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ESSENTIALLY INVERTIBLE MEASURABLE OPERATORS AFFILIATED TO A SEMIFINITE VON NEUMANN ALGEBRA AND COMMUTATORS

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Abstract: Suppose that a von Neumann operator algebra $\mathscr{M}$ acts on a Hilbert space $\mathscr{H}$ and τ is a faithful normal semifinite trace on $\mathscr{M}$. If Hermitian operators $X, Y \in S(\mathscr{M},\tau)$ are such that $-X \leq Y \leq X$ and Y is τ-essentially invertible then so is X. Let $0 < p \leq 1$. If a p-hyponormal operator $A \in S(\mathscr{M},\tau)$ is right τ-essentially invertible then A is τ-essentially invertible. If a p-hyponormal operator $A \in \mathscr{B}(\mathscr{H})$ is right invertible then A is invertible in $\mathscr{B}(\mathscr{H})$. If a hyponormal operator $A \in S(\mathscr{M},\tau)$ has a right inverse in $S(\mathscr{M},\tau)$ then A is invertible in $S(\mathscr{M},\tau)$. If $A, T \in \mathscr{M}$ and $\mu_t(A^n)^{\frac{1}{n}} \to 0$ as $n \to \infty$ for every $t > 0$ then AT (TA) has no right (left) τ-essential inverse in $S(\mathscr{M},\tau)$. Suppose that $\mathscr{H}$ is separable and $\dim \mathscr{H} = \infty$. A right (left) essentially invertible operator $A \in \mathscr{B}(\mathscr{H})$ is a commutator if and only if the right (left) essential inverse of A is a commutator.

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Introduction

Suppose that a von Neumann operator algebra $\mathscr{M}$ acts on a Hilbert space $\mathscr{H}$ and τ is a faithful normal semifinite trace on $\mathscr{M}$ with $\tau(I) = +\infty$. This article continues the study of the invertibility and τ-essential invertibility of τ-measurable operators which was initiated in [1, 2]; the study of the commutators of τ-measurable operators which was carried out in [3, 4]; and the study of the block projection operator $\mathscr{P}_m$ which was implemented in [5]. The following results were obtained: If Hermitian operators $X, Y \in S(\mathscr{M},\tau)$ are such that $-X \leq Y \leq X$ and Y is τ-essentially invertible then so is X (Theorem 1). Let $0 < p \leq 1$. If a p-hyponormal operator $A \in S(\mathscr{M},\tau)$ is right τ-essentially invertible then A is τ-essentially invertible (Corollary 5). If a p-hyponormal operator $A \in \mathscr{B}(\mathscr{H})$ is right invertible then A is invertible in $\mathscr{B}(\mathscr{H})$ (Theorem 2). If a hyponormal operator $A \in S(\mathscr{M},\tau)$ has a right inverse in $S(\mathscr{M},\tau)$ then A is invertible in $S(\mathscr{M},\tau)$ (Theorem 3). If $A, T \in \mathscr{M}$ and $\mu_t(A^n)^{\frac{1}{n}} \to 0$ as $n \to \infty$ for every $t > 0$ then the operator AT (TA) has no right (left) τ-essential inverse in $S(\mathscr{M},\tau)$ (Theorem 4). In particular, if $A, T \in \mathscr{B}(\mathscr{H})$ and A is quasinilpotent then AT (TA) has no right (left) essential inverse (Corollary 12). Let $\mathscr{H}$ be separable and $\dim \mathscr{H} = \infty$. A right (left) essentially invertible operator $A \in \mathscr{B}(\mathscr{H})$ is a commutator if and only if the right (left) essential inverse of A is a commutator (Corollary 14).

1. Notations and Definitions

Let $\mathscr{M}$ be a von Neumann operator in a Hilbert space $\mathscr{H}$, let $\mathscr{M}^{\mathrm{pr}}$ be the projection lattice ($P = P^2 = P^*$) in $\mathscr{M}$, and let I be the identity of $\mathscr{M}$, while $P^{\perp} = I - P$ for $P \in \mathscr{M}^{\mathrm{pr}}$ and $\mathscr{M}^+$ is the cone of positive elements in $\mathscr{M}$.

A mapping $\varphi : \mathscr{M}^+ \to [0, +\infty]$ is called a *trace* if $\varphi(X + Y) = \varphi(X) + \varphi(Y)$ and $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathscr{M}^+$, $\lambda \geq 0$ (moreover, $0 \cdot (+\infty) \equiv 0$), and $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathscr{M}$. A trace φ is called *faithful* if $\varphi(X) > 0$ for all $X \in \mathscr{M}^+$, $X \neq 0$; *semifinite* if $\varphi(X) = \sup\{\varphi(Y) : Y \in \mathscr{M}^+, Y \leq X,$

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$\varphi(Y) < +\infty\}$ for every $X \in \mathscr{M}^+$; *normal* if $X_i \nearrow X$ $(X_i, X \in \mathscr{M}^+) \Rightarrow \varphi(X) = \sup \varphi(X_i)$ (see [6, Chapter V, § 2]).

