Positivity
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Positivity

Trace inequalities for Rickart C^*-algebras

Airat Bikchentaev[1]

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Abstract
Rickart C^*-algebras are unital and satisfy polar decomposition. We proved that if a unital C^*-algebra $\mathcal{A}$ satisfies polar decomposition and admits "good" faithful tracial states then $\mathcal{A}$ is a Rickart C^*-algebra. Via polar decomposition we characterized tracial states among all states on a Rickart C^*-algebra. We presented the triangle inequality for Hermitian elements and traces on Rickart C^*-algebra. For a block projection operator and a trace on a Rickart C^*-algebra we proved a new inequality. As a corollary, we obtain a sharp estimate for a trace of the commutator of any Hermitian element and a projection. Also we give a characterization of traces in a wide class of weights on a von Neumann algebra.

Keywords Hilbert space · Polar decomposition · von Neumann algebra · C^*-algebra · Weight · trace

Mathematics Subject Classification 46L05 · 46L30 · 47C15

1 Introduction

Dimension functions and traces on C^*-algebras are fundamental tools in the operator theory and its applications. Therefore, they have been actively studied in recent decades, see [26,27,35,41,46,50,51]. Here we study traces on Rickart C^*-algebras. A Rickart C^*-algebra is a C^*-algebra within which the right annihilator of any element equals the principal right ideal generated by some projection. These algebras were introduced by Rickart [49] and were named $\aleph_1$-AW*-algebras by Kaplansky [45]. Today Rickart C^*-algebras and their Jordan counterparts are actively studied objects, see [2,4,6,7,30,39,40,47,52], and [57]. In Rickart C^*-algebras left projections are equivalent to right projections [1]. These algebras are unital and satisfy polar decomposition ([3, Corollary 3.5], [38, Corollary 7.4]). Recall that a C^*-algebra $\mathcal{A}$

✉ Airat Bikchentaev
Airat.Bikchentaev@kpfu.ru

1 Kazan Federal University, 18 Kremlyovskaya str., Kazan, Russia 420008

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satisfies polar decomposition [8], if for any $X \in \mathcal{A}$, there exists a partial isometry U such that $X = U|X|$. Here we prove that if a unital C^*-algebra $\mathcal{A}$ satisfies polar decomposition and admits "good" faithful tracial states then $\mathcal{A}$ is a Rickart C^*-algebra (Theorem 3.11). Polar decomposition allows us to characterize tracial states among all states on a Rickart C^*-algebra $\mathcal{A}$ by the inequality $\varphi(UPU^*) \le \varphi(P)$ for all partial isometries $U \in \mathcal{A}$ and projections $P \in \mathcal{A}$ (Theorem 3.9). The characterization of traces on C^*-algebras is an urgent problem and attracts the attention of a large group of researchers, see [9–13,32,36,43,48,54]. We also presented characterizations of the traces in a broad class of weights on von Neumann algebras (Lemma 3.5, Theorem 3.10). As a consequence from it we give characterization of tracial states among all states on arbitrary C^*-algebra $\mathcal{A}$ by inequality $\varphi(|X|) \le \varphi(Y)$ for all Hermitian $X, Y \in \mathcal{A}$ with $-Y \le X \le Y$ (Theorem 3.6, Remark 3.7), cf. with Gardner's characterization of tracial states by the inequality $|\varphi(X)| \le \varphi(|X|)$ for all $X \in \mathcal{A}$ in [36].

Originally studied by Gohberg and Krein in [37], the block projection operators admit a natural extension to the setting of quasi-normed ideals and noncommutative integration. Let $n \ge 2$ and projections $P_1, \ldots, P_n \in \mathcal{A}$ be such that $P_1 + \cdots + P_n = I$. Define a block projection operator $\mathcal{P}_n \colon \mathcal{A} \to \mathcal{A}$ by the formula $\mathcal{P}_n(X) = \sum_{k=1}^{n} P_k X P_k$ for all $X \in \mathcal{A}$. For a trace φ on a Rickart C^*-algebra $\mathcal{A}$ and for all Hermitian $X \in \mathcal{A}$ we proved the inequality $\varphi(|X - \mathcal{P}_n(X)|) \le C_n \varphi(|X|)$, where $C_n = 2 - 2^{2-n}$ for all $n \ge 2$ (Theorem 3.12). As its consequence we obtain a sharp estimate for a trace of the commutator of any Hermitian element and a projection (Corollary 3.13). Block projection operators on a von Neumann algebras (and on algebras of measurable with respect to semifinite normal traces operators) were investigated in [14,31]. In this case we established several uniform submajorization inequalities for block projection operators in [25].

2 Preliminaries

2.1 Weights and traces on C^*-algebras

A C^*-algebra is a complex Banach $*$-algebra $\mathcal{A}$ such that $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{A}$. For a C^*-algebra $\mathcal{A}$ by $\mathcal{A}^{\mathrm{pr}}$, $\mathcal{A}^{\mathrm{sa}}$ and $\mathcal{A}^+$ we denote its subsets of projections ($A = A^* = A^2$), Hermitian elements ($A^* = A$) and positive elements, respectively. If $A \in \mathcal{A}$, then $|A| = \sqrt{A^*A} \in \mathcal{A}^+$. If $A \in \mathcal{A}^{\mathrm{sa}}$ then $A_+ = (|A| + A)/2$ and $A_- = (|A| - A)/2$ lie in $\mathcal{A}^+$ and $A = A_+ - A_-$, $A_+A_- = 0$. If I is the unit of the algebra $\mathcal{A}$ and $P \in \mathcal{A}^{\mathrm{pr}}$ then $P^\perp = I - P$.

A mapping $\varphi : \mathcal{A}^+ \to [0, +\infty]$ is called *a weight* on a C^*-algebra $\mathcal{A}$, if $\varphi(X+Y) = \varphi(X)+\varphi(Y)$, $\quad \varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{A}^+$, $\lambda \ge 0$ (moreover, $0 \cdot (+\infty) \equiv 0$). For a weight φ define

$$\mathfrak{M}_\varphi^+ = \{X \in \mathcal{A}^+ \colon \ \varphi(X) < +\infty\}, \quad \mathfrak{M}_\varphi^{\mathrm{sa}} = \mathrm{lin}_{\mathbb{R}} \mathfrak{M}_\varphi^+, \quad \mathfrak{M}_\varphi = \mathrm{lin}_{\mathbb{C}} \mathfrak{M}_\varphi^+.$$

The restriction $\varphi|_{\mathfrak{M}_\varphi^+}$ can always be extended by linearity to a functional on $\mathfrak{M}_\varphi$, which we denote by the same letter φ. Such an extension allows us to identify finite weights (i.e., $\varphi(X) < +\infty$ for all $X \in \mathcal{A}^+$) with positive functionals on $\mathcal{A}$. A positive linear functional φ on $\mathcal{A}$ with $\|\varphi\| = 1$ is called *a state*. A weight φ is called *faithful*, if $\varphi(X) = 0 \ \ (X \in \mathcal{A}^+) \Rightarrow X = 0$; *a trace*, if $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{A}$. A trace φ on a C^*-algebra $\mathcal{A}$ is called *semifinite*, if $\varphi(A) = \sup\{\varphi(B): \ B \in \mathcal{A}^+, \ B \leq A, \ \varphi(B) < +\infty\}$ for every $A \in \mathcal{A}^+$.

