Methods of Analytical Regularization in the Spectral Theory of Open Waveguides

(Invited Paper)

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***Abstract*—The theory of electromagnetic eigenwaves propagating on open dielectric and metallic waveguides is reviewed. The main steps of the theoretical approach based on analytical regularization of the problem are outlined and discussed. Generalized eigenwave problems for lossless dielectric waveguides are considered more comprehensively as examples of such approach. Some of the unsolved problems and the directions of future research are pointed out too.**

I. Introduction

The theory of electromagnetic eigenwaves (also known as natural waves, normal waves, and travelling waves) able to propagate on open dielectric and metallic waveguides is still cannot be considered as complete. Commonly it is supposed that the field components of such a wave depend on the longitudinal coordinate x_3 and time t as $\exp(i(\beta x_3 - kt/c))$ where β is the propagation constant (or wavenumber), k is the free-space wavenumber, and c is the free-space propagation velocity. Only in several simple cases, such as dielectric slab and coaxially-layered circular dielectric fiber, it is possible to study the eigenwaves explicitly. Such a study brings into consideration a variety of waves: proper surface or guided waves, leaky waves, complex surface waves, etc., differing by the field behavior as a function of coordinates. However, if the fiber cross-section is arbitrary, or if additional perfectly electric conducting (PEC) elements are present, as in the microstrip or slot lines, the theory meets certain problems. There are several important questions to be answered already at the stage of the mathematical formulation of the problem of eigenwaves. Clearly, it should be a sort of eigenvalue problem for the wavenumber β. However, what should be the domain of the variation of the eigenvalue parameter? In what class should one seek the wave field components as a function of coordinates in the cross-section and along the waveguide? How we can accurately compute them?

Section II is a summary of results related to the theory of propagation of time-harmonic electromagnetic waves on arbitrarily shaped open waveguides with compact cross-section. Most of them are also contained in the contributed publications of the authors [1]–[38]. In Section III, we consider more comprehensively generalized eigenwave problems for two types of lossless dielectric waveguides: a “step-index” waveguide with a smooth cross-sectional boundary and an inhomogeneous, in cross-section, waveguide. Section IV is devoted to a mini-review of unsolved problems and future directions of research.

II. Brief overview of main points

A. Start from the excitation problem

It is impossible to come to a reasonably general formulation of the eigenwave problem from any other starting point than the problem of the time-harmonic excitation ($\sim \exp(ikt/c)$), where $k > 0$) of an open waveguide by the elementary electric and magnetic current sources (Dirac dipoles). This is the same as determining the open waveguide Green’s functions. Here, a necessary assumption should be made, and finally verified, that arbitrary field can be presented as a convolution with the Green’s functions.

B. Fourier transform

By virtue of infinite length of a regular waveguide along the x_3-axis, the Fourier transformation with the kernel $\exp(i\beta x_3)$ is a natural instrument of bringing the problem consideration to the two-dimensional (2D) space, for the field transforms as a function of cross-section $x = (x_1, x_2)$ and integration parameter β. Here, another necessary assumption appears that the eigenwave fields are no more than the slow-growth functions of x_3; hence, the Fourier integrals should be interpreted in terms of distributions. It is necessary to distinguish between the open waveguides, whose elements have a compact cross-section (embedded in free space) and those of noncompact cross-section, for example, compact open waveguides embedded into a flat-layered medium, whose cross-section has infinite boundaries. Although two cases have much in common, the latter one is more complicated. In the former case, the Fourier transform enables one to reduce the dimensionality of excitation problem: Fourier-images $F(x, \beta)$ of the field components must satisfy the Helmholtz equation $[\Delta + k^2\varepsilon(x)\mu(x) - \beta^2]F = 0$ in the 2D open domain with the boundary conditions (for the PEC and impedance elements) and transmission conditions (for the dielectric elements) given at the bounded curves.

C. Analytic continuation

The use of Fourier transform naturally brings a necessity of analytic continuation of the field Fourier-images $F(x, \beta)$,

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from the real values of parameter β to the complex domain. This complex domain is uniquely determined by the Green's function of the mentioned above 2D Helmholtz equation, $(i/4)H_0^{(1)}(\sqrt{k^2\varepsilon_\infty - \beta^2}|x-y|)$, and is common to all open waveguides of compact cross-section. This is the infinite-sheet Riemann surface Λ of the function $\ln\sqrt{k^2\varepsilon_\infty - \beta^2}$, where ε_∞ is the dielectric constant of environment.