An operator in $\mathscr{H}$ (not necessarily bounded or densely defined) is called *affiliated to a von Neumann algebra* $\mathscr{M}$ if it commutes with every unitary operator in the commutator subalgebra $\mathscr{M}'$ of $\mathscr{M}$. Throughout the sequel, τ is a faithful normal semifinite trace on $\mathscr{M}$. A closed operator X affiliated to $\mathscr{M}$ and having an everywhere dense domain $\mathscr{D}(X)$ in $\mathscr{H}$ is called *τ-measurable* if for every $\varepsilon > 0$ there exists $P \in \mathscr{M}^{\mathrm{pr}}$ such that $P\mathscr{H} \subset \mathscr{D}(X)$ and $\tau(P^{\perp}) < \varepsilon$. The set $S(\mathscr{M}, \tau)$ of all τ-measurable operators is an algebra under passage to the adjoint operator, multiplication by scalars, and the operations of strong addition and multiplication obtained by closing the usual operations (see [7, Chapter IX]). Given a family $\mathscr{L} \subset S(\mathscr{M}, \tau)$, denote by $\mathscr{L}^+$ and $\mathscr{L}^{\mathrm{h}}$ the positive and Hermitian parts of $\mathscr{L}$ respectively. The partial order in $S(\mathscr{M}, \tau)^{\mathrm{h}}$, generated by the proper cone $S(\mathscr{M}, \tau)^+$, will be denoted by $\leq$. If $X \in S(\mathscr{M}, \tau)$ and $X = U|X|$ is the polar decomposition of X then $U \in \mathscr{M}$ and $|X| = \sqrt{X^*X} \in S(\mathscr{M}, \tau)^+$.

Denote the *rearrangement* of $X \in S(\mathscr{M}, \tau)$ by $\mu_t(X)$, i.e., $\mu_t(X)$ is a nonincreasing right continuous function $\mu(X)\colon (0, \infty) \to [0, \infty)$ defined by the formula

$$\mu_t(X) = \inf\{\|XP\| : P \in \mathscr{M}^{\mathrm{pr}},\ \tau(P^{\perp}) \leq t\}, \quad t > 0.$$

The set of τ-compact operators $S_0(\mathscr{M}, \tau) = \{X \in S(\mathscr{M}, \tau) : \lim_{t\to\infty} \mu_t(X) = 0\}$ is an ideal in $S(\mathscr{M}, \tau)$. The set of elementary operators $\mathscr{F}(\mathscr{M}, \tau) = \{X \in \mathscr{M} : \exists s > 0\ (\mu_t(X) = 0\ \forall t > s)\}$ is an ideal in $\mathscr{M}$.

Lemma 1 [8]. *Let $X, Y \in S(\mathscr{M}, \tau)$. Then*
(i) $\mu_t(X) = \mu_t(|X|) = \mu_t(X^*)$ *for all* $t > 0$;
(ii) $\mu_t(\lambda X) = |\lambda| \mu_t(X)$ *for all* $\lambda \in \mathbb{C}$ *and* $t > 0$;
(iii) *if* $|X| \leq |Y|$ *then* $\mu_t(X) \leq \mu_t(Y)$ *for all* $t > 0$;
(iv) *if* $X \in \mathscr{M}$ *then* $\lim_{t\to+0} \mu_t(X) = \sup_{t>0} \mu_t(X) = \|X\|$;
(v) $\mu_{t+s}(X + Y) \leq \mu_t(X) + \mu_s(Y)$ *for all* $t, s > 0$;
(vi) $\mu_t(|X|^{\alpha}) = \mu_t(X)^{\alpha}$ *for all* $\alpha > 0$ *and* $t > 0$.

Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathscr{M}$ with $\tau(I) = +\infty$. An operator $A \in S(\mathscr{M}, \tau)$ is called *right* (*left*) *τ-essentially invertible* if $I - AB \in S_0(\mathscr{M}, \tau)$ (respectively, $I - BA \in S_0(\mathscr{M}, \tau)$) for some operator $B \in S(\mathscr{M}, \tau)$. An operator $A \in S(\mathscr{M}, \tau)$ is called *τ-essentially invertible* if there exists $B \in S(\mathscr{M}, \tau)$ such that $I - AB, I - BA \in S_0(\mathscr{M}, \tau)$ [2]. The following conditions are equivalent for $A \in S(\mathscr{M}, \tau)^{\mathrm{h}}$: (i) A is right τ-essentially invertible; (ii) A is left τ-essentially invertible; and (iii) A is τ-essentially invertible (see [2, Corollary 3.10]). An operator $A \in S(\mathscr{M}, \tau)$ is τ-essentially invertible if and only if A is simultaneously left and right τ-essentially invertible (see [2, Theorem 3.9]). We have the decompositions (see [9])

$$S(\mathscr{M}, \tau) = S_0(\mathscr{M}, \tau) + \mathscr{M}, \quad S(\mathscr{M}, \tau)^+ = S_0(\mathscr{M}, \tau)^+ + \mathscr{M}^+ \tag{1}$$

(i.e., every operator $A \in S(\mathscr{M}, \tau)$ has the form $A = A_1 + A_2$ with $A_1 \in S_0(\mathscr{M}, \tau)$, $A_2 \in \mathscr{M}$). Hence, the (left or right) τ-essential invertibility of A is equivalent to (left or right) the τ-essential invertibility of A_2.

An operator $A \in S(\mathscr{M}, \tau)$ is called *p-hyponormal* for some $0 < p \leq 1$ if $(A^*A)^p \geq (AA^*)^p$; and *p-cohyponormal*, if A^* is p-hyponormal. Every p-hyponormal operator $A \in S(\mathscr{M}, \tau)$ is paranormal (see [10, Theorem 4.4]), i.e., $2|A|^2 \leq \lambda^{-1}|A^2|^2 + \lambda I$ for all $\lambda > 0$. If a paranormal operator $A \in S(\mathscr{M}, \tau)$ has an inverse $A^{-1} \in \mathscr{M}$ then A^{-1} is also paranormal (see [11, Theorem 2(iii)]). If $A \in S(\mathscr{M}, \tau)$ is hyponormal and $(\lambda I + A)^{-1} \in \mathscr{M}$ for some $\lambda \in \mathbb{C}$ then $(\lambda I + A)^{-1}$ is hyponormal (see [11, Proposition 2]).