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$, $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all linear bounded operators on $\mathcal{H}$. The strong operator topology (i.e., *so*-topology) is the locally convex topology generated by seminorms $X \in \mathcal{B}(\mathcal{H}) \mapsto \|X\xi\|, \xi \in \mathcal{H}$. By Gelfand–Naimark theorem every C^*-algebra is isometrically isomorphic to a concrete C^*-algebra of operators on a Hilbert space $\mathcal{H}$ [29, II.6.4.10]. By the commutant of a set $\mathcal{X} \subset \mathcal{B}(\mathcal{H})$ we mean the set

$$\mathcal{X}' = \{Y \in \mathcal{B}(\mathcal{H}): \ \ XY = YX \ \ \text{for all} \ \ X \in \mathcal{X}\}.$$

A $*$-subalgebra $\mathcal{A}$ of the algebra $\mathcal{B}(\mathcal{H})$ is said to be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, if $\mathcal{A} = \mathcal{A}''$. A weight φ on von Neumann algebra $\mathcal{A}$ is called *normal*, if $X_i \nearrow X \ \ (X_i, X \in \mathcal{A}^+) \Longrightarrow \varphi(X) = \sup \varphi(X_i)$; *semifinite*, if the set $\mathfrak{M}_\varphi$ is ultraweakly dense in $\mathcal{A}$ (see [56, Definition VII.1.1]). Let φ be a weight on a C^*-algebra $\mathcal{A}$. Let us define the seminorm

$$\|A\|_\varphi = \inf\{\varphi(A_1 + A_2): \ \ A = A_1 - A_2, \ \ A_1, A_2 \in \mathfrak{M}_\varphi^+\}$$

on the real vector space $\mathfrak{M}_\varphi^{\text{sa}}$. If φ be a faithful normal semifinite weight on a von Neumann algebra $\mathcal{A}$ then the function $A \mapsto \|A\|_\varphi \ \ (A \in \mathfrak{M}_\varphi^{\text{sa}})$ is a norm on $\mathfrak{M}_\varphi^{\text{sa}}$ [53, Corollary 15.5]. Using Upmeier's results [59], it is actually proved in [5, Theorem 1.4.2] that a weight on a von Neumann algebra $\mathcal{A}$ is a trace if and only if $\varphi(SAS) = \varphi(A)$ for any $A \in \mathcal{A}^+$ and a symmetry $S \in \mathcal{A}^{\text{sa}}$.

2.2 Representations of C^*-algebras

The universal representation of a C^*-algebra $\mathcal{A}$ is the pair

$$\{\pi, \mathfrak{H}\} = \sum_{\varphi \in \mathcal{S}(\mathcal{A})}^{\oplus} \{\pi_\varphi, \mathfrak{H}_\varphi\},$$

where $\mathcal{S}(\mathcal{A})$ is the set of all states on $\mathcal{A}$, $(\pi_\varphi, \mathfrak{H}_\varphi)$ is the Gelfand–Naimark–Segal representation of a C^*-algebra $\mathcal{A}$, associated with φ. In this case the von Neumann algebra $\mathcal{M} = \pi(\mathcal{A})''$, generated by $\pi(\mathcal{A})$, is called *the universal enveloping von Neumann algebra* of C^*-algebra $\mathcal{A}$ [55, Chap. III, Definition 2.3].

Let φ be a positive linear functional on a C^*-algebra $\mathcal{A}$ and π be the universal representation of $\mathcal{A}$. By construction of π an arbitrary state on $\mathcal{A}$ turns into a vector state on $\pi(\mathcal{A})$, hence it is extended to normal state on the universal enveloping algebra $\mathcal{M} = \pi(\mathcal{A})''$. Hence for φ there exists a positive normal functional $\widehat{\varphi}$ on the universal enveloping von Neumann algebra such that $\widehat{\varphi}(\pi(A)) = \varphi(A) \ \ (A \in \mathcal{A}^+)$.

A representation with a trace of a C^*-algebra $\mathcal{A}$ is a pair (π, ν) with the following properties:

(i) π is a nondegenerate representation of a C^*-algebra $\mathcal{A}$ on some Hilbert space;
(ii) ν is a faithful normal trace on the von Neumann algebra $\pi(\mathcal{A})''$;
(iii) $\pi(\mathcal{A}) \cap \mathfrak{N}_\nu$ generate the von Neumann algebra $\pi(\mathcal{A})''$, where

$$\mathfrak{N}_\nu = \{A \in \pi(\mathcal{A})'' : \nu(A^*A) < +\infty\}.$$

Let $\mathcal{A}$ be a C^*-algebra, (π, ν) be a representation of $\mathcal{A}$ with a trace. Then ν is semifinite and $\varphi = (\nu \circ \pi)|_{\mathcal{A}^+}$ is a lower semicontinuous trace on $\mathcal{A}$. Conversely, let φ be a lower semicontinuous semifinite trace on $\mathcal{A}$. Then there exists a representation of $\mathcal{A}$ with a trace (π, ν) ([34, 6.6]), which is called *associated with* φ, and ν is called a *natural trace*. In this case, the relation

$$\nu(\pi(A)) = \varphi(A) \quad \text{for all} \quad A \in \mathcal{A}^+ \tag{1}$$

holds (see [34, Proposition 6.6.5 (i)]).

Lemma 2.1 *Let φ be a lower semicontinuous semifinite trace on a C^*-algebra $\mathcal{A}$, (π, ν) and $\mathcal{M}$ be defined as above.*

(i) *If $A \in \mathcal{A}^+$ and a number $p > 0$, then $\pi(A^p) = \pi(A)^p$.*
(ii) *If $A \in \mathcal{A}$, then $|\pi(A)| = \pi(|A|)$.*
(iii) *If $A \in \mathfrak{M}_\varphi$, then $\pi(A) \in \mathfrak{M}_\nu$ and $\nu(\pi(A)) = \varphi(A)$.*

Proof (i). Recall that if $A \in \mathcal{A}^+$, then $\pi(A) \in \mathcal{M}^+$. **Step 1**. For a rational number $p > 0$ the assertion can easily be deduced from the relation

$$\pi(XY) = \pi(X)\pi(Y) \quad \text{for all} \quad X, Y \in \mathcal{A}. \tag{2}$$

Step 2. For an irrational number $p > 0$ we choose a sequence $\{p_n\}_{n=1}^\infty$ of positive rational numbers convergent to p. Applying $\|\cdot\|$-continuity of the mapping

$$x \mapsto A^x \quad (x > 0; \quad \text{an element} \quad A \in \mathcal{A}^+ \quad \text{is fixed})$$

and $\|\cdot\|$-continuity of the representation (π, ν), and taking into account Step 1 we obtain the required assertion.

(ii). Follows by (2), the relation $\pi(X^*) = \pi(X)^*$ for all $X \in \mathcal{A}$ and item (i) with $p = 1/2$.