D. Reichardt condition

On the mentioned Riemann surface Λ, it is the Reichardt condition that serves as analytic continuation of the 2D Sommerfeld radiation condition (for $|\beta| < k\sqrt{\varepsilon_\infty}$) and the exponential decay condition (for $|\beta| > k\sqrt{\varepsilon_\infty}$) from the real axis of the "physical" sheet to all complex β. Due to this condition, but also due to the transmission-type conditions at the boundaries of dielectric elements, if they are present, these 2D problems for the analytic continuations of the field Fourier-images are non-selfadjoint ones. Note that this condition permits the Fourier-images to grow exponentially with $|x_3| \to \infty$ if β is located at the sheets other than the "physical" sheet of Λ. Nevertheless, Reichardt condition guarantees the uniqueness of solution provided that β is not an eigenvalue.

E. Analytical regularization

For a wide class of open waveguides, 2D boundary-value problems for the Fourier images can be converted to the canonical Fredholm operator equations, $(I+A)X = B$, where I is identity operator, A is a compact operator, and X and B are linked to the Fourier images of the scattered and incident field, respectively, via some linear operators. Here, operator A is a meromorphic function of β on Λ, k, and all geometrical and material parameters of the waveguide. Such a conversion is based commonly on the analytical regularization of the singular integral equations equivalent to the original boundary value problem. Here, the Reichardt condition guarantees that arbitrary-source field can be represented as a convolution with the 2D Green's functions and their normal derivatives for any complex β.

F. Fredholm-Steinberg theorems

Once a regularization has been done, one can use the theory of Fredholm in the form generalized by Steinberg for the operators depending on parameters. The results are as follows: it is possible to prove the existence of the bounded resolvent, and hence, the existence of the Fourier transforms, as no more than meromorphic functions of β on Λ. The poles have no finite accumulation points on Λ. They can be of only finite multiplicity. They are piece-wise continuous functions of the geometry and piecewise-analytic functions of k and material parameters. The continuity or analyticity can be lost only at such a value of parameter that two or more poles coalesce. The poles can appear or disappear only at the boundary of the domain of meromorphicity: at infinity and in the branch points $\beta = \pm k\sqrt{\varepsilon_\infty}$. The residues of the poles of the Fourier images satisfy certain 2D source-free boundary-value problem (i.e. eigenvalue problem) for the mentioned above Helmholtz equation, with the spectral parameter β located on Λ.

G. Generalized eigenwave problems

The latter circumstance leads to a conclusion that the eigenvalue problems about the natural waves of an open waveguide can be studied independently of the excitation problems. However, in view of the presented above chain of considerations, it should be formulated in a generalized sense. Namely, it should admit complex β on Λ and include the Reichardt condition at the infinity in the cross-section. In so doing one gets a universal framework to study all types of known eigenwaves: surface waves, leaky waves, complex surface waves, etc., and hence trace the transitions of each wave from one type to another under variations of non-spectral parameters.

H. Discreteness of eigenwave spectrum

For a wide class of virtually all realistic configurations of open waveguides, the mentioned above generalized eigenwave problems admit analytical regularization and are equivalently reducible to a homogeneous Fredholm operator equation of the second kind $[I + A(\beta)]X = 0$. The set of eigenvalues of β on Λ forms the spectrum of the operator $I + A(\beta)$ and coincides with the spectrum of generalized eigenwaves of the open waveguide. As one can see, the latter is purely discrete on Λ. In particular, this enables one to conclude that the surface waves, whose wavenumbers are located on the finite interval $k\sqrt{\varepsilon_\infty} < |\beta| < k \sup \sqrt{\varepsilon}$ of the real axis of the "physical" sheet, can be only of finite number.