An operator $A \in S(\mathscr{M}, \tau)$ is called a *commutator* if $A = XY - YX$ for some $X, Y \in S(\mathscr{M}, \tau)$. Operators $A, B \in S(\mathscr{M}, \tau)$ are *similar* if there exists an invertible operator $T \in S(\mathscr{M}, \tau)$ with $T^{-1} \in S(\mathscr{M}, \tau)$ such that $A = TBT^{-1}$.

Given $P_k = P_k^2 \in \mathscr{M}$, $k = 1, \dots, m$, with $P_1 + \dots + P_m = I$, define (cf. [5]) the generalized block projection operator $\mathscr{P}_m : S(\mathscr{M}, \tau) \to S(\mathscr{M}, \tau)$ by the formula

$$\mathscr{P}_m(A) = \sum_{k=1}^{m} P_k A P_k, \quad A \in S(\mathscr{M}, \tau).$$

If $\mathscr{M} = \mathscr{B}(\mathscr{H})$ is the $*$-algebra of all bounded linear operators in $\mathscr{H}$ and $\tau = \mathrm{tr}$ is a canonical trace then $S(\mathscr{M}, \tau)$, $S_0(\mathscr{M}, \tau)$ and $\mathscr{F}(\mathscr{M}, \tau)$ coincide with $\mathscr{B}(\mathscr{H})$, the ideal $\mathfrak{S}_\infty(\mathscr{H})$ of compact operators, and the ideal $\mathscr{F}(\mathscr{H})$ of finite-rank operators in $\mathscr{H}$ respectively; the right (left) τ-essential invertibility coincides with the classical right (left) essential invertibility which was studied, for example, in [12, § 14]. We have $\mu_t(X) = \sum_{n=1}^{\infty} s_n(X)\chi_{[n-1,n)}(t)$, $t > 0$, where $\{s_n(X)\}_{n=1}^{\infty}$ is the sequence of the s-numbers of a compact operator X and χ_A is the indicator of a set $A \subset \mathbb{R}$.

2. The Main Results

Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathscr{M}$ with $\tau(I) = +\infty$.

Lemma 2. *If $n \in \mathbb{N}$, $n \geq 2$, and $A_1, \dots, A_n \in S(\mathscr{M}, \tau)$ are right τ-essentially invertible with right τ-essential inverses $B_1, \dots, B_n$ respectively then the product $A_1 \cdots A_n$ is right τ-essentially invertible with right τ-essential inverse $B_n \cdots B_1$. Conversely, if $A_1, A_2 \in S(\mathscr{M}, \tau)$ and A_1A_2 is right τ-essentially invertible then so is A_1.*

Proof. Proceed by induction. Let $n = 2$. Then

$$I - A_1A_2 \cdot B_2B_1 = I - A_1B_1 + A_1(I - A_2B_2)B_1 \in S_0(\mathscr{M}, \tau).$$

Suppose that the claim holds for $n = m$. Then for $n = m + 1$ we have

$$I - A_1 \cdots A_n \cdot B_n \cdots B_1 = I - A_1B_1 + A_1(I - A_2 \cdots A_n \cdot B_n \cdots B_2)B_1 \in S_0(\mathscr{M}, \tau)$$

by the induction assumption for the collection of operators $A_2, \dots, A_n$.

If $A_1, A_2 \in S(\mathscr{M}, \tau)$ and B is a right τ-essential inverse for A_1A_2 then $I - A_1 \cdot A_2B = I - A_1A_2 \cdot B \in S_0(\mathscr{M}, \tau)$; i.e., A_2B is a right τ-essential inverse for A_1. □

In particular, if $A \in S(\mathscr{M}, \tau)$ is right τ-essentially invertible and $A = U|A|$ is the polar decomposition then the partial isometry U is also right τ-essentially invertible. Lemma 2 implies

Lemma 3. *If $A \in S(\mathscr{M}, \tau)$ is right τ-essentially invertible then so is A^n for all $n \in \mathbb{N}$, with $n \geq 2$. Conversely, if A^n is right τ-essentially invertible for some $n \geq 2$ then so is A.*

Note that Lemma 3 easily implies a new proof of Proposition 3.15 in [2].

Lemma 4. *Let $A \in S(\mathscr{M}, \tau)^+$ and $0 < p < +\infty$. Then the following are equivalent:*
(i) *A is τ-essentially invertible;*
(ii) *A^p is τ-essentially invertible.*

Proof. (i)⇒(ii): Let $B \in S(\mathscr{M}, \tau)$ be such that $I - AB \in S_0(\mathscr{M}, \tau)$.

Step 1. Let $0 < p < 1$. From

$$I - A^p \cdot A^{1-p}B = I - AB \in S_0(\mathscr{M}, \tau)$$

it follows that A^p is right τ-essentially invertible. Therefore, A^p is τ-essentially invertible by [2, Corollary 3.10].

Step 2. Let $1 < p < +\infty$. If $p \in \mathbb{N}$ then Lemma 3 works. Let $p \notin \mathbb{N}$ and let $n \in \mathbb{N}$ be such that $n - 1 < p < n$. The operator $X = A^n$ is right τ-essentially invertible by Lemma 3. Therefore, $A^p = X^{p/n}$ is τ-essentially invertible by Step 1 since $0 < \frac{p}{n} < 1$.