(iii). For $A \in \mathfrak{M}_\varphi$ there exist the sets $\{\lambda_k\}_{k=1}^n \subset \mathbb{C}$ and $\{A_k\}_{k=1}^n \subset \mathfrak{M}_\varphi^+$ such that $A = \sum_{k=1}^n \lambda_k A_k$. By (1) we have $\pi(A_k) \in \mathfrak{M}_\nu^+$ for all $k = 1, \ldots, n$. Therefore, $\pi(A) = \sum_{k=1}^n \lambda_k \pi(A_k) \in \mathfrak{M}_\nu$ and

$$\nu(\pi(A)) = \sum_{k=1}^n \lambda_k \nu(\pi(A_k)) = \sum_{k=1}^n \lambda_k \varphi(A_k) = \varphi(A)$$

by the correctness of the linear extension of the trace ν to $\mathfrak{M}_\nu$. □

3 Traces on Rickart C*-algebras

Lemma 3.1 *Let $\mathcal{A}$ be a Rickart C^*-algebra and $X \in \mathcal{A}^{\mathrm{sa}}$, $Y \in \mathcal{A}^{\mathrm{sa}}$. If $-Y \le X \le Y$ then $2|X| \le Y + UYU$ for some unitary element $U \in \mathcal{A}^{\mathrm{sa}}$.*

Proof Since a Rickart C^*-algebra $\mathcal{A}$ is unital [52, Lemma 2.2] and every element $Z \in \mathcal{A}^{\mathrm{sa}}$ possesses a well defined support projection [28], we can literally repeat the proof of [14, Theorem 1]. □

Lemma 3.2 (cf. [14, Corollary 1]) *Let $\mathcal{A}$ be a Rickart C^*-algebra. Then for any finite set $\{A_k\}_{k=1}^n \subset \mathcal{A}^{\mathrm{sa}}$ there exists a unitary element $U \in \mathcal{A}^{\mathrm{sa}}$ such that*

$$|A_1 + \cdots + A_n| \le \frac{|A_1| + \cdots + |A_n| + U(|A_1| + \cdots + |A_n|)U}{2}.$$

Proof We sum the inequalities

$$-|A_k| \le A_k \le |A_k|, \quad k = 1, \ldots, n,$$

term by term and obtain $-|A_1| - \cdots - |A_n| \le A_1 + \cdots + A_n \le |A_1| + \cdots + |A_n|$. Now for a pair of elements

$$X = A_1 + \cdots + A_n \in \mathcal{A}^{\mathrm{sa}}, \quad Y = |A_1| + \cdots + |A_n| \in \mathcal{A}^{+}$$

we apply Lemma 3.1. □

Theorem 3.3 *Let φ be a trace on a Rickart C^*-algebra $\mathcal{A}$. For any finite set $\{A_k\}_{k=1}^n \subset \mathcal{A}^{\mathrm{sa}}$ we have*

$$\varphi\left(\left|\sum_{k=1}^{n} A_k\right|\right) \le \sum_{k=1}^{n} \varphi(|A_k|).$$

Corollary 3.4 *Let φ be a trace on a Rickart C^*-algebra $\mathcal{A}$. If $X \in \mathcal{A}^{\mathrm{sa}}$, $Y \in \mathcal{A}^{+}$ with $-Y \le X \le Y$, then $\varphi(|X|) \le \varphi(Y)$. In particular, if $A, B \in \mathcal{A}^{\mathrm{sa}}$ then $\varphi(|AB+BA|) \le \varphi(A^2 + B^2)$.*

Proof By Lemma 3.1 there exists a unitary element $U \in \mathcal{A}^{\mathrm{sa}}$ such that $2|X| \le Y + UYU$. Hence by monotonocity of a trace φ on the cone $\mathcal{A}^{+}$ we have

$$2\varphi(|X|) = \varphi(2|X|) \le \varphi(Y + UYU) = \varphi(Y) + \varphi(UYU) = 2\varphi(Y).$$

If $A, B \in \mathcal{A}^{\mathrm{sa}}$ then $-A^2 - B^2 \le AB + BA \le A^2 + B^2$ by the inequalities $(A \pm B)^2 \ge 0$. □

Lemma 3.5 *Let a weight φ on a von Neumann algebra $\mathcal{A}$ be a) normal and semifinite or b) finite. Then the following conditions are equivalent:*

(i) $\varphi(|X|) \le \varphi(Y)$ *for all* $X \in \mathcal{A}^{\mathrm{sa}}$, $Y \in \mathcal{A}^{+}$ *with* $-Y \le X \le Y$;
(ii) $\varphi(|AB + BA|) \le \varphi(A^2 + B^2)$ *for all* $A, B \in \mathcal{A}^{\mathrm{sa}}$;
(iii) φ *is a trace.*

Proof (i)⇒(iii). For arbitrary $A, B \in \mathcal{A}^{\mathrm{sa}}$ we have $-|A| \le A \le |A|, -|B| \le B \le |B|$. We sum these inequalities term by term and obtain

$$-|A| - |B| \le A + B \le |A| + |B| \quad \text{for all} \quad A, B \in \mathcal{A}^{\mathrm{sa}}.$$

Now by (i) we have the inequality $\varphi(|A + B|) \le \varphi(|A| + |B|) = \varphi(|A|) + \varphi(|B|)$ for all $A, B \in \mathcal{A}^{\mathrm{sa}}$ and φ is a trace by Theorem 2 of [54] in the case a) (Theorem 1 of [54] in the case b)).

(ii)⇒(i). For arbitrary $X \in \mathcal{A}^{\mathrm{sa}}$, $Y \in \mathcal{A}^{+}$ with $-Y \le X \le Y$ there exist $A \in \mathcal{A}^{\mathrm{sa}}$, $B \in \mathcal{A}^{+}$ such that $X = AB + BA$, $Y = A^2 + B^2$ by [18, Lemma 1].

The implications (iii)⇒(i), (iii)⇒(ii) are established in Corollary 3.4. □

Theorem 3.6 *Let a positive linear functional φ on a C^*-algebra $\mathcal{A}$ meet one of the following conditions:*

(i) $\varphi(|X|) \le \varphi(Y)$ *for all* $X \in \mathcal{A}^{\mathrm{sa}}$, $Y \in \mathcal{A}^{+}$ *with* $-Y \le X \le Y$;
(ii) $\varphi(|AB + BA|) \le \varphi(A^2 + B^2)$ *for all* $A, B \in \mathcal{A}^{\mathrm{sa}}$.

Then the functional φ is tracial.

Proof Note first that if φ is a tracial functional on $\mathcal{A}$ then conditions (i) and (ii) are met, see Remark 3.7 below. Assume now that condition i) is met.

Consider the universal enveloping von Neumann algebra of a C^*-algebra $\mathcal{A}$ [55, III.2]. Let π be the corresponding universal representation of a C^*-algebra $\mathcal{A}$ and $\widehat{\varphi}$ be a positive normal functional on $\mathcal{M} = \pi(\mathcal{A})''$ such that $\widehat{\varphi}(\pi(A)) = \varphi(A)$ for all $A \in \mathcal{A}$. Consider the operators $\widehat{X} \in \mathcal{M}^{\mathrm{sa}}$ and $\widehat{Y} \in \mathcal{M}^{+}$ with $-\widehat{Y} \le \widehat{X} \le \widehat{Y}$. It follows by Kaplansky density theorem that there exist bounded nets $\{X_\alpha\}$ from $\pi(\mathcal{A})^{\mathrm{sa}}$ and $\{Y_\alpha\}$ from $\pi(\mathcal{A})^{+}$, which *so*-converge to $\widehat{X}$, $\widehat{Y}$. Let $\{H_\alpha\}$ and $\{K_\alpha\}$ be such that

$$X_\alpha = \pi(H_\alpha) \quad \text{and} \quad Y_\alpha = \pi(K_\alpha).$$

We can assume that $H_\alpha \in \mathcal{A}^{\mathrm{sa}}$, $K_\alpha \in \mathcal{A}^{+}$ and $-K_\alpha \le H_\alpha \le K_\alpha$. Indeed, let $Z = \pi(A) \in \pi(\mathcal{A})^{+}$. If

$$A = B + iC \quad (B, C \in \mathcal{A}^{\mathrm{sa}}; \quad i \in \mathbb{C}, i^2 = -1),$$

then $\pi(A) = \pi(B) + i\pi(C)$ and $\pi(C) = 0$, hence $Z = \pi(B)$. On the other hand, we have $Z^{1/2} = \pi(A')$ for some $A' \in \mathcal{A}^{\mathrm{sa}}$. Now

$$Z = Z^{1/2}Z^{1/2} = \pi(A')\pi(A') = \pi(A'^2) \quad \text{and} \quad A'^2 \ge 0.$$

Hence we can assume that $A = A'^2 \in \mathcal{A}^{+}$.