I. Symmetry of eigenwave spectrum

Some properties of the spectrum of eigenwaves can be deduced directly from the formulation of the generalized eigenwave problem. It is verified directly that on any open waveguide the eigenvalue wavenumbers $\pm\beta$ form symmetric pairs on Λ. Moreover, on the lossles waveguides, they form conjugate quartets $\pm\beta$, $\pm\overline{\beta}$ on Λ. Hence, it is enough to study them only in one quadrant of each Riemann sheet.

J. Free of spectrum domain

Using the vector Green's formula, it has been shown that on Λ there exists a non-empty domain, which is free of the spectrum of eigenwaves. This domain depends on the type of the open waveguide. If it contains only PEC elements but has no material (dielectric or magnetic) elements, this domain includes the whole "physical" sheet of Λ; in a lossless dielectric waveguide, it includes the intervals $|\beta| < k\sqrt{\varepsilon_\infty}$ and $|\beta| \geq k \sup \sqrt{\varepsilon}$ of the real axis of the "physical" sheet; in the lossy case this whole real axis is free of spectrum, etc.

K. Orthogonality and power flux

The vector Green's formula, applied to the eigenwave field, enables one to prove the orthogonality of the surface waves and the complex surface waves, in the power sense. However if the wavenumber β is not located on the "physical" sheet of Λ, this proof fails. The Green's formula is also an instrument to study the properties of the power flux associated with a generalized eigenwave. For example, it shows that any complex surface wave on a lossless open waveguide can be only hybrid (i.e., has all six components of the electromagnetic field) and does not carry power, as its total flux in the cross-section is identical zero. Another important conclusion is that, on open waveguide, a surface wave can carry the power not only in the direction of its propagation; the opposite direction is allowed, although only for the hybrid waves. Still other conclusion is that the analyticity of each spectrum point as a function of k enables one to validate the concept of the group velocity.

L. Radiation condition in 3D excitation problem

Strictly speaking, in the original 3D problem of the elementary-source excitation of an arbitrary open waveguide, the classical Sommerfeld condition of radiation is not valid for the extraction of unique solution. The reason is the presence of infinite along x_3 boundaries, and hence possible presence of surface waves able to carry the power to infinity along the waveguide without attenuation. In view of the mentioned above results of study of the 2D problem for the field Fourier images, one can formulate a modified condition of radiation, adapted to the open waveguide case. It has the form of asymptotic request to the far-field behavior that explicitly involves the surface waves. This condition guarantees uniqueness of the 3D problem solution and validates the early assumption that arbitrary-source field can be presented as a convolution with the Green's functions. Here, one comes to a necessity of taking account of the direction of the power flux (or, equivalently, the sign of the group velocity) associated with each surface wave. The modified radiation condition enables one to formulate the Principle of Radiation as "*No waves bringing power from infinity, in the scattered field*". It is only if the losses are introduced in the waveguide elements that the modified radiation condition is reduced to the Sommerfeld one, as then no surface waves exist. In that case the Principle of Radiation is reduced to conventional form, "*No waves propagating from infinity, in the scattered field*".

III. Generalized eigenwave roblems for lossless dielectric waveguides

A. Generalized natural waves of a step-index dielectric waveguide

Let the three-dimensional space be occupied by an isotropic source-free medium, and let the permittivity be prescribed as a positive real-valued function $\varepsilon = \varepsilon(x)$ independent of the longitudinal coordinate and equal to a constant $\varepsilon_\infty > 0$ outside a cylinder. In this section we consider the generalized natural waves of a step-index optical fiber and suppose that the permittivity is equal to a constant $\varepsilon_+ > \varepsilon_\infty$ inside the cylinder. The axis of the cylinder is parallel to the longitudinal coordinate and its cross section is a bounded domain Ω_i with a twice continuously differentiable boundary γ (see Fig. 1). The domain Ω_i is a subset of a circle with radius R_0. Denote by Ω_e the unbounded domain $\Omega_e = \mathrm{R}^2 \setminus \overline{\Omega}_i$, by U the space of complex-valued continuous and continuously differentiable in $\overline{\Omega}_i$ and $\overline{\Omega}_e$, twice continuously differentiable in Ω_i and Ω_e functions, and by Λ the Riemann surface of the function $\ln \chi_\infty(\beta)$, where $\chi_\infty = \sqrt{k^2\varepsilon_\infty - \beta^2}$. Here k is a given wavenumber. Denote by Λ_0 the principal ("physical") sheet of this Riemann surface specified by the following conditions: $\mathrm{Im}\chi_\infty(\beta) \ge 0$ and $-\pi/2 < \arg\chi_\infty(\beta) < 3\pi/2$.

