S is a right semi-Artinian ring.*

Proof. (1). There exists a projective right R-module P' such that there exists an isomorphism $R^n \cong P \oplus P'$ for some positive integer n. Therefore, for some idempotent $e \in \mathrm{M}_n(R)$, we have the ring isomorphism $e\mathrm{M}_n(R)e \cong S$. Since the property to be a right V-ring is invariant in the sense of Morita, $\mathrm{M}_n(R)$ is a right V-ring. Then it follows from Theorem 4.2 that S is a right V-ring.

The proofs of assertions of (2), (3), and (4) are similar to the proof of (1) and use of Theorems 3.2, 4.4, and 3.1, respectively. □

Corollary 4.7. *Let $K = K_n(R\colon \{\eta_{ikj}\})$ be a formal matrix ring over R of order n. Then*

(1) *K is a right V-ring if and only if R is a right V-ring and $\{\eta_{ikj}\} \subset U(R)$*;
(2) *K is a right semi-Artinian ring if and only if R is a right semi-Artinian ring*;
(3) *K is a right* max*-ring if and only if R is a right* max*-ring*;
(4) *K is a right SV-ring if and only if R is a right SV-ring and $\{\eta_{ikj}\} \subset U(R)$.*

If M is a right R-module, then we denote by

$$\begin{pmatrix} R & \mathrm{Hom}_R(M,R) \\ M & \mathrm{End}_R(M) \end{pmatrix}$$

the formal matrix ring such that the bimodule homomorphisms $\varphi\colon \mathrm{Hom}_R(M,R) \otimes_{\mathrm{End}_R(M)} M \to R$, $\psi\colon M \otimes_R \mathrm{Hom}_R(M,R) \to \mathrm{End}_R(M)$ are defined by the relations $m\otimes f \mapsto \bigl(m' \mapsto mf(m')\bigr)$, $f\otimes m \mapsto f(m)$, respectively.

Corollary 4.8. *Let R be a right SV-ring and let M be a finitely generated right R-module. Then the following conditions are equivalent*:

(1) $K = \begin{pmatrix} R & \mathrm{Hom}_R(M,R) \\ M & \mathrm{End}_R(M) \end{pmatrix}$ *is a right SV-ring*;

(2) *M is a projective right R-module.*

Proof. (1) $\Longrightarrow$ (2). It follows from Theorem 4.4 that the module M is $\mathrm{Hom}_R(M,R)$-regular. Then it follows from [24, Corollary 1.7] that M is a projective module.

(2) $\Longrightarrow$ (1). It follows from [5, Theorem 2.7] that R is a regular ring. Since the module M is finitely generated, it follows from the assumption of (2) that $\mathrm{Hom}_R(M,R)$ is a projective left R-module. Consequently, it follows from [24, Theorem 2.8] that the right R-module M and the left R-module $\mathrm{Hom}_R(M,R)$ are regular. It follows from the canonical isomorphism of right R-modules $M \cong \mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R)$ that the module M is $\mathrm{Hom}_R(M,R)$-regular and the module $\mathrm{Hom}_R(M,R)$ is M-regular. Then it follows from Corollary 4.6 and Theorem 4.4 that K is a right SV-ring. □

5. Clean Rings and the Isomorphism Problem for Formal Matrix Rings

A ring A is said to be *clean* if any element of A is a sum of an invertible element and an idempotent. A ring R is said to be *strongly clean* (*uniquely strongly clean*) if every element r of R can be represented (respectively, uniquely represented) in the form $r = e + u$, where $e = e^2$, $u \in U(R)$, $eu = ue$. A ring R is said to be *strongly nil-clean* if every element r of R can be written in the form $r = e + n$, where $e = e^2$, n is a nilpotent element, and $en = ne$. It follows from [9, Propositions 2.5 and 2.6 and Corollaries 3.11 and 3.26] that every strongly nil-clean ring is a uniquely strongly clean ring.

Theorem 5.1. *Let F be a field. Then a formal matrix ring $K = K_n(F\colon \{\eta_{ikj}\})$ is nil-clean if and only if $F \cong F_2$.*

Proof. It is easy to show that $K/J(K) \cong \mathrm{M}_{n_1}(F) \times \cdots \times \mathrm{M}_{n_k}(F)$, where $n_1 + \cdots + n_k = n$. Then the assertion of the original theorem follows from [6, Theorem 3] and [9, Theorem 3.15]. □

Theorem 5.2. *Let R be an arbitrary ring. Then a formal matrix ring $K = K_n(R\colon \{\eta_{ikj}\})$ is strongly nil-clean if and only if R is a strongly nil-clean ring and every element of the set $\{\eta_{iji} \mid 1 \le i,j \le n,\ i \ne j\}$ is nilpotent.*

Proof. $\Longrightarrow$. Since every strongly nil-clean ring is a uniquely strongly clean ring, it follows from [8, Corollary 18] that $K/J(K)$ is a Boolean ring and, consequently, $\{\eta_{iji} \mid 1 \le i,j \le n,\ i \ne j\} \subset J(R)$. It follows from [9, Corollary 3.26] that R is a strongly nil-clean ring. In addition, it follows from [9, Corollary 3.17] that $J(R)$ is a nil-ideal.

$\Longleftarrow$. It is clear that

$$J(K) = \begin{pmatrix} J(R) & R & \dots & R \\ R & J(R) & \dots & R \\ \dots & \dots & \dots & \dots \\ R & R & \dots & J(R) \end{pmatrix}.$$

By [9, Corollary 3.17] $J(R)$ is a nil-ideal and every element of the set $\{\eta_{iji} \mid 1 \le i,j \le n,\ i \ne j\}$ is nilpotent. Therefore, it is easy to see that every matrix in $J(K)$ is a nilpotent. Since

$$K/J(K) \cong R/J(R) \times \cdots \times R/J(R),$$

we have that $K/J(K)$ is a strongly nil-clean ring, and it follows from [9, Corollary 3.22] that K is a strongly nil-clean ring. □

The following hypothesis is proved in [20] for $n = 2$.