Note that $\widehat{Y}-\widehat{X}, \widehat{Y}+\widehat{X} \in \mathcal{M}^+$. We proved that there exist nets $\{T_\alpha\}$ and $\{S_\alpha\}$ in $\mathcal{A}^+$ such that $\pi(T_\alpha) \to \widehat{Y}-\widehat{X}$ and $\pi(S_\alpha) \to \widehat{Y}+\widehat{X}$ in the so-topology. Put

$$H_\alpha = \frac{T_\alpha - S_\alpha}{2}, \quad K_\alpha = \frac{T_\alpha + S_\alpha}{2}.$$

Thus, there exist bounded nets $\{H_\alpha\}$ in $\mathcal{A}^{\mathrm{sa}}$ and $\{K_\alpha\}$ in $\mathcal{A}^+$ such that $-K_\alpha \le H_\alpha \le K_\alpha$ and $\pi(H_\alpha) \to \widehat{X}$ and $\pi(K_\alpha) \to \widehat{Y}$ in the so-topology. Note that $-\pi(K_\alpha) \le \pi(H_\alpha) \le \pi(K_\alpha)$. By item (i) of Lemma 3.5 we have

$$\widehat{\varphi}(|\pi(H_\alpha)|) \le \widehat{\varphi}(\pi(K_\alpha)).$$

We now take into account so-continuity of the functional calculus, pass to the limit in the so-topology in the latter inequality and obtain

$$\widehat{\varphi}(|\widehat{X}|) \le \widehat{\varphi}(\widehat{Y}).$$

By item (i) of Lemma 3.5 $\widehat{\varphi}$ is a tracial functional on $\mathcal{M}$. Now for all $X, Y \in \mathcal{A}$ we have

$$\begin{aligned}\varphi(XY) &= \widehat{\varphi}(\pi(XY)) = \widehat{\varphi}(\pi(X)\pi(Y)) = \\ &= \widehat{\varphi}(\pi(Y)\pi(X)) = \widehat{\varphi}(\pi(YX)) = \varphi(YX).\end{aligned}$$

Thus, φ is a tracial functional on $\mathcal{A}$. □

Remark 3.7 Let φ be a lower semicontinuous semifinite trace on a C^*-algebra $\mathcal{A}$ [34, 6.1.1]. Then $\varphi(|X|) \le \varphi(Y)$ for all $X \in \mathcal{A}^{\mathrm{sa}}$, $Y \in \mathcal{A}^+$ with $-Y \le X \le Y$. In particular, $\varphi(|AB+BA|) \le \varphi(A^2+B^2)$ for all $A, B \in \mathcal{A}^{\mathrm{sa}}$. Indeed, let (π, ν) be a representation of the C^*-algebra $\mathcal{A}$, assosiated with the trace φ. Then for a "natural" faithful normal trace ν on a von Neumann algebra $\mathcal{M} = \pi(\mathcal{A})''$ relation (1) holds. For $X \in \mathcal{A}^{\mathrm{sa}}$, $Y \in \mathcal{A}^+$ with $-Y \le X \le Y$ we have $-\pi(Y) \le \pi(X) \le \pi(Y)$. Then by item (ii) of Lemma 2.1 and Lemma 3.5 we obtain

$$\varphi(|X|) = \nu(\pi(|X|)) = \nu(|\pi(X)|) \le \nu(\pi(Y)) = \varphi(Y).$$

Corollary 3.8 *Let a C^*-algebra $\mathcal{A}$ be such that $|X| \le Y$ for all $X \in \mathcal{A}^{\mathrm{sa}}$, $Y \in \mathcal{A}^+$ with $-Y \le X \le Y$. Then the algebra $\mathcal{A}$ is abelian.*

Proof Any positive functional φ on $\mathcal{A}$ is subject to the inequality of item (i) of Theorem 3.6. It implies that any positive functional on $\mathcal{A}$ is tracial, i. e. $\varphi(XY) = \varphi(YX)$ holds for all elements $X, Y \in \mathcal{A}$. Since the set of all positive linear functionals separate points of the algebra $\mathcal{A}$, from the latter condition we have $XY = YX$ for all $X, Y \in \mathcal{A}$. Therefore the C^*-algebra $\mathcal{A}$ is abelian. □

Theorem 3.9 *For a positive linear functional φ on a Rickart C^*-algebra $\mathcal{A}$ the following conditions are equivalent:*

(i) *φ is tracial;*
(ii) *$\varphi(UPU^*) \le \varphi(P)$ for all partial isometries $U \in \mathcal{A}$ and $P \in \mathcal{A}^{\mathrm{pr}}$.*

Proof (i)$\Rightarrow$(ii). Let $A \in \mathcal{A}$ and $P \in \mathcal{A}^{\mathrm{pr}}$. By the inequality $A^*A \le \|A\|^2 I$ we obtain $PA^*AP \le P \cdot \|A\|^2 I \cdot P = \|A\|^2 P$ and by monotonocity of the functional φ on $\mathcal{A}^+$ and its homogeneity we have

$$\varphi(APA^*) = \varphi(AP \cdot PA^*) = \varphi(PA^*AP) \le \|A\|^2 \varphi(P).$$

(ii)$\Rightarrow$(i). Let $A \in \mathcal{A}$ with $\|A\| = 1$ be arbitrary and $U \in \mathcal{A}$ be a partial isometry. We can choose convex combinations of projections

$$X_n = \sum_{k=1}^{m_n} \lambda_k^{(n)} P_k^{(n)}, \quad \lambda_k^{(n)} > 0, \ k = 1, \ldots, m_n, \quad \sum_{k=1}^{m_n} \lambda_k^{(n)} = 1, \quad \{P_k\}_{k=1}^{m_n} \subset \mathcal{A}^{\mathrm{pr}}, \quad n \in \mathbb{N},$$

so that $X_n \xrightarrow{\|\cdot\|} A^*A$ as $n \to \infty$, see the implication (R)$\Rightarrow$(CP) in Theorem 6.1.2 [28]. Hence

$$UX_nU^* \xrightarrow{\|\cdot\|} UA^*AU^* \quad \text{as} \quad n \to \infty$$

by continuity of the product operation in $\mathcal{A}$. Note that

$$\varphi(UX_nU^*) = \sum_{k=1}^{m_n} \lambda_k^{(n)} \varphi(UP_k^{(n)}U^*) \le \sum_{k=1}^{m_n} \lambda_k^{(n)} \varphi(P_k^{(n)}) = \varphi(X_n)$$

for all $n \in \mathbb{N}$. We pass here to the limit as $n \to \infty$, and by the automatic continuity of the positive linear functional φ conclude that

$$\varphi(UA^*AU^*) \le \varphi(A^*A).$$

Let U be the partial isometry from the polar decomposition $A = U|A|$ of the element A. Then $|A^*| = U|A|U^*$ and $U^*A = |A|$. Therefore, $UA^*AU^* = AA^*$ and $\varphi(AA^*) \le \varphi(A^*A)$.