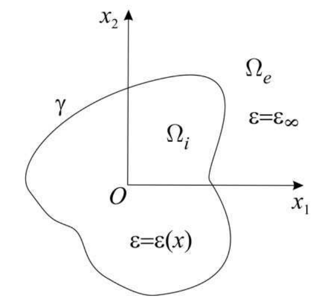

Fig. 1. A schematic waveguide's cross-section.

A nonzero vector $\{\mathrm{E},\mathrm{H}\} \in U^6$ is referred to as a generalized eigenvector (or eigenwave) of the problem corresponding to an eigenvalue $\beta \in \Lambda$ if the following relations are valid [16]: the differential equations which follows from Maxwell's equations for $x \in \mathrm{R}^2 \setminus \gamma$

$$\mathrm{rot}_\beta \mathrm{E} = i\omega\mu_0 \mathrm{H}, \quad \mathrm{rot}_\beta \mathrm{H} = -\, i\omega\varepsilon_0\varepsilon \mathrm{E}, \tag{1}$$

the transmission conditions for $x \in \gamma$

$$\nu \times \mathrm{E}^+ = \nu \times \mathrm{E}^-, \tag{2}$$

$$\nu \times \mathrm{H}^+ = \nu \times \mathrm{H}^-, \tag{3}$$

and the Reichardt radiation condition

$$\begin{bmatrix} \mathrm{E} \\ \mathrm{H} \end{bmatrix} = \sum_{l=-\infty}^{\infty} \begin{bmatrix} \mathrm{A}_l \\ \mathrm{B}_l \end{bmatrix} H_l^{(1)}(\chi_\infty r) \exp(il\varphi), \qquad |x| \ge R_0. \tag{4}$$

Here ω is the radian frequency; ε_0, μ_0 are the free-space dielectric and magnetic constants, respectively; differential operator rot_β is obtained from the standard operator by replacing the generating waveguide line derivative with $i\beta$ multiplication and $H_l^{(1)}(z)$ is the Hankel function of the first kind and index l; r and φ are the polar coordinates of the point x.

Theorem 1 (see [24]). *The imaginary axis* I *and the real axis* R *of the sheet* Λ_0 *except the set*

$$G = \left\{\beta \in \mathrm{R} : k\sqrt{\varepsilon_\infty} < |\beta| < k\sqrt{\varepsilon_+}\right\}$$

are free of the eigenvalues of problem (1)–(4). *Surface and complex eigenwaves correspond to real eigenvalues* $\beta \in G$ *and complex eigenvalues* $\beta \in \Lambda_0$, *respectively. Leaky eigenwaves correspond to complex eigenvalues* β *belonging to an "improper" sheet of* Λ *for which* $\mathrm{Im}\chi_\infty(\beta) < 0$ and $-\pi/2 < \arg\chi_\infty(\beta) < 3\pi/2$.

Theorem 1 generalizes the well-known results on the spectrum localization of a step-index circular dielectric waveguide which were obtained by the separation of variables method.

We use representation of eigenvectors of problem (1)–(4) in the form of single-layer potentials u and v (see [15]):

$$\mathrm{E}_1 = \frac{i}{k^2\varepsilon - \beta^2}\left(\mu_0\omega\frac{\partial v}{\partial x_2} + \beta\frac{\partial u}{\partial x_1}\right),$$

$$\mathrm{E}_2 = \frac{-i}{k^2\varepsilon - \beta^2}\left(\mu_0\omega\frac{\partial v}{\partial x_1} - \beta\frac{\partial u}{\partial x_2}\right), \quad \mathrm{E}_3 = u, \tag{5}$$

$$\mathrm{H}_1 = \frac{i}{k^2\varepsilon - \beta^2}\left(\beta\frac{\partial v}{\partial x_1} - \varepsilon_0\varepsilon\omega\frac{\partial u}{\partial x_2}\right),$$