(ii)⇒(i): For the operator $Y = A^p$, we have $A = Y^{1/p}$. It suffices to apply (i)⇒(ii) to the pair of X and $X^{1/p}$. □

Theorem 1. *Suppose that $X \in S(\mathscr{M}, \tau)^+$ and $Y \in S(\mathscr{M}, \tau)^{\mathrm{h}}$ are such that*

$$-X \leq Y \leq X. \tag{2}$$

If Y is τ-essentially invertible then so is X.

Proof. Let $Z \in S(\mathscr{M},\tau)$ be such that $I - YZ \in S_0(\mathscr{M},\tau)$. Condition (2) and [13, Corollary 3.15(i)] give

$$Y = X^{1/2} T X^{1/2} \tag{3}$$

for some $T \in \mathscr{M}^{\mathrm{h}}$ with $\|T\| \le 1$. Then

$$I - X^{1/2} \cdot T X^{1/2} Z = I - YZ \in S_0(\mathscr{M},\tau)$$

and $X^{1/2}$ is right τ-essentially invertible. By Lemma 3, $X = (X^{1/2})^2$ is right τ-essentially invertible. Therefore, X is τ-essentially invertible by [2, Corollary 3.10]. □

Remark 1. If $X \in \mathscr{B}(\mathscr{H})^+$ and $Y \in \mathscr{B}(\mathscr{H})^{\mathrm{h}}$ satisfy condition (2) and Y is invertible then X is invertible [14, Corollary 2]. Representation (3) and [15, Lemma 3.5(ii)] yield a new proof of this fact.

Corollary 1. *The set $\{X \in S(\mathscr{M},\tau)^+ : X$ is τ-essentially invertible$\}$ is a subcone of $S(\mathscr{M},\tau)^+$.*

Corollary 2. *If $X \in S(\mathscr{M},\tau)^{\mathrm{h}}$ then the following are equivalent:*
(i) *X is τ-essentially invertible;*
(ii) *$|X|$ is τ-essentially invertible.*

Proof. (i)⇒(ii): Follows from the inequality $-|X| \le X \le |X|$ and Theorem 1.

(ii)⇒(i): Let $Z \in S(\mathscr{M},\tau)$ be such that $I - |X|Z \in S_0(\mathscr{M},\tau)$. Let $X = X_+ - X_-$ be the Jordan decomposition into the positive and negative parts with $X_+X_- = 0$ and $|X| = X_+ + X_-$. Let the projection $P \in \mathscr{M}^{\mathrm{pr}}$ be the support of X_+. For $U = P - P^{\perp}$ we have $U^2 = I$, $U|X| = X$, and

$$U(I - |X|Z)U = I - X \cdot ZU \in S_0(\mathscr{M},\tau);$$

i.e., X is τ-essentially invertible. □

Corollary 3. *Let $A, B \in S(\mathscr{M},\tau)$, $X_1 = A^*A + B^*B$, and $Y_1 = A^*B + B^*A$; for $A \ge 0$ let $X_2 = A + BAB^*$ and $Y_2 = AB^* + BA$; for $A, B \in S(\mathscr{M},\tau)^{\mathrm{h}}$ and $0 \le t \le 1$ let $X_3 = A^2 + B^2$ and $Y_3 = t^{1/2}A + (1-t)^{1/2}B$; for $A, B \ge 0$ let $X_4 = A + B$ and $Y_4 = A - B$. If Y_k is τ-essentially invertible then also X_k is τ-essentially invertible, $k = 1, 2, 3, 4$.*

Proof. For $k = 1$ the claim follows from the inequalities

$$(A \pm B)^*(A \pm B) \ge 0$$

and Theorem 1. For $k = 2$ the claim follows from the inequalities

$$(A^{1/2} \pm BA^{1/2})^*(A^{1/2} \pm BA^{1/2}) \ge 0$$

and Theorem 1. For $k = 3$ the claim follows from the inequality

$$|t^{1/2}A + (1-t)^{1/2}B| \le (A^2 + B^2)^{1/2}$$

(see the proof of Proposition 3.5 in [13]), Corollary 2, and Theorem 1. □

Corollary 4. *Let $A \in S(\mathscr{M},\tau)$ and $X \in \mathscr{M}$ with $\|X\| \le 1$. If XA^pX^* is τ-essentially invertible for some $0 < p \le 1$ then so is the operator XAX^*.*

Proof. Since the function $f(t) = t^p$ $(0 < p \le 1)$ is operator monotone on the half-axis $[0, +\infty)$, we have $XA^pX^* \le (XAX^*)^p$ by Hansen's inequality proved in [16] for bounded operators but extendable easily to τ-measurable operators. Therefore, $(XAX^*)^p$ is τ-essentially invertible by Theorem 1. Hence, XAX^* is τ-essentially invertible by Lemma 4. □

Corollary 5. *If an operator $A \in S(\mathscr{M},\tau)$ p-hyponormal for some $0 < p \leq 1$ is right τ-essentially invertible then so is A.*

Proof. An operator $X \in S(\mathscr{M},\tau)$ is right τ-essentially invertible if and only if XX^* is τ-essentially invertible (see [2, Corollary 3.3]). Thus, AA^* is τ-essentially invertible. By Lemma 4, $(AA^*)^p$ is τ-essentially invertible. Since $(A^*A)^p \geq (AA^*)^p$; therefore, $(A^*A)^p$ is τ-essentially invertible by Theorem 1. Now, by Lemma 4, A^*A is τ-essentially invertible. Hence, A is τ-essentially invertible by [2, Theorem 3.2]. Consequently, A is τ-essentially invertible by [2, Theorem 3.9]. □