Now we consider the element A^* instead of A in the foregoing proof, apply the equality $(A^*)^* = A$, and analogously obtain $\varphi(A^*A) \le \varphi(AA^*)$. ◻

Theorem 3.10 *Let φ be a normal semifinite weight on a von Neumann algebra $\mathcal{A}$ such that $\varphi(QPQ) \le \varphi(P)$ for all $P, Q \in \mathcal{A}^{\mathrm{pr}}$. Then φ is a trace.*

Proof It follows by item (iv) of Theorem 3.4 in [11] that for every projection $T \in \mathcal{A}^{\mathrm{pr}}$ with $\varphi(T) < \infty$ the reduced weight φ_T on the reduced algebra $T\mathcal{A}T$ is a trace. Hence φ is a trace by [54, Lemma 2]. ◻

For other trace characterizations see [15–17,17,19–24,58] and references therein.

If A is a bounded operator on the Hilbert space $\mathcal{H}$ and $0 \le A \le I$, then $\{A^{1/n}\}$ is a monotone increasing sequence of operators whose strong-operator limit is the projection on the closure of the range of A [44, Lemma 5.1.5].

Theorem 3.11 *Let a unital C^*-algebra $\mathcal{A}$ satisfy polar decomposition. Assume that for every $A \in \mathcal{A}$, $\|A\| = 1$, with the polar decomposition $A = U|A|$ there exists a faithful tracial functional φ_A on $\mathcal{A}$ with the following property:*

$$\varphi_A(P - |A|^{1/n}) \to 0 \quad as \quad n \to \infty, \tag{3}$$

where $P = U^*U$. Then $\mathcal{A}$ is a Rickart C^*-algebra.

Proof Let A, P, φ_A be given as in the formulation of the theorem, and $A_n = |A|^{1/n}$ for all $n \in \mathbb{N}$. Then $P \in \mathcal{A}^{\mathrm{pr}}$, $P|A| = |A|$ and $0 \le A_n \le A_{n+1} \le P$ for all $n \in \mathbb{N}$. Let (π, ν) be the representation of a C^*-algebra $\mathcal{A}$, associated with a trace φ_A (see [34, 6.6.4]). Then relation (1) holds for the "natural" faithful normal semifinite trace ν on the von Neumann algebra $\mathcal{M} = \pi(\mathcal{A})''$, generated by $\pi(\mathcal{A})$ (see [34, A.60]). Hence $\nu(I) = \varphi_A(I) < +\infty$. For operators

$$B_n = \pi(A_n) = \pi(|A|)^{1/n}, \quad n \in \mathbb{N},$$

we have $0 \le B_n \le B_{n+1} \le I$ for all $n \in \mathbb{N}$. Hence by Vigier theorem (see, for example, [44, Lemma 5.1.4]) there exists

$$Q = \sup_{n \ge 1} B_n = \lim_{n \to \infty} B_n \in \mathcal{M}^{\mathrm{pr}},$$

where the limit is taken in the so-topology on $\mathcal{M}$. The projection Q is the support projection of the operator $\pi(|A|)$ and $\pi(|A|) = Q\pi(|A|)$. Let us show that $Q = \pi(P)$. We have

$$\pi(P)Q = \pi(P) \lim_{n \to \infty} B_n = \lim_{n \to \infty} \pi(P)B_n = \lim_{n \to \infty} \pi(P|A|)\pi(|A|^{1/n-1}) =$$
$$= \lim_{n \to \infty} B_n = Q,$$

i.e., $Q \le \pi(P)$. Obviously, by (1) we obtain

$$\varphi_A(P - A_n) = \nu(\pi(P - A_n)) = \nu(\pi(P) - \pi(A_n)) = \nu(\pi(P) - B_n) =$$
$$= \nu(\pi(P) - Q + Q - B_n) = \nu(\pi(P) - Q) + \nu(Q - B_n), \quad n \in \mathbb{N}.$$

Via the so-continuity of the normal functional ν we have $\nu(Q - B_n) \to 0$ as $n \to \infty$. Therefore, $\nu(\pi(P) - Q) = 0$ via (3). Since ν is faithful, we obtain $Q = \pi(P)$.

Let us prove that $\mathcal{A}$ is a weakly Rickart C^*-algebra (see [52, Definition 3.2]). Assume that $X \in \mathcal{A}$ with $AX = 0$. Let us show that $PX = 0$. Since $AXX^*A^* = 0$, we have

$$\pi(AXX^*A^*) = \pi(A)\pi(X)\pi(X)^*\pi(A)^* = 0. \tag{4}$$

Since the von Neumann algebra $\mathcal{M}$ is a weakly Rickart C^*-algebra, via (4) we obtain

$$0 = Q\pi(X)\pi(X)^*Q = \pi(P)\pi(X)\pi(X)^*\pi(P) = \pi(PXX^*P).$$

Now by (1) we have $0 = \nu(\pi(PXX^*P)) = \varphi_A(PXX^*P)$ and, since φ_A is faithful, we obtain $PXX^*P = |X^*P|^2 = 0$. Therefore, $\||X^*P|^2\| = \||X^*P|\|^2 = \|X^*P\|^2 = 0$ and $X^*P = 0$. Thus $PX = (X^*P)^* = 0$ and $\mathcal{A}$ is a weakly Rickart C^*-algebra. Since $\mathcal{A}$ is unital, it is a Rickart C^*-algebra by [8, Section 4, Theorem 1]. □

Theorem 3.12 *Let φ be a trace on a Rickart C^*-algebra $\mathcal{A}$, $A \in \mathcal{A}^{\text{sa}}$, $n \geq 2$ and $P_1, \dots, P_n \in \mathcal{A}^{\text{pr}}$ be with $P_1 + \cdots + P_n = I$, $\mathcal{P}_n(A) = \sum_{k=1}^n P_k A P_k$. Then we have*

(i) $\varphi(|A - \mathcal{P}_n(A)|) \leq C_n \varphi(|A|)$, *where $C_n = 2 - 2^{2-n}$ for all $n \geq 2$;*
(ii) $\varphi(|\mathcal{P}_n(A)|) \leq \varphi(\mathcal{P}_n(|A|))$ *for all $n \geq 2$.*

Proof (i). It is clear that $P_k P_m = 0$ for $k \neq m$, where $k, m = 1, \dots, n$. Since $A \in \mathcal{A}^{\text{sa}}$ and $\mathcal{P}_n(A) \in \mathcal{A}^{\text{sa}}$, we conclude that $A - \mathcal{P}_n(A) \in \mathcal{A}^{\text{sa}}$.