$$\mathrm{H}_2 = \frac{i}{k^2\varepsilon - \beta^2}\left(\beta\frac{\partial v}{\partial x_2} + \varepsilon_0\varepsilon\omega\frac{\partial u}{\partial x_1}\right), \quad \mathrm{H}_3 = v, \tag{6}$$

$$\begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = \frac{i}{4}\int_\gamma H_0^{(1)}\left(\chi_{+/\infty}(\beta)\,|x-y|\right)\begin{bmatrix} f_{+/\infty}(y) \\ g_{+/\infty}(y) \end{bmatrix} dl(y), \tag{7}$$

where $x \in \Omega_{i/e}$ and unknown densities $f_{+/\infty}$ and $g_{+/\infty}$ belong to the space of Hölder continuous functions $C^{0,\alpha}$. Original problem (1)–(4) is reduced [16] by single-layer potential representation (5)–(7) to a nonlinear eigenvalue problem for a set of singular integral equations on boundary γ. This problem has the following operator form in the Banach space $W = (C^{0,\alpha})^4$:

$$A(\beta)w \equiv (I + B(\beta))w = 0, \tag{8}$$

Here I is the identical operator and $B(\beta): W \to W$ is a compact operator consisting particularly of the boundary singular integral operators $L : C^{0,\alpha} \to C^{1,\alpha}$ and $S : C^{0,\alpha} \to C^{0,\alpha}$ defined by the following relationships:

$$Lp = -\frac{1}{2\pi}\int_0^{2\pi} \ln|\sin\frac{t-\tau}{2}|p(\tau)d\tau, \quad t \in [0, 2\pi], \tag{9}$$

$$Sp = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{ctg}\frac{\tau - t}{2}p(\tau)d\tau + \frac{i}{2\pi}\int_0^{2\pi} p(\tau)d\tau, \quad t \in [0, 2\pi]. \tag{10}$$

Original problem (1)–(4) is spectrally equivalent [24] to problem (8) with a restriction. Namely, suppose that $w \in W$ is an eigenvector of the operator-valued function $A(\beta)$ corresponding to an eigenvalue $\beta \in \Lambda_0 \setminus D$, where

$$D = \{\beta \in \mathrm{I}\} \cup \{\beta \in \mathrm{R} : \beta^2 < k^2\varepsilon_\infty\}.$$

Then using this vector we can construct the densities of the single-layer potential representation (5)–(7) of an eigenwave $\{\mathrm{E}, \mathrm{H}\} \in U^6$ of problem (1)–(4) corresponding to the same eigenvalue β. On the other side, any eigenwave of (1)–(4) corresponding to an eigenvalue $\beta \in \Lambda_0 \setminus D$ can be represented in the form of single-layer potentials. The densities of these potentials constitute an eigenvector $w \in W$ of the operator-valued function $A(\beta)$ corresponding to the same eigenvalue β.

Theorem 2 (see [24]). *For each $\beta \in \{\beta \in R : \beta^2 \geq k^2\varepsilon_+\}$ the operator $A(\beta)$ has a bounded inverse operator. The set of all eigenvalues β of the operator-valued function $A(\beta)$ can be only a set of isolated points on Λ. Each eigenvalue β depends continuously on $\omega > 0$, $\varepsilon_+ > 0$, and $\varepsilon_\infty > 0$ and can appear and disappear only at the boundary of Λ, i.e. at $\beta = \pm k\sqrt{\varepsilon_\infty}$ and at infinity.*

Theorem 2 generalizes the known results on the dependence of the propagation constants β of a step-index circular dielectric waveguide on wavenumber k and permittivity ε.

Describe a projection method [16] for numerical solution of problem (8). Denote by N the set of all integers. We represent the approximate eigenvector of the operator-valued function $A(\beta)$ in the form

$$w_n = (w_n^{(j)})_{j=1}^4, \quad w_n^{(j)}(t) = \sum_{k=-n}^{n} \alpha_k^{(j)} \exp(ikt),$$

where, $n \in N$, $j = 1, 2, 3, 4$, and look for unknown coefficients $\alpha_k^{(j)}$ by the Galerkin method

$$\int_0^{2\pi} (Aw_n)^{(j)}(t)\exp(-ikt)dt = 0, \; k = -n, ..., n, \; j = 1, 2, 3, 4.$$