Passing to adjoint operators, we obtain

Corollary 6. *If an operator $A \in S(\mathscr{M},\tau)$ p-cohyponormal for some $0 < p \leq 1$ is left τ-essentially invertible then A is τ-essentially invertible.*

Corollary 7. *For a normal operator $A \in S(\mathscr{M},\tau)$ the following are equivalent:*
(i) *A is left τ-essentially invertible;*
(ii) *A is right τ-essentially invertible;*
(iii) *A is τ-essentially invertible.*

Proposition 1. *Let $A = U|A|$ be the polar decomposition of $A \in S(\mathscr{M},\tau)$. Then $(I - UU^*)(I - AB) = I - UU^*$ for all $B \in S(\mathscr{M},\tau)$.*

Proof. Recall that $UU^*U = U$. □

Corollary 8. *Let $A = U|A|$ be the polar decomposition of $A \in S(\mathscr{M},\tau)$. If A is left τ-essentially invertible then $I - UU^* \in \mathscr{F}(\mathscr{M},\tau)$.*

Corollary 9. *Suppose that $P \in \mathscr{M}^{\mathrm{pr}}$ is the support of a τ-essentially invertible operator $A \in S(\mathscr{M},\tau)^{\mathrm{h}}$. Then $P^{\perp} \in \mathscr{F}(\mathscr{M},\tau)$.*

Theorem 2. *If $A \in \mathscr{B}(\mathscr{H})$ p-hyponormal for some $0 < p \leq 1$ has a right inverse in $\mathscr{B}(\mathscr{H})$ then A is invertible in $\mathscr{B}(\mathscr{H})$.*

Proof. Given some nonzero operator $X \in \mathscr{B}(\mathscr{H})$, put $X_1 = X/\|X\|$. Then, by Hansen's inequality [16], we have

$$X^*(AA^*)^pX \leq X^*(A^*A)^pX = \|X\|^2X_1^*(A^*A)^pX_1 \leq \|X\|^2(X_1^*A^*AX_1)^p.$$

Therefore, if $AX = 0$ ($\Leftrightarrow AX_1 = 0$) then $(A^*A)^{p/2}X = 0$ and

$$AA^*X = (AA^*)^{1-p/2}(AA^*)^{p/2}X = 0.$$

Hence, $X^* \cdot AA^*X = |A^*X|^2 = 0$ and $A^*X = 0$.

Suppose that $AB = I$ for some operator $B \in \mathscr{B}(\mathscr{H})$. Then $A(BA - I) = (AB - I)A = 0$. Thus, $A^*(BA - I) = 0$ (we put $X = BA - I$); and, by the equality $B^*A^* = I$, we obtain

$$BA - I = B^*A^*(BA - I) = 0. \tag{4}$$

The theorem is proved. □

Corollary 10. *If an operator $A \in \mathscr{B}(\mathscr{H})$ p-cohyponormal for some $0 < p \leq 1$ has a left inverse in $\mathscr{B}(\mathscr{H})$ then A is invertible in $\mathscr{B}(\mathscr{H})$.*

Theorem 3. *If a hyponormal operator $A \in S(\mathscr{M},\tau)$ has a right inverse in $S(\mathscr{M},\tau)$ then A is invertible in $S(\mathscr{M},\tau)$.*

Proof. If $X \in S(\mathscr{M},\tau)$ then

$$|A^*X|^2 = X^*AA^*X \leq X^*A^*AX \leq X^*A^*AX = |AX|^2.$$

Therefore, if $AX = 0$ then $A^*X = 0$.

Suppose that $AB = I$ for some $B \in S(\mathscr{M},\tau)$. Then $A(BA-I) = (AB-I)A = 0$ and $A^*(BA-I) = 0$ (we put $X = BA - I$) and (4) holds. □

Corollary 11. *If a cohyponormal operator $A \in S(\mathscr{M},\tau)$ has a left inverse in $S(\mathscr{M},\tau)$ then A is invertible in $S(\mathscr{M},\tau)$.*

Remark 2. (i) If $A, B \in S(\mathscr{M},\tau)$ and $AB = I + X$ for some $X \in S_0(\mathscr{M},\tau)$ then $BA = (BA)^2 + Y$ for some $Y \in S_0(\mathscr{M},\tau)$. Indeed, $B(AB - I) = (BA - I)B \in S_0(\mathscr{M},\tau)$ and $(BA - I)BA \in S_0(\mathscr{M},\tau)$. If $\tau(I) = +\infty$ then $BA \notin S_0(\mathscr{M},\tau)$: if we assume that $BA \in S_0(\mathscr{M},\tau)$ then $A \cdot BA \cdot B = (AB)^2 = I + 2X + X^2 \in S_0(\mathscr{M},\tau)$; a contradiction.

(ii) If $A, B \in S(\mathscr{M},\tau)$ and $AB = I$ then $BA = (BA)^2$, i.e., BA is an idempotent. If $\tau(I) = +\infty$ then $BA \notin S_0(\mathscr{M},\tau)$.