Step 1. Consider $A \in \mathcal{A}^+$ and $n \geq 2$. By [14, Lemma 2] we have the representation

$$\mathcal{P}_n(A) = \frac{1}{2^{n-1}} \sum_{k=1}^{2^{n-1}} S_k A S_k, \tag{5}$$

where the unitaries $S_k \in \mathcal{A}^{\text{sa}}$, $k = 1, \dots, 2^{n-1}$, have the form $P_1 \pm P_2 \pm \cdots \pm P_n$. Denote $P_1 + P_2 + \cdots + P_n = I$ by $S_{2^{n-1}}$. Then

$$A - \mathcal{P}_n(A) = \sum_{k=1}^{2^{n-1}-1} \frac{1}{2^{n-1}} (A - S_k A S_k)$$

and by Theorem 3.3 we obtain

$$\begin{aligned}
\varphi(|A - \mathcal{P}_n(A)|) &= \frac{1}{2^{n-1}} \varphi\Big(\Big| \sum_{k=1}^{2^{n-1}-1} (A - S_k A S_k) \Big|\Big) \leq \frac{1}{2^{n-1}} \sum_{k=1}^{2^{n-1}-1} \varphi(|A - S_k A S_k|) \leq \\
&\leq \frac{1}{2^{n-1}} \sum_{k=1}^{2^{n-1}-1} (\varphi(A) + \varphi(S_k A S_k)) = \frac{2(2^{n-1} - 1)}{2^{n-1}} \varphi(A) = \\
&= (2 - 2^{2-n}) \varphi(A).
\end{aligned}$$

Step 2. Let $n \geq 2$, $A \in \mathcal{A}^{\text{sa}}$ and $A = A_+ - A_-$ be the Jordan decomposition into positive and negative parts with $A_+ A_- = 0$ and $A_+ + A_- = |A|$. For arbitrary $X, Y \in \mathcal{A}^{\text{sa}}$ we have $\varphi(|X + Y|) \leq \varphi(|X|) + \varphi(|Y|)$, see Theorem 3.3. Therefore by Step 1 for A_+ and A_- we obtain

$$\begin{aligned}
\varphi(|A - \mathcal{P}_n(A)|) &\leq \varphi(|A_+ - \mathcal{P}_n(A_+)|) + \varphi(|A_- - \mathcal{P}_n(A_-)|) \leq \\
&\leq C_n \varphi(A_+) + C_n \varphi(A_-) = C_n \varphi(|A|)
\end{aligned}$$

with $C_n = 2 - 2^{2-n}$ for all $n \geq 2$.

Finally, for every number $n \geq 2$ we consider the one-dimensional projection $A^{(n)} \in \mathbb{M}_n(\mathbb{C})$, which in an orthonormal basis $\{\xi_1, \dots, \xi_n\} \subset \mathbb{C}^n$ has the form $a_{ij}^{(n)} = 1/n$ for all $i, j = 1, \ldots, n$ and let $\varphi = \mathrm{tr}$ be the canonical trace, $\langle\cdot,\cdot\rangle$ be a scalar product on $\mathbb{C}^n$. For projections $P_i = \langle\cdot, \xi_i\rangle\xi_i$, $i = 1, \ldots, n$ we have $A^{(n)} - \mathcal{P}_n(A^{(n)}) = A^{(n)} - \frac{1}{n}I = \left(1 - \frac{1}{n}\right)A^{(n)} - \frac{1}{n}A^{(n)\perp}$, hence $|A^{(n)} - \mathcal{P}_n(A^{(n)})| = \left(1 - \frac{1}{n}\right)A^{(n)} + \frac{1}{n}A^{(n)\perp}$ and

$$\begin{aligned}\varphi(|A^{(n)} - \mathcal{P}_n(A^{(n)})|) &= \left(1 - \frac{1}{n}\right)\varphi(A^{(n)}) + \frac{1}{n}\varphi(A^{(n)\perp})\\ &= \left(1 - \frac{1}{n}\right)\cdot 1 + \frac{1}{n}\cdot(n-1) = 2 - \frac{2}{n}\end{aligned}$$

for all $n \geq 2$. In particular, $C_2 = 1$ is the sharp constant. Thus, for the best possible constants $\tilde{C}_n$ in the inequalities $\varphi(|A - \mathcal{P}_n(A)|) \leq C_n\varphi(|A|)$ we have the estimates $2 - 2n^{-1} \leq \tilde{C}_n \leq 2 - 2^{2-n}$ for all $n \geq 3$.

(ii). Since $-P_k|A|P_k \leq P_kAP_k \leq P_k|A|P_k$ for all $k = 1, \ldots, n$, we have $-\mathcal{P}_n(|A|) \leq \mathcal{P}_n(A) \leq \mathcal{P}_n(|A|)$ for all $n \geq 2$, and $\varphi(|\mathcal{P}_n(A)|) \leq \varphi(\mathcal{P}_n(|A|))$ by Corollary 3.4. □

Note that

$$|PAP^\perp + P^\perp AP| = |PAP^\perp - P^\perp AP|, \quad [A, P] = -(PAP^\perp - P^\perp AP). \tag{6}$$

The formula $S = 2P - I$ defines a one-to-one correspondence between symmetries $S \in \mathcal{A}^{\mathrm{sa}}$ and projections $P \in \mathcal{A}^{\mathrm{pr}}$. Then by (6) we have

$$|A - SAS| = 2|PAP^\perp + P^\perp AP| = 2|[A, P]|.$$

Corollary 3.13 *Let φ be a trace on a Rickart C^*-algebra $\mathcal{A}$, $A \in \mathcal{A}^{\mathrm{sa}}$ and $P \in \mathcal{A}^{\mathrm{pr}}$. Then $\varphi(|[A, P]|) \leq \varphi(|A|)$.*

For $\mathcal{A} = \mathbb{M}_2(\mathbb{C})$ and $P = \mathrm{diag}(1, 0)$, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we have $|[A, P]| = |A| = I$.

Remark 3.14 Let $\mathcal{A}$ be a von Neumann algebra and $A \in \mathcal{A}^{\mathrm{pr}}$. In this case, the assertion of Corollary 3.13 was proved by another method in Theorem 2 of [21]. Assume also that $P = \mathcal{P}_n(A) \in \mathcal{A}^{\mathrm{pr}}$. Via representation (5) the projection P is a convex combination of projections S_kAS_k, $k = 1, \ldots, 2^{n-1}$. Since $\mathcal{A}^{\mathrm{pr}}$ belongs to the set $\mathrm{ext}\{X \in \mathcal{A}^+ : \|X\| \leq 1\}$ of the extreme points of the positive part of the unit ball of $\mathcal{A}$ [44, Chapter 2, 2.8.14], we infer that $A = S_kPS_k$ for all $k = 1, \ldots, 2^{n-1}$.

Proposition 3.15 *For all $A \in \mathbb{M}_m(\mathbb{C})$ and $P \in \mathbb{M}_m(\mathbb{C})^{\mathrm{pr}}$ we have a determinant relation $|\det(A - \mathcal{P}_2(A))| = |\det([A, P])|$.*

Proof Since $|\det(X)| = \det(|X|)$ for all $X \in \mathbb{M}_m(\mathbb{C})$, by (6) we have

$$\begin{aligned}|\det(A - \mathcal{P}_2(A))| &= \det(|A - \mathcal{P}_2(A)|) = \det(|PAP^\perp + P^\perp AP|) =\\ &= \det(|PAP^\perp - P^\perp AP|) = \det(|[A, P]|) = |\det([A, P])|.\end{aligned}$$

□

Theorem 3.16 *Let $\mathcal{A} = \mathbb{M}_m(\mathbb{C})$ and a unitary element $A \in \mathcal{A}$ be such that for some $n \leq m$ the element $U = \mathcal{P}_n(A)$ is also unitary. Then* $\det(U) = \det(A)$.