Functions $\exp(ikt)$ are orthogonal eigenfunctions of the singular integral operators $L : C^{0,\alpha} \to C^{1,\alpha}$ and $S : C^{0,\alpha} \to C^{0,\alpha}$ corresponding to the following eigenvalues:

$$\lambda_m^{(L)} = \{\ln 2 \text{ if } m = 0, \; (2|m|)^{-1} \text{ if } m \neq 0\},$$

$$\lambda_m^{(S)} = \{i \text{ if } m = 0, \; i\,\mathrm{sign}(m) \text{ if } m \neq 0\}$$

for the operators L and S respectively. Hence, the action of the main (singular) parts of the integral operators in (8) on the basis functions is expressed explicitly.

Denote by W_n^T the set of all trigonometric polynomials of the orders up to n. Denote by $W_n \subset W$ the space of the elements $w_n = (w_n^{(j)})_{j=1}^4$ where $w_n^{(j)} \in W_n^T$. Using the Galerkin method for numerical solution of problem (8) we get a finite-dimensional nonlinear spectral problem

$$A_n(\beta)w_n = 0, \quad A_n : W_n \to W_n. \tag{11}$$

Theorem 3 (see [25]). *If β_0 belongs to the spectrum $\sigma(A)$ of the operator-valued function $A(\beta)$, then there exists a sequence $\{\beta_n\}_{n\in N}$, $\beta_n \in \sigma(A_n)$, such that $\beta_n \to \beta_0$, $n \in N$. If $\{\beta_n\}_{n\in N}$, $\beta_n \in \sigma(A_n)$, is a sequence such that $\beta_n \to \beta_0 \in \Lambda$, then $\beta_0 \in \sigma(A)$. If $\beta_n \in \sigma(A_n)$, $A_n(\beta_n)w_n = 0$, and $\beta_n \to \beta_0 \in \Lambda$, $w_n \to w_0$, $n \in N$, $\|w_n\| = 1$, then $\beta_0 \in \sigma(A)$ and $A(\beta_0)w_0 = 0$, $\|w_0\| = 1$.*

B. Generalized natural waves of an inhomogeneous waveguide

In this section we consider the generalized natural waves of an inhomogeneous optical fiber without a sharp boundary [22]. Let the permittivity ε belong to the space $C^2(\mathrm{R}^2)$ of twice continuously differentiable in R^2 functions. Denote by ε_+ the maximum of the function ε in the domain Ω_i. Suppose that $\varepsilon_+ > \varepsilon_\infty > 0$. A nonzero vector $\{\mathrm{E}, \mathrm{H}\} \in (C^2(\mathrm{R}^2))^6$ is referred to as a generalized eigenvector (or eigenwave) of the problem corresponding to an eigenvalue $\beta \in \Lambda$ if the following relations are valid [31]: differential equations (1) for all $x \in \mathrm{R}^2$ and radiation condition (4).

Theorem 4 (see [31]). *The imaginary axis* I *and the real axis* R *of the sheet Λ_0 except the set G are free of eigenvalues of the problem* (1), (4). *Surface and complex eigenwaves correspond to real eigenvalues $\beta \in G$ and complex eigenvalues $\beta \in \Lambda_0$, respectively. Leaky eigenwaves correspond to complex eigenvalues β belonging to an "improper" sheet of Λ.*

It is proved in [31] that if vector $\{\mathrm{E}, \mathrm{H}\} \in (C^2(\mathrm{R}^2))^6$ is an eigenvector of problem (1), (4) corresponding to an eigenvalue $\beta \in \Lambda$, then

$$\mathrm{E}(x) = k^2\int_{\Omega_i} (\varepsilon(y) - \varepsilon_\infty)\Phi(\beta; x, y)\mathrm{E}(y)dy +$$

$$\mathrm{grad}_\beta \int_{\Omega_i} \left(\mathrm{E}, \varepsilon^{-1}\mathrm{grad}\varepsilon\right)(y)\Phi(\beta; x, y)dy, \quad x \in \mathrm{R}^2, \tag{12}$$

$$\mathrm{H}(x) = -i\omega\varepsilon_0 \mathrm{rot}_\beta \int\limits_{\Omega_i} (\varepsilon(y) - \varepsilon_\infty)\Phi(\beta; x, y)\mathrm{E}(y)dy, \quad x \in \mathrm{R}^2,$$
$$\Phi(\beta; x, y) = \frac{i}{4} H_0^{(1)} \left(\chi_\infty(\beta) \left|x - y\right|\right). \tag{13}$$