Lemma 5. *Let $X \in S(\mathscr{M},\tau)$ and let a real $x > 0$ be such that $\mu_t(X) \geq x$ for all $t > 0$. Then $\mu_t(X + Y) \geq x$ for all $t > 0$ and $Y \in S_0(\mathscr{M},\tau)$.*

Proof. Suppose that the following inequalities hold: $\mu_s(X+Y) \leq x - 2\varepsilon$, $\mu_s(Y) < \varepsilon$ for some $s > 0$ and $\varepsilon > 0$. Then, by Lemma 1(v),(ii), we have

$$x \leq \mu_{2s}(X) = \mu_{2s}((X + Y) - Y) \leq \mu_s(X + Y) + \mu_s(Y) < x - \varepsilon;$$

a contradiction. □

Theorem 4. *Suppose that τ is a faithful normal semifinite trace on a von Neumann algebra $\mathscr{M}$, $A, T \in \mathscr{M}$, and $\mu_t(A^n)^{\frac{1}{n}} \to 0$ as $n \to \infty$ for every $t > 0$. Then AT (TA) has no right (left) τ-essential inverse in $S(\mathscr{M},\tau)$.*

Proof. It suffices to show that AT (TA) has no right (left) τ-essential inverse in $\mathscr{M}$; see expansion (1).

Step 1. Suppose that AT has a τ-essentially inverse in $\mathscr{M}$, i.e., there exists $X \in \mathscr{M}$ such that $AT \cdot X = I + K$ with $K \in S_0(\mathscr{M},\tau)$. Then

$$ATX \cdot X^*T^*A^* = I + K_1 \geq 0, \tag{5}$$

where $K_1 = KK^* + K + K^* \in S_0(\mathscr{M},\tau)^{\mathrm{h}}$. Clearly,

$$TXX^*T^* = |X^*T^*|^2 \leq \|X^*T^*\|^2 \cdot I = \|TX\|^2 \cdot I. \tag{6}$$

From (5) and (6) we have

$$\|TX\|^2 \cdot AA^* \geq I + K_1 \geq 0. \tag{7}$$

Multiplying both sides of (7) from the left by A and from the right by A^* and appreciating (7), we obtain

$$\|TX\|^2 \cdot A^2A^{*2} \geq AA^* + AK_1A^* \geq \|TX\|^{-2} \cdot I + \|TX\|^{-2}K_1 + AK_1A^*,$$

i.e., $\|TX\|^4 \cdot A^2A^{*2} \geq I + K_2$ and $K_2 = K_1 + \|TX\|^2 \cdot AK_1A^* \in S_0(\mathscr{M},\tau)^{\mathrm{h}}$. Continuing this process, by induction we obtain

$$\|TX\|^{2n} \cdot A^nA^{*n} \geq I + K_n, \tag{8}$$

where $K_n = K_{n-1} + \|TX\|^2 \cdot AK_{n-1}A^* \in S_0(\mathscr{M},\tau)^{\mathrm{h}}$ for $n = 2, 3, \ldots$. Since $K_n \geq -|K_n|$ for all $n \in \mathbb{N}$, from (7) and (8) we have $\|TX\|^{2n} \cdot A^nA^{*n} + |K_n| \geq I$ for all $n \in \mathbb{N}$. Lemma 1(iii) implies that

$$\mu_t(\|TX\|^{2n} \cdot A^nA^{*n} + |K_n|) \geq \mu_t(I) = 1 \quad \text{for all } n \in \mathbb{N},\ t > 0.$$

Applying Lemma 5 with $X = \|TX\|^{2n} \cdot A^nA^{*n} + |K_n|$, $Y = -|K_n|$, and $x = 1$, we see that

$$\mu_t(\|TX\|^{2n} \cdot A^nA^{*n}) \geq 1 \quad \text{for all } n \in \mathbb{N},\ t > 0.$$

By Lemma 1(i),(ii),(vi), $\|TX\|^{2n}\mu_t(A^n)^2 \geq 1$ for all $n \in \mathbb{N}$ and $t > 0$. Therefore, $\mu_t(A^n)^2 \geq \|TX\|^{-2n}$ for all $n \in \mathbb{N}$ and $t > 0$. Since $\lambda \mapsto \lambda^{\frac{1}{2n}}$, with $\lambda \in \mathbb{R}^+$, is a monotone real function, we get $\mu_t(A^n)^{\frac{1}{2n}} \geq \|TX\|^{-1}$ for all $n \in \mathbb{N}$ and $t > 0$; a contradiction.

Step 2. Recall that AT has a left τ-essentially inverse in $\mathscr{M}$, i.e., there exists an operator $X \in \mathscr{M}$ such that $X \cdot TA = I + K$ with $K \in S_0(\mathscr{M},\tau)$. Passing to adjoint operators, we infer that $A^*T^* \cdot X^* = I + K^*$ with $K \in S_0(\mathscr{M},\tau)$. Note that, by Lemma 1(i), $\mu_t(A^{*n})^{\frac{1}{n}} = \mu_t(A^{n*})^{\frac{1}{n}} = \mu_t(A^n)^{\frac{1}{n}} \to 0$ as $n \to \infty$ for every $t > 0$. In view of Step 1, we again obtain a contradiction. The theorem is proved. □

Corollary 12. *Suppose that $A, T \in \mathscr{B}(\mathscr{H})$ and A is quasinilpotent. Then AT (TA) has no right* (*left*) *essentially inverse operator.*

Proof. It is known that $A \in \mathscr{B}(\mathscr{H})$ is quasinilpotent if and only if $\|A^n\|^{\frac{1}{n}} \to 0$ as $n \to \infty$ (see, for example, [17]). For every $B \in \mathscr{B}(\mathscr{H})$ we have (recall that $\tau = \mathrm{tr}$)

$$\|B\| = s_1(B) = \mu_t(B) \quad \text{for } t \in (0, 1);$$

therefore, $\mu_t(A^n)^{\frac{1}{n}} = \|A^n\|^{\frac{1}{n}} \to 0$ as $n \to \infty$ for $t \in (0, 1)$. $\square$

Remark 3. Recall [17, p. 208] that the integration operator

$$(Jf)(s) = \int_0^s f(t)\, dt$$

on the Hilbert space $L_2(0, 1)$ is not nilpotent but quasinilpotent.