Proof Via representation (5), see also [14, Lemma 2], the unitary U is a convex combination of unitaries $S_k A S_k$, $k = 1, \dots, 2^{n-1}$. Since the unitaries from $\mathcal{A}$ belong to the set $\operatorname{ext}\{X \in \mathcal{A} : \|X\| \leq 1\}$ of the extreme points of the unit ball of $\mathcal{A}$ [55, Chap. I, Theorem 10.2], we infer that $U = S_k A S_k$ for all $k = 1, \dots, 2^{n-1}$. Then we apply theorem on determinant of matrix product, since $\det(S_k) \in \{-1, 1\}$ for all $k = 1, \dots, 2^{n-1}$. □

Proposition 3.17 *Consider a separable Hilbert space $\mathcal{H}$, an operator $A \in \mathcal{B}(\mathcal{H})^{\mathrm{sa}}$, a vector $\xi \in \mathcal{H}$ and a number $\varepsilon > 0$. Then there exists a finite-dimensional projection $P \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ such that $P\xi = \xi$ and $\|[A, P]\|_2 < \varepsilon$, where $\| \cdot \|_2$ is the Hilbert–Schmidt norm.*

Proof Lemma of the Weyl–von Neumann theorem proof (see, for example, [42, Lemma 14.11]) shows us that there exists a finite-dimensional projection $P \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ such that $P\xi = \xi$ and $\|P^{\perp} A P + P A P^{\perp}\|_2 < \varepsilon$. We note that $\|X\|_2 = \||X|\|_2$ for all Hilbert–Schmidt operators $X \in \mathcal{B}(\mathcal{H})$ and apply relations (6). □

Lemma 3.18 *Let φ be a weight on a C^*-algebra $\mathcal{A}$, $A \in \mathfrak{M}_{\varphi}^{\mathrm{sa}}$ and $B \in \mathcal{A}^{\mathrm{sa}}$.*

(i) *We have $|\varphi(A)| \leq \|A\|_{\varphi} \leq \varphi(|A|)$.*
(ii) *If $-A \leq B \leq A$ then $B \in \mathfrak{M}_{\varphi}^{\mathrm{sa}}$ and $\|B\|_{\varphi} \leq \|A\|_{\varphi}$.*

Proof (i). For all $A_1, A_2 \in \mathfrak{M}_{\varphi}^{+}$ with $A_1 - A_2 = A$ we have $|\varphi(A)| \leq \varphi(A_1 + A_2)$. Passing to infimum over all such A_1, A_2, we obtain $|\varphi(A)| \leq \|A\|_{\varphi}$. The inequality $\|A\|_{\varphi} \leq \varphi(|A|)$ follows by the Jordan decomposition $A = A_+ - A_-$ with $A_+ A_- = 0$ and $|A| = A_+ + A_-$.

(ii). The relation $0 \leq A + B \leq 2A$ yields that $A + B \in \mathfrak{M}_{\varphi}^{+}$ and $B = (A + B) - A \in \mathfrak{M}_{\varphi}^{\mathrm{sa}}$. Since $\|X\|_{\varphi} = \varphi(X)$ for every $X \in \mathfrak{M}_{\varphi}^{+}$ we have

$$\left\| \frac{A+B}{2} \right\|_{\varphi} = \varphi\Big(\frac{A+B}{2}\Big), \quad \left\| \frac{A-B}{2} \right\|_{\varphi} = \varphi\Big(\frac{A-B}{2}\Big).$$

Therefore via the triangle inequality and additivity of φ we obtain

$$\|B\|_{\varphi} = \left\| \frac{B+A}{2} - \frac{A-B}{2} \right\|_{\varphi} \leq \left\| \frac{B+A}{2} \right\|_{\varphi} + \left\| \frac{A-B}{2} \right\|_{\varphi} = \\ = \varphi\Big(\frac{B+A}{2}\Big) + \varphi\Big(\frac{A-B}{2}\Big) = \varphi(A) = \|A\|_{\varphi}.$$

□

Theorem 3.19 *Let $A \in \mathcal{B}(\mathcal{H})^{\mathrm{sa}}$, $P \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ and $S = 2P - I$. Then for all $\delta > 0$ we have $-\delta I - \delta^{-1}|[A, P]|^2 \leq A - SAS \leq \delta I + \delta^{-1}|[A, P]|^2$.*

Proof By the Comb–Simon separating inequality [33, Lemma 3.24] for all $\delta > 0$ we have

$$A \geq PAP - \delta^{-1} PAP^{\perp}AP + P^{\perp}(A - \delta I)P^{\perp},$$
$$A \geq P^{\perp}AP^{\perp} - \delta^{-1} P^{\perp}APAP^{\perp} + P(A - \delta I)P.$$

Adding these inequalities term by term, and taking into account the equalities

$$2PAP + 2P^{\perp}AP^{\perp} = A + SAS, \quad PAP^{\perp}AP + P^{\perp}APAP^{\perp} = |[A, P]|^2$$

we obtain $A \geq SAS - \delta I - \delta^{-1}|[A, P]|^2$. Multiplying both sides of this inequality on the left and right by the symmetry S, we have $SAS \geq A - \delta I - \delta^{-1}|[A, P]|^2$. □

By item (ii) of Lemma 3.18 and Theorem 3.19 we have

Corollary 3.20 *Let φ be a state on a unital C^*-algebra $\mathcal{A}$, $A \in \mathcal{A}^{\mathrm{sa}}$, $P \in \mathcal{A}^{\mathrm{pr}}$ and $S = 2P - I$. Then for all $\delta > 0$ we have $\|A - SAS\|_{\varphi} \leq \delta + \delta^{-1}\varphi(|[A, P]|^2)$.*

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References

1. Ara, P.: Left and right projections are equivalent in Rickart C^*-algebras. J. Algebra **120**(2), 433–448 (1989)
2. Ara, P.: K-theory for Rickart C^*-algebras. K-Theory **5**(3), 281–292 (1991)
3. Ara, P., Goldstein, D.: A solution of the matrix problem for Rickart C^*-algebras. Math. Nachr. **164**, 259–270 (1993)
4. Ara, P., Goldstein, D.: Rickart C^*-algebras are σ-normal. Arch. Math. (Basel) **65**(6), 505–510 (1995)
5. Ayupov, S.A.: Classification and representation of ordered Jordan algebras (Russian). “Fan”, Tashkent (1986)
6. Ayupov, Sh.A., Arzikulov, F.N.: Jordan counterparts of Rickart and Baer $*$-algebras. Uzbek. Mat. Zh. **1**, 13–33 (2016)
7. Ayupov, Sh.A., Arzikulov, F.N.: Jordan counterparts of Rickart and Baer $*$-algebras, II. São Paulo. J. Math. Sci. **13**(1), 27–38 (2019)
8. Berberian, S.K.: Baer *-rings. Springer, Berlin (1972)
9. Bikchentaev, A.M.: On a property of L_p-spaces on semifinite von Neumann algebras. Math. Notes **64**(1–2), 159–163 (1998)
10. Bikchentaev, A.M., Tikhonov, O.E.: Characterization of the trace by monotonicity inequalities. Linear Algebra Appl. **422**(1), 274–278 (2007)
11. Bikchentaev, A.M.: Commutativity of projections and trace characterization on von Neumann algebras. Sib. Math. J. **51**(6), 971–977 (2010)
12. Bikchentaev, A.M.: Commutation of projections and trace characterization on von Neumann algebras. II. Math. Notes **89**(3–4), 461–471 (2011)
13. Bikchentaev, A.M.: The Peierls-Bogoliubov inequality in C^*-algebras and characterization of tracial functionals. Lobachevskii J. Math. **32**(3), 175–179 (2011)
14. Bikchentaev, A.M.: A block projection operator in normed ideal spaces of measurable operators. Russ. Math. (Iz. VUZ) **56**(2), 75–79 (2012)
15. Bikchentaev, A.M.: Commutativity of operators and characterization of traces on C^*-algebras. Dokl. Math. **87**(1), 79–82 (2013)
16. Bikchentaev, A.M.: Commutation of projections and characterization of traces on von Neumann algebras. III. Int. J. Theor. Phys. **54**(12), 4482–4493 (2015)

