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CHARACTERIZATION OF TRACES ON C^*-ALGEBRAS: A SURVEY

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ABSTRACT. The paper is a survey on problems of trace characterization on C^*-algebras. We present trace characterization in terms of inequalities for modulus of operators and of noncommutative integration theory (triangle, Hölder, Cauchy–Schwarz–Buniakowski and Young), and statistical mechanics (Golden–Thompson, Peierls–Bogoliubov and Araki–Lieb–Thirring). Some of the results contain applications to characterization of C^*-algebras commutativity problem. Theorem 19, part of Corollary 6 (the equivalence (ii) and (iii)) and Corollary 7 are new.

1. INTRODUCTION

We assume (essentially, to simplify the exposition) that all C^*-algebras considered in the paper are unital, the unit element being denoted by I. If $\mathcal{A}$ is a C^*-algebra, then by $\mathcal{A}^{\text{sa}}$ and $\mathcal{A}^+$ we denote the subsets of self-adjoint and positive elements, respectively. The modulus $(X^*X)^{1/2}$ of $X \in \mathcal{A}$ is written as $|X|$.

By $\mathcal{M}^{\text{pr}}$ we denote the set of projections in a von Neumann algebra $\mathcal{M}$. Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators on a Hilbert space $\mathcal{H}$. We define the projection $P \wedge Q$ (the infimum of P and Q in the projection lattice $\mathcal{B}(\mathcal{H})^{\text{pr}}$) as $(P \wedge Q)(\mathcal{H}) = P(\mathcal{H}) \cap Q(\mathcal{H})$; $P \vee Q = I - (I - P) \wedge (I - Q)$ is the projection onto $\overline{\text{Lin}\,(P\mathcal{H} \cup Q\mathcal{H})}$. By $s_r(A)$ we denote the support projection of $A \in \mathcal{B}(\mathcal{H})^+$. The spectrum of an operator $A \in \mathcal{B}(\mathcal{H})$ is denoted by $\sigma(A)$. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be *hyponormal*, if $A^*A \geq AA^*$ [20, Chapter 16, 160].

In what follows, M_n stands for the algebra of $n \times n$ complex matrices.

Definition 1. A *weight* on C^*-algebra $\mathcal{A}$ is a mapping $\varphi : \mathcal{A}^+ \to [0, +\infty]$ such that

$$\varphi(X + Y) = \varphi(X) + \varphi(Y) \text{ for all } X, Y \in \mathcal{A}^+,$$
$$\varphi(\lambda X) = \lambda\varphi(X) \text{ for } X \in \mathcal{A}^+ \text{ and } \lambda > 0,$$

by convention, $0 \cdot (+\infty) = 0$.

The most important special kinds of weights (or states) on a C^*-algebra are the traces, which play a crucial role in several places in the structure theory.

Definition 2. A *trace* on C^*-algebra $\mathcal{A}$ is a weight τ on $\mathcal{A}$ satisfying $\tau(X^*X) = \tau(XX^*)$ for all $X \in \mathcal{A}$.

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