Using the integral representation (12) for $x \in \Omega_i$ we obtain a nonlinear eigenvalue problem for an IE in Ω_i which can be written in the operator form

$$A(\beta)\mathrm{F} \equiv (I - B(\beta))\mathrm{F} = 0, \tag{14}$$

where the operator $B(\beta) : (L_2(\Omega_i))^3 \to (L_2(\Omega_i))^3$ corresponds to the right side of the integral representation (12) for $x \in \Omega_i$. For any $\beta \in \Lambda$ the operator $B(\beta)$ is compact [31].

It is proved in [31] that original problem (1), (4) is spectrally equivalent to problem (14) with a restriction. Namely, suppose that vector $\{\mathrm{E}, \mathrm{H}\} \in (C^2(\mathrm{R}^2))^6$ is the eigenwave of problem (1), (4) corresponding to an eigenvalue $\beta \in \Lambda$. Then $\mathrm{F} = \mathrm{E} \in [L_2(\Omega_i)]^3$ is an eigenvector of the operator-valued function $A(\beta)$ corresponding to the same eigenvalue β. Suppose that $\mathrm{F} \in [L_2(\Omega_i)]^3$ is an eigenvector of the operator-valued function $A(\beta)$ corresponding to an eigenvalue $\beta \in \Lambda$ and that the same number β is not an eigenvalue of the following problem:

$$\left[\Delta + \left(k^2\varepsilon - \beta^2\right)\right] u = 0, \quad x \in \mathrm{R}^2, \quad u \in C^2(\mathrm{R}^2), \tag{15}$$

$$u = \sum_{l=-\infty}^{\infty} a_l H_l^{(1)} \left(\chi_\infty r\right) \exp\left(il\varphi\right), \quad r \geq R_0. \tag{16}$$

Let $\mathrm{E} = B(\beta)\mathrm{F}$ and $\mathrm{H} = (i\omega\mu_0)^{-1}\mathrm{rot}_\beta \mathrm{E}$ for $x \in \mathrm{R}^2$. Then $\{\mathrm{E}, \mathrm{H}\} \in (C^2(\mathrm{R}^2))^6$ and $\{\mathrm{E}, \mathrm{H}\}$ is an eigenvector of original problem (1), (4) corresponding to the same eigenvalue β.

Theorem 5(see [31]).*For each $\beta \in \{\beta \in R : \beta^2 \geq k^2\varepsilon_+\}$ the operator $A(\beta)$ has a bounded inverse. The set of all eigenvalues β of the operator-valued function $A(\beta)$ can be only a set of isolated points on Λ. Each eigenvalue β depends continuously on $\omega > 0$, $\varepsilon_+ > 0$, and $\varepsilon_\infty > 0$ and can appear and disappear only at the boundary of Λ, i.e. at $\beta = \pm k\sqrt{\varepsilon_\infty}$ and at infinity.*

Similar to Theorems 3, 4, and 5 results for integrated optical guides are obtained in [28] and [29].

Scalar problem (15), (16) is a problem on eigenwaves of a inhomogeneous optical fiber in weakly guiding approximation. In fiber optics, a weakly guiding fiber is one where the difference between ε_+ and ε_∞ is very small (typically less than 1% of ε_∞). The statements similar to Theorems 4 and 5 for scalar problem (15), (16) are proved in [23].

Initial problem (15), (16) for surface waves is reduced to a linear eigenvalue problem for an integral operator with a real-valued symmetric weakly singular kernel. The existence of the spectrum of this operator are proved in [37].

The collocation method for numerical approximation of weakly singular domain integral operators associated with problem (15), (16) is proposed in [34] and is developed in [35], [36]. The statement similar to Theorem 3 concerning convergence of the collocation method is proved in [37].

The statements similar to Theorems 1 and 2 for the scalar problem in weakly guiding approximation are proved in [20].