17. Bikchentaev, A.M.: Inequality for a trace on a unital C^*-algebra. Math. Notes **99**(4), 487–491 (2016)
18. Bikchentaev, A.M.: On operator monotone and operator convex functions. Russ. Math. (Iz. VUZ) **60**(5), 61–65 (2016)
19. Bikchentaev, A.M.: Differences of idempotents in C^*-algebras. Sib. Math. J. **58**(2), 183–189 (2017)
20. Bikchentaev, A.M.: Differences of idempotents in C^*-algebras and the quantum Hall effect. Theoret. Math. Phys. **195**(1), 557–562 (2018)
21. Bikchentaev, A.M.: Trace and differences of idempotents in C^*-algebras. Math. Notes **105**(5–6), 641–648 (2019)
22. Bikchentaev, A.M.: Metrics on projections of the von Neumann algebra associated with tracial functionals. Sib. Math. J. **60**(6), 952–956 (2019)
23. Bikchentaev, A.M., Abed, S.A.: Projections and traces on von Neumann algebras. Lobachevskii J. Math. **40**(9), 1260–1267 (2019)
24. Bikchentaev, A.M.: Inequalities for determinants and characterization of the trace. Sib. Math. J. **61**(2), 248–254 (2020)
25. Bikchentaev, A.: Sukochev, F.: Inequalities for the block projection operators. J. Funct. Anal. **280**(7), article 108851 (2021)
26. Blackadar, B.E., Cuntz, J.: The structure of stable algebraically simple C^*-algebras. Am. J. Math. **104**(4), 813–822 (1982)
27. Blackadar, B., Handelman, D.: Dimension functions and traces on C^*-algebras. J. Funct. Anal. **45**(3), 297–340 (1982)
28. Blackadar, B.: Projections in C^*-algebras, in: R.S. Doran (Ed.), C^*-Algebras: 1943–1993, A Fifty Year Celebration, in: Contemporary Mathematics, vol. 167, pp. 115–129, Amer. Math. Soc., Providence, R.I. (1994)
29. Blackadar, B.: Operator algebras, theory of C^*-algebras and von Neumann algebras, in: Encyclopaedia of Mathematical Sciences, in: Operator Algebras and Non-commutative Geometry, III, vol. 122, Springer-Verlag, Berlin (2006)
30. Busqué, C.: K-theoretic triviality for Rickart C^*-algebras and $\aleph_0$-continuous regular rings. Proc. Am. Math. Soc. **101**(1), 24–28 (1987)
31. Chilin, V., Krygin, A., Sukochev, F.: Extreme points of convex fully symmetric sets of measurable operators. Integr. Equ. Oper. Theory **15**(2), 186–226 (1992)
32. Cho, K., Sano, T.: Young's inequality and trace. Linear Algebra Appl. **431**(8), 1218–1222 (2009)
33. Cycon, H.L., Froese, R.G., Kirsch, W., Simon, B.: Schrödinger operators with application to quantum mechanics and global geometry. Texts and Monographs in Physics. Springer Study Edition. Springer, Berlin (1987)
34. Dixmier, J.: C^*-algebras and their representations (Russian). Translated from the French. "Nauka", Moscow (1974)
35. Elliott, G.A., Robert, L., Santiago, L.: The cone of lower semicontinuous traces on a C^*-algebra. Am. J. Math. **133**(4), 969–1005 (2011)
36. Gardner, L.T.: An inequality characterizes the trace. Can. J. Math. **31**(6), 1322–1328 (1979)
37. Gohberg, I.C., Kreĭn, M.G.: Introduction to the Theory of Linear Nonselfadjoint Operators. Transl. Mathem. Monographs, Vol. 18, Amer. Math. Soc., Providence, R.I. (1969)
38. Goldstein, D.: Polar decomposition in Rickart C^*-algebras. Publ. Mat. **39**(1), 5–21 (1995)
39. Goldstein, DSh., Katz, A.A.: On the algebra of measurable operators affiliated to a Rickart ordered $*$-algebra. Far East J. Math. Sci. (FJMS) **24**(1), 45–54 (2007)
40. Goldstein, D.Sh., Katz, A.A.: On a local structure in Rickart algebras—definitions and basic properties. Int. J. Funct. Anal. Oper. Theory Appl. **2**(2), 133–168 (2010)
41. Haagerup, U.: Quasitraces on exact C^*-algebras are traces. C. R. Math. Acad. Sci. Soc. R. Can. **36**(2–3), 67–92 (2014)
42. Halmos, P.R., Sunder, V.S.: Bounded integral operators on L^2 spaces. Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], 96. Springer Berlin-New York (1978)
43. Hoa, D.T., Osaka, H., Toan, H.M.: On generalized Powers-Størmer's inequality. Linear Algebra Appl. **438**(1), 242–249 (2013)
44. Kadison, R.V., Ringrose, J.R.: Fundamentals of the theory of operator algebras. Vol. I. Elementary theory. Graduate Studies in Mathematics, 15. Amer. Math. Soc., Providence, R.I. (1997)
45. Kaplansky, I.: Projections in Banach algebras. Ann. Math. **53**(2), 235–249 (1951)
46. Lord, S., Sukochev, F., Zanin, D.: Singular Traces. Theory and Applications, de Gruyter Studies in Mathematics, Vol. 46 (2013)

47. Menal, P., Moncasi, J.: Lifting units in self-injective rings and an index theory for Rickart C^*-algebras. Pac. J. Math. **126**(2), 295–329 (1987)
48. Petz, D., Zemánek, J.: Characterizations of the trace. Linear Algebra Appl. **111**, 43–52 (1988)
49. Rickart, C.E.: Banach algebras with an adjoint operation. Ann. Math. **47**(3), 528–550 (1946)
50. Robert, L.: On the comparison of positive elements of a C^*-algebra by lower semicontinuous traces. Indiana Univ. Math. J. **58**(6), 2509–2515 (2009)
51. Rørdam, M.: Fixed-points in the cone of traces on a C^*-algebra. Trans. Am. Math. Soc. **371**(12), 8879–8906 (2019)
52. Saitô, K., Wright, J.D.M.: On defining AW*-algebras and Rickart C*-algebras. Q. J. Math. **66**(3), 979–989 (2015)
53. Sherstnev, A.N.: Methods of bilinear forms in non-commutative measure and integral theory. Russian), Fizmatlit, Moscow (2008)
54. Stolyarov, A.I., Tikhonov, O.E., Sherstnev, A.N.: Characterization of normal traces on von Neumann algebras by inequalities for the modulus. Math. Notes **72**(3–4), 411–416 (2002)
55. Takesaki, M.: Theory of operator algebras. I. Encyclopaedia of Mathematical Sciences, 124. Operator Algebras and Non-commutative Geometry, 5. Springer-Verlag, Berlin (2002)
56. Takesaki, M.: Theory of Operator Algebras, Operator Algebras and Non-Commutative Geometry, 6. v. II, Encyclopaedia Math. Sci., 125, Springer-Verlag, New York (2003)
57. Tanaka, R.: C^*-algebras with weak* angelic dual balls. Proc. Am. Math. Soc. Ser. B **7**, 127–137 (2020)
58. Tikhonov, O.E.: Subadditivity inequalities in von Neumann algebras and characterization of tracial functionals. Positivity **9**(2), 259–264 (2005)
59. Upmeier, H.: Automorphism groups of Jordan C^*-algebras. Math. Z. **176**(1), 21–34 (1981)