Hypothesis. Let R be a commutative local ring. If every element of the set $\{\eta_{iji} \mid 1 \le i, j \le n,\ i \ne j\}$ is nilpotent, then the formal matrix ring $K = K_n(R\colon \{\eta_{ikj}\})$ is strongly clean.

The isomorphism problem for formal matrix rings has been intensively studied lately; the problem has the following formulation. For two given families $\{\beta_1, \beta_2, \dots, \beta_n\}$ and $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ of elements of the commutative ring R, find conditions under which we have the isomorphism

$$\mathbb{M}_{\beta_1,\beta_2,\dots,\beta_n}(R) \cong \mathbb{M}_{\gamma_1,\gamma_2,\dots,\gamma_n}(R).$$

The study of the isomorphism problem is initiated in [15]. The following theorem is proved in this paper.

Theorem 5.3. *Let R be a commutative ring and let s and t be two elements in R such that at least one of the elements is not a zero-divisor. The rings K_s and K_t are isomorphic to each other if and only if there exist an invertible element $v \in R$ and an automorphism α of the ring R such that $t = v\alpha(s)$.*

The isomorphism problem is studied in [2, 3, 17, 19–21]. Below, we provide results for formal matrix rings of the form $\mathbb{M}_{\beta_1,\beta_2,\dots,\beta_n}(R)$ obtained in [2].

Theorem 5.4. *Let R be a commutative ring, $n \ge 3$, $\beta, \gamma_1, \dots, \gamma_n \in R$, and let $\operatorname{ann}_R(\beta) \subseteq J(R)$. Then*

$$\mathbb{M}_{\underbrace{\beta, 0, \dots, 0}_{n}}(R) \cong \mathbb{M}_{\gamma_1,\gamma_2,\dots,\gamma_n}(R)$$

if and only if $\gamma_i = \alpha(\beta)v_i a_i$ for all $i = \overline{1,n}$, where $\alpha \in \operatorname{Aut}(R)$, $v_i \in U(R)$, and $1 = a_1 + a_2 + \cdots + a_n$ is the decomposition of the identity element into a sum of orthogonal idempotents a_i.

Corollary 5.5. *Let R be a commutative ring, $n \ge 3$, $\beta, \gamma_1, \dots, \gamma_n \in R$ and let*

$$\mathbb{M}_{\underbrace{\beta, 0, \dots, 0}_{n}}(R) \cong \mathbb{M}_{\gamma_1,\gamma_2,\dots,\gamma_n}(R).$$

Then the following assertions hold.

1. *If β is not a zero-divisor in the ring R, then there exist $\alpha \in \operatorname{Aut}(R)$, $v_1, \dots, v_n \in U(R)$, and the decomposition of the identity element $1 = a_1 + a_2 + \cdots + a_n$ such that $\gamma_i = \alpha(\beta)v_i a_i$, $i = \overline{1,n}$.*
2. *If R is a domain, then there exist $\alpha \in \operatorname{Aut}(R)$ and $v \in U(R)$ such that $\gamma_i = \alpha(\beta)v$ and $\gamma_j = 0$ if $i \ne j$ for some $1 \le i \le n$.*

Theorem 5.6. *Let R be a commutative ring, $n \ge 3$, $\beta, \gamma_1, \dots, \gamma_n \in R$, and $\operatorname{ann}_R(\beta^2) \subseteq J(R)$. Then*

$$\mathbb{M}_{\underbrace{\beta, \beta, \dots, \beta}_{n}}(R) \cong \mathbb{M}_{\gamma_1,\gamma_2,\dots,\gamma_n}(R)$$

if and only if $\gamma_i = \alpha(\beta)v_i$ for all $i = \overline{1,n}$, where $\alpha \in \operatorname{Aut}(R)$ and $v_i \in U(R)$.

Theorem 5.7. *Let R be a commutative ring such that $Z(R) \subseteq J(R)$, $n \ge 3$, and $\beta, \gamma_1, \dots, \gamma_n \in R$. Then*

$$\mathbb{M}_{\underbrace{\beta, \beta, \dots, \beta}_{n}}(R) \cong \mathbb{M}_{\gamma_1,\gamma_2,\dots,\gamma_n}(R)$$

if and only if $\gamma_i = \alpha(\beta)v_i$ for all $i = \overline{1,n}$, where $\alpha \in \operatorname{Aut}(R)$ and $v_i \in U(R)$.

Corollary 5.8 ([21, Theorem 18]). *Let R be a commutative ring with $Z(R) \subseteq J(R)$ and let $n \ge 3$. Then*

$$\mathbb{M}_{\underbrace{\beta, \beta, \dots, \beta}_{n}}(R) \cong \mathbb{M}_{\gamma,\gamma,\dots,\gamma}(R)$$

if and only if $\gamma = \alpha(\beta)v$, where $\alpha \in \operatorname{Aut}(R)$ and $v \in U(R)$.

Hypothesis. Let R be a division ring and let $K = K_n(R\colon \{\eta_{ikj}\})$ be an arbitrary formal matrix ring over R. Then we have the isomorphism

$$K \cong K_n(R\colon \{\theta_{ikj}\}),$$

where $\{\theta_{ikj}\} \subset \{0, 1\}$.

This study is supported by the Russian Science Foundation (project 16-11-10013).

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