Remark 4. Let τ be a faithful semifinite trace on a von Neumann algebra $\mathscr{M}$. If $A \in \mathscr{M}$ is quasinilpotent; then, by Lemma 1(iv), $\mu_t(A^n)^{\frac{1}{n}} \to 0$ as $n \to \infty$ for every $t > 0$.

Proposition 2. *If $A, B \in S(\mathscr{M}, \tau)$ are similar then A is left τ-essentially invertible* (*right τ-essentially invertible or τ-essentially invertible*) *if and only if B is left τ-essentially invertible* (*right τ-essentially invertible or τ-essentially invertible*).

Proof. Suppose that A is left τ-essentially invertible, i.e., $I - XA \in S_0(\mathscr{M}, \tau)$ for some $X \in S(\mathscr{M}, \tau)$. We have

$$I - T^{-1}XT \cdot B = T^{-1}(I - X \cdot TBT^{-1})T \in S_0(\mathscr{M}, \tau).$$

The rest is obvious. $\square$

Theorem 5. *Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathscr{M}$ with $\tau(I) = +\infty$. Suppose that $A \in S(\mathscr{M}, \tau)$ has a right* (*left*) *τ-essentially inverse operator $B \in S(\mathscr{M}, \tau)$, i.e., $I - AB \in S_0(\mathscr{M}, \tau)$* (*respectively, $I - BA \in S_0(\mathscr{M}, \tau)$*). *If $A = \lambda I + X$ with some $\lambda \in \mathbb{C} \setminus \{0\}$ and $X \in S_0(\mathscr{M}, \tau)$ then $B = \lambda^{-1}I + Y$ with some $Y \in S_0(\mathscr{M}, \tau)$.*

Proof. Let $B \in S(\mathscr{M}, \tau)$ be a right τ-essential inverse for $A \in S(\mathscr{M}, \tau)$. We have

$$Z = I - AB = I - \lambda B - XB \in S_0(\mathscr{M}, \tau),$$

which yields $\lambda B = I - XB - Z$; i.e., $B = \lambda^{-1}I - \lambda^{-1}(XB + Z) = \lambda^{-1}I + Y$. Obviously, $Y = -\lambda^{-1}(XB + Z) \in S_0(\mathscr{M}, \tau)$ since $X, Z \in S_0(\mathscr{M}, \tau)$. $\square$

Corollary 13. *Suppose that $\dim \mathscr{H} = \infty$, an operator $A \in \mathscr{B}(\mathscr{H})$ has a right* (*left*) *essential inverse $B \in \mathscr{B}(\mathscr{H})$ and is the sum of a compact and a nonzero scalar operator. Then B cannot be a commutator.*

Proof. The sum of a continuous and a nonzero scalar operator in an infinite-dimensional Hilbert space cannot be a commutator [18, Chapter 19, Corollary to Problem 182]. $\square$

Corollary 14. *Suppose that $\mathscr{H}$ is separable and $\dim \mathscr{H} = \infty$. A right* (*left*) *essentially invertible operator $A \in \mathscr{B}(\mathscr{H})$ is a commutator if and only if the right essential inverse A_r^{-1}* (*respectively, the left inverse A_l^{-1}*) *is a commutator.*

Proof. If $\mathscr{H}$ is separable and infinite-dimensional then an operator $A \in \mathscr{B}(\mathscr{H})$ is a commutator if and only if it is representable as the sum of a nonzero scalar and a compact operator (see [18, Chapter 19, Problem 182]). Let $A, A_r^{-1} \in \mathscr{B}(\mathscr{H})$ and $Z = I - AA_r^{-1} \in \mathfrak{S}_\infty(\mathscr{H})$.

Let $A = \lambda I + X$ with some $\lambda \in \mathbb{C} \setminus \{0\}$ and $X \in \mathfrak{S}_\infty(\mathscr{H})$. Then $I - (\lambda I + X)A_r^{-1} = Z$ and $A_r^{-1} = \lambda^{-1}I + Y$ with $Y = -\lambda^{-1}\big(Z + XA_r^{-1}\big) \in \mathfrak{S}_\infty(\mathscr{H})$.

Let $A_r^{-1} = \lambda I + X$ with some $\lambda \in \mathbb{C} \setminus \{0\}$ and $X \in \mathfrak{S}_\infty(\mathscr{H})$. Then $I - A(\lambda I + X) = Z$ and $A = \lambda^{-1}I + Y$ with $Y = -\lambda^{-1}(Z + AX) \in \mathfrak{S}_\infty(\mathscr{H})$. The rest is obvious. $\square$

Proposition 3. *Suppose that $\mathscr{H}$ is separable and $\dim \mathscr{H} = \infty$ and $A \in \mathscr{B}(\mathscr{H})$. If $\mathscr{P}_m(A)$ is a commutator then so is A.*

Proof. Suppose that $B \in \mathscr{B}(\mathscr{H})$ is not a commutator; i.e., $B = \lambda I + X$ with some $\lambda \in \mathbb{C} \setminus \{0\}$ and $X \in \mathfrak{S}_\infty(\mathscr{H})$. Then $\mathscr{P}_m(B) = \lambda I + Y$ with $Y = \sum_{k=1}^m P_k X P_k \in \mathfrak{S}_\infty(\mathscr{H})$, i.e., $\mathscr{P}_m(B)$ is not a commutator. □

