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MATHEMATICS

Trace and Integrable Operators Affiliated with a Semifinite von Neumann Algebra

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Abstract—New properties of the space of integrable (with respect to the faithful normal semifinite trace) operators affiliated with a semifinite von Neumann algebra are found. A trace inequality for a pair of projections in the von Neumann algebra is obtained, which characterizes trace in the class of all positive normal functionals on this algebra. A new property of a measurable idempotent are determined. A useful factorization of such an operator is obtained; it is used to prove the nonnegativity of the trace of an integrable idempotent. It is shown that if the difference of two measurable idempotents is a positive operator, then this difference is a projection. It is proved that a semihyponormal measurable idempotent is a projection. It is also shown that a hyponormal measurable tripotent is the difference of two orthogonal projections.

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This paper continues the author's study initiated in papers [1–3]; we use the notation and terminology of these papers. In Section 2, we establish new properties of the space $L_1(\mathcal{M}, \tau)$ of integrable (with respect to the trace τ) operators affiliated with the semifinite von Neumann algebra $\mathcal{M}$. We show that if A and B are a hyponormal and a cohyponormal τ-measurable operator, respectively, and $AB \in L_1(\mathcal{M}, \tau)$, then $BA \in L_1(\mathcal{M}, \tau)$ and $\|BA\|_1 \le \|AB\|_1$; moreover, $\tau(AB) = \tau(BA)$, and for self-adjoint A and B, we have $\tau(AB) = \tau(BA) \in \mathbb{R}$. We prove that if $A \in L_1(\mathcal{M}, \tau)$, then $\tau(A^*) = \overline{\tau(A)}$. We obtain a trace inequality for a pair of projections in $\mathcal{M}$, which characterizes trace in the class of all positive normal functionals on $\mathcal{M}$.

In Section 3, we establish new properties of a τ-measurable idempotent ($A = A^2$). We obtain a useful factorization of such an operator; using it, we prove that $\tau(A) \in \mathbb{R}^+$ for an idempotent $A \in L_1(\mathcal{M}, \tau)$. Therefore, if $A, A^2 \in L_1(\mathcal{M}, \tau)$ and $A = A^3$, then $\tau(A) \in \mathbb{R}$. We show that if the difference of two τ-measurable idempotents is a positive operator, then this difference is a projection. We prove that a semihyponormal τ-measurable idempotent is a projection. We also show that a hyponormal τ-measurable tripotent ($A = A^3$) is the difference of two orthogonal projections.

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1. NOTATION AND DEFINITIONS

Suppose that $\mathcal{M}$ is a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$, $\mathcal{M}^{\text{pr}}$ is the lattice of projections on $\mathcal{M}$, I is the identity of $\mathcal{M}$, $P^\perp = I - P$ for $P \in \mathcal{M}^{\text{pr}}$, and $\mathcal{M}^+$ is the cone of positive elements in $\mathcal{M}$. If $P, Q \in \mathcal{M}^{\text{pr}}$, then the projection $P \wedge Q$ is defined by $(P \wedge Q)\mathcal{H} = P\mathcal{H} \cap Q\mathcal{H}$ and $P \vee Q = (P^\perp \wedge Q^\perp)^\perp$ is the projection onto $\overline{\text{Lin}(P\mathcal{H} \cup Q\mathcal{H})}$.

A mapping $\varphi\colon \mathcal{M}^+ \to [0, +\infty]$ is called a trace if $\varphi(X + Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$, $\lambda \ge 0$ (it is assumed that $0 \cdot (+\infty) \equiv 0$), and $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$. A trace φ is said to be faithful if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$; it is semifinite if $\varphi(X) = \sup\{\varphi(Y)\colon Y \in \mathcal{M}^+, Y \le X, \varphi(Y) < +\infty\}$ for each $X \in \mathcal{M}^+$; and it is normal if $X_i \nearrow X$ ($X_i, X \in \mathcal{M}^+$) $\Rightarrow \varphi(X) = \sup \varphi(X_i)$. For a trace φ, we set $\mathfrak{M}_\varphi^+ = \{X \in \mathcal{M}^+\colon \varphi(X) < +\infty\}$ and $\mathfrak{M}_\varphi = \text{lin}_{\mathbb{C}} \mathfrak{M}_\varphi^+$. The restriction $\varphi \mid \mathfrak{M}_\varphi^+$ admits a well-defined extension by linearity to a functional on $\mathfrak{M}_\varphi$, which we denote by the same letter φ.

An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be affiliated with a von Neumann algebra $\mathcal{M}$ if it commutes with any unitary operator in the commutator subalgebra $\mathcal{M}'$ of $\mathcal{M}$. A self-adjoint operator is affiliated with $\mathcal{M}$ if and only if all projections in its spectral decomposition of unity belong to $\mathcal{M}$.

In what follows, τ is a faithful normal semifinite trace on $\mathcal{M}$. A closed operator X affiliated with $\mathcal{M}$ whose domain $\mathfrak{D}(X)$ is dense in $\mathcal{H}$ is said to be τ-measurable if, for any $\varepsilon > 0$, there exists a $P \in \mathcal{M}^{\text{pr}}$ such that