The statement similar to Theorem 3 for a scalar problem in weakly guiding approximation is proved in [17]. A collocation method as an alternative to Galerkin methods [16], [26] is proposed in [38].

IV. Directions of Future Research

A. Existence of natural waves

Non-emptiness of the spectrum of generalized eigenwaves is the hardest point of the theory of open waveguides. It can be proved in the "local sense" based on the operator generalization of the Rusche theorem and explicit existence of eigenwaves of certain canonic open waveguides, as the zeros of well known special functions or their combinations. This once again needs a regularized form of the eigenvalue operator equation. However, to complete this proof to a "global" existence, one needs some guaranty that a finite change of non-spectral parameter cannot kick all the eigenvalues off to infinity or annihilate them in the branch point. This proof needs additional work. In particular case of surface waves of lossless dielectric waveguides the operators are self-adjoint. In this case the existence has been proven for several problems accurately by the methods of the spectral theory of compact self-adjoint operators in [19], [27], [30], [32], [33].

B. Multiple poles and "associated" natural waves

Unlike hollow closed waveguides, for the open waveguides it is not possible to prove the simple character of the poles of the field component Fourier-images, and hence the generalized eigenvalues. This is similar to the impedance-wall and multilayer closed waveguides. Hence, multiple poles, of finite multiplicity M, can exist. In such case, besides of the "parent" natural wave propagating as $\exp(i\beta x_3)$, a finite chain of the "associated" natural waves appears that propagate as $x_3^m \exp(i\beta x_3)$, $m = 1, 2, \ldots M - 1$. This consideration validates the initial assumption, at the early stage of analysis of the excitation problem, that the field functions should be considered in the class of slow-growth functions of x_3.

C. Defect of the model

A close view at the mentioned in Section II 3D radiation condition reveals one intrinsic defect of the original model of the time-harmonic excitation of a lossless infinite open waveguide. If the parameters of a lossless waveguide and k are such that "associated" surface waves exist, then it appears not possible to apply even the modified condition of radiation. The reason is that in this situation both the "parent" surface wave and the "associated" waves have zero total power flux in the infinite cross-section domain. Hence it is impossible to select the proper sign of the eigenwave propagation constant (wavenumber) that ensures solution uniqueness.

D. Similarity between h and k as eigenparameters

From the formulation of the generalized eigenwave problem, one can notice that the parameters β and k enter it in very similar manner. Indeed, one can also study this problem for the k-eignenvalues, with $\beta > 0$. Much of the above theory is valid in this case as well. For instance, the domain of analytic continuation in k is the same Riemann surface Λ. The same

Reichardt condition and the same analytical regularization approach bring us to the conclusions about the discreteness of the k-spectrum on Λ and about the properties of eigenvalues as a function of β. However, the mentioned similarity is not total, and hence the other properties of the spectrum of generalized eigenoscillations are to be studied in more detail. Analysis of such a "dual" eigenvalue problem appears to be a natural stage if one studies more general problem of the excitation of an open waveguide by a δ-pulse or other time-dependent source distributed along the x_3 axis as $\exp(i\beta x_3)$, with $\beta = \text{const}$.

E. Extentions and unsolved problems

There are several possible directions of the extension of the developed theory, each of them being associated with a separate class of problems. The results obtained for the regular open waveguides can be generalized to the regular-periodic open waveguides. This needs the use of a generalized version of the Fourier transform approach exploiting additionally the Floquet-expansions in x_3, in the image domain. Then it is possible to see that the domain of analytic continuation in β is the Riemann surface of the function

$$\sum_{m=-\infty}^{+\infty} \ln(k - \beta - 2\pi m/p)(k + \beta + 2\pi m/p),$$

where p is the period along x_3. The other direction of work is the theory of open waveguides with non-compact cross-section, such as microstripline on infinitely wide dielectric substrate. Here, the approach of the double Fourier transform should be used. The study of the Green's functions and radiation condition should apparently bring into consideration the surface waves of two types: those which are associated with the strip and are deformed by the presence of the substrate, and those which are associated with the substrate and are deformed by the presence of the strip. Another interesting direction is the theory of the open waveguide bends and branchings. Here, the key problem is the one of a terminated (semi-infinite) open waveguide in free space.

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