Proposition 4. *Let τ be a-faithful normal semifinite trace on a von Neumann algebra $\mathscr{M}$ with $\tau(I) = +\infty$. An operator of the form $\lambda I + X$ with $\lambda \in \mathbb{C} \setminus \{0\}$ and $X \in S_0(\mathscr{M}, \tau)$ cannot be a commutator of $A, B \in S(\mathscr{M}, \tau)$ with $B = C + Y$, where $C^2 = C \in S(\mathscr{M}, \tau)$ and $Y \in S_0(\mathscr{M}, \tau)$.*

Proof. By way of contradiction, suppose that there are $\lambda \in \mathbb{C} \setminus \{0\}$, $X \in S_0(\mathscr{M}, \tau)$, and $A, B \in S(\mathscr{M}, \tau)$ with $B = C + Y$, where $C^2 = C$ and $Y \in S_0(\mathscr{M}, \tau)$ such that $\lambda I + X = AB - BA$. Multiplying both sides of this equality from the left and the right by an idempotent $C \in S(\mathscr{M}, \tau)$, we obtain $\lambda C + CXC = CAYC - CYAC$. Since $X, Y \in S_0(\mathscr{M}, \tau)$ and $\lambda \neq 0$, we have $C \in S_0(\mathscr{M}, \tau)$. Consequently, $B \in S_0(\mathscr{M}, \tau)$ and $\lambda I + X \in S_0(\mathscr{M}, \tau)$; a contradiction. □

The following are equivalent for a matrix $X \in \mathbb{M}_n(\mathbb{C})$:
(i) X is unitarily equivalent to a matrix with zero diagonal;
(ii) $\operatorname{tr}(X) = 0$;
(iii) X is a commutator;
(iv) $\operatorname{tr}(|I + zX|) \geq n$ for all $z \in \mathbb{C}$;
(v) $\mathscr{P}_m(X)$ is a commutator.

Proof. Demonstration of (i)⇔(ii) can be found in [19, Chapter II, Problem 209]; of (ii)⇔(iii), in [18, Chapter 19, Problem 182]; and of (ii)⇔(iv), in [20, Theorem 4.8]. The equivalence (ii)⇔(v) follows since $\operatorname{tr}(X) = \operatorname{tr}(\mathscr{P}_m(X))$ for all $X \in \mathbb{M}_n(\mathbb{C})$ and $m \leq n$ (see [21, Lemma 1]). The Jacobi formula $\det(e^X) = e^{\operatorname{tr}(X)}$ holds for all matrices $X \in \mathbb{M}_n(\mathbb{C})$ (see [22, Theorem 2.12]). Therefore, if $\operatorname{tr}(X) = 0$ then $\det(e^X) = 1$.

Theorem 6. *For an invertible commutator $A \in \mathbb{M}_n(\mathbb{C})$ with spectrum $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$, the inverse matrix A^{-1} is a commutator if and only if*

$$\lambda_1 + \dots + \lambda_n = \lambda_1^{-1} + \dots + \lambda_n^{-1} = 0. \tag{9}$$

Proof. The claim follows from the equivalence (i)⇔(iii) and the coincidence of the spectral trace and the matrix trace [19, Chapter II, Theorem 201]. In particular, an invertible matrix $A \in \mathbb{M}_2(\mathbb{C})$ is a commutator if and only if so is A^{-1}. □

Corollary 15. *For an invertible commutator $A \in \mathbb{M}_3(\mathbb{C})$, the inverse matrix A^{-1} is a commutator if and only if the spectrum $\sigma(A) = \{\lambda_1, \lambda_2, \lambda_3\}$ of A consists of the cubic roots of* $\det(A)$.

Proof. Necessity: We have (9) for $n = 3$. Multiplying both sides of the equality $\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} = 0$ by $\det(A) = \lambda_1\lambda_2\lambda_3$, we get $\lambda_2\lambda_3 + \lambda_1\lambda_3 + \lambda_1\lambda_2 = 0$. Considering that $(\lambda_1 + \lambda_2 + \lambda_3)^2 = 0$, this yields

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0. \tag{10}$$

The system $\begin{cases} \lambda_1 = -\lambda_2 - \lambda_3, \\ \lambda_1^2 = -\lambda_2^2 - \lambda_3^2 \end{cases}$ implies that $\lambda_2^2 + \lambda_3^2 + 2\lambda_2\lambda_3 = -\lambda_2^2 - \lambda_3^2$, i.e., $\lambda_2^2 + \lambda_3^2 + \lambda_2\lambda_3 = 0$. It follows from (10) that $\lambda_1^2 = \lambda_2\lambda_3$. By symmetry, $\lambda_2^2 = \lambda_1\lambda_3$, $\lambda_3^2 = \lambda_1\lambda_2$. Since $\frac{\lambda_1^2}{\lambda_2^2} = \frac{\lambda_2}{\lambda_1}$ and $\frac{\lambda_2^2}{\lambda_3^2} = \frac{\lambda_3}{\lambda_2}$, we obtain $\lambda_1^3 = \lambda_2^3 = \lambda_3^3$.

Sufficiency: Suppose that $A \in \mathbb{M}_3(\mathbb{C})$ has the spectrum $\sigma(A) = \{\lambda_1, \lambda_2, \lambda_3\}$ consisting of the cubic roots of $\det(A) \neq 0$. Then $\lambda_1 + \lambda_2 + \lambda_3 = 0$ and A is a commutator. The spectrum $\sigma(A^{-1}) = \{\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}\}$ consists of the cubic roots of $\det(A^{-1}) = \det(A)^{-1}$. Therefore, $\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} = 0$, and A^{-1} is a commutator. □

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