Proceedings of the International Conference DAYS on DIFFRACTION 2014, pp. 118–123 

# An inverse eigenvalue problem of the theory of optical waveguides

**Karchevskii E.M.**, Spiridonov A.O.
Department of Applied Mathematics, Kazan Federal University, Russia; e-mail: ekarchev@yandex.ru

**We present a new method for calculation of permittivities of dielectric materials using optical fiber's propagation constants measurements. Our numerical algorithm is based on approximate solution of a nonlinear nonselfadjoint inverse eigenvalue problem for a system of weakly singular integral equations. We prove that it is enough to measure propagation constants of fundamental eigenmode only at two frequencies for reconstruction of dielectric constants of core and cladding of a waveguide.**

## 1 Introduction

Inverse eigenvalue problems [1] arise in a remarkable variety of applications, including system and control theory, geophysics, molecular spectroscopy, particle physics, structure analysis, and so on. An inverse eigenvalue problem concerns the reconstruction of a physical system from prescribed spectral data. The involved spectral data may consist of the complete or only partial information of eigenvalues or eigenvectors. Particularly, permittivity determination problems from measurements of propagation constants were investigated firstly for rectangular waveguides by separation of variables method in [2]. For waveguides of arbitrary cross section such problems can be set up as inverse eigenvalue problems of the theory of optical waveguides [3] on the base of integral formulations.

Two mathematical models of optical waveguides were investigated in details by integral equation methods: step-index waveguides [4]–[6] and waveguides without a sharp boundary [7]–[9]. A review of modern results in this field is given in [10]. We use a mathematical model of a weakly-guiding step-index arbitrarily shaped optical fiber. We prove that it is enough to measure propagation constants of fundamental eigenmode at two distinct frequencies for the reconstruction of unknown dielectric constants of core and cladding of the fiber. We also present a new numerical algorithm for dielectric constants calculation based on an approximate solution of a nonlinear nonselfadjoint inverse eigenvalue problem for a system of weakly singular integral equations.

## 2 Eigenmodes of dielectric waveguides

Let us consider an optical fiber as a regular cylindrical dielectric waveguide in a free space. The cross section of the waveguide's core is a bounded domain $\Omega_i$ with a twice continuously differentiable boundary $\gamma$ (see Fig. 1). The axis of the cylinder is parallel to the $x_3$-axis. Let $\Omega_e = \mathbb{R}^2 \setminus \overline{\Omega}_i$ be the unbounded domain of the cladding. Let the permittivity be prescribed as a positive piecewise constant function $\varepsilon$ which is equal to a constant $\varepsilon_\infty$ in the domain $\Omega_e$ and to a constant $\varepsilon_+ > \varepsilon_\infty$ in the domain $\Omega_i$.

Eigenvalue problems of the dielectric waveguide theory [3] are formulated on the base of the set of homogeneous Maxwell equations

$$\operatorname{rot}\mathcal{E} = -\mu_0 \frac{\partial \mathcal{H}}{\partial t}, \quad \operatorname{rot}\mathcal{H} = \varepsilon_0 \varepsilon \frac{\partial \mathcal{E}}{\partial t}. \qquad (1)$$

Nontrivial solutions of set (1) which have the form

$$\begin{bmatrix} \mathcal{E} \\ \mathcal{H} \end{bmatrix}(x, x_3, t) = \operatorname{Re}\left( \begin{bmatrix} \mathrm{E} \\ \mathrm{H} \end{bmatrix}(x) e^{i(\beta x_3 - \omega t)} \right) \qquad (2)$$

are called the eigenmodes of the waveguide. Here positive $\omega$ is the radian frequency, $\beta$ is the propagation constant.

In forward eigenvalue problems the permittivity is known and it is necessary to calculate longitudinal wavenumbers $k = \omega\sqrt{\varepsilon_0 \mu_0}$ and propagation constants $\beta$ such that there exist eigenmodes. The

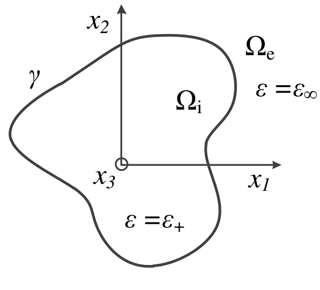


Figure 1: The cross-section of a cylindrical dielectric waveguide in a free-space

eigenmodes have to satisfy a transparency condition at the boundary $\gamma$ and a condition at infinity.

In inverse eigenvalue problems it is necessary to reconstruct the unknown permittivitie $\varepsilon$ by some information on natural eigenmodes which exist for some eigenvalues $k$ and $\beta$. The main question is how many observations of natural eigenmodes are enough for unique and stable reconstruction of the permittivity.

The domain $\Omega_e$ is unbounded. Therefore, it is necessary to formulate a condition at infinity for complex amplitudes E and H of eigenmodes. Let us confine ourselves to the investigation of the surface modes only. The propagation constants $\beta$ of surface modes are real and belong to the interval $G = (k\varepsilon_\infty, k\varepsilon_+)$. The amplitudes of surface modes satisfy the following condition:

$$\begin{bmatrix} \mathrm{E} \\ \mathrm{H} \end{bmatrix} = e^{-\sigma r} O\left(\frac{1}{\sqrt{r}}\right), \quad r = |x| \to \infty. \tag{3}$$

Here $x = (x_1, x_2)$, $\sigma = \sqrt{\beta^2 - k^2\varepsilon_\infty} > 0$ is the transverse wavenumber in the cladding.

Denote by $\chi = \sqrt{k^2\varepsilon_+ - \beta^2}$ the transverse wavenumber in the waveguide's core. Under the weakly guidance approximation the original problem is reduced as in [4] to the calculation of numbers $\chi$ and $\sigma$ such that there exist nontrivial solutions of the Helmholtz equations

$$\Delta u + \chi^2 u = 0, \quad x \in \Omega_i, \tag{4}$$
$$\Delta u - \sigma^2 u = 0, \quad x \in \Omega_e, \tag{5}$$

which satisfy the transparency conditions

$$u^+ = u^-, \quad \frac{\partial u^+}{\partial \nu} = \frac{\partial u^-}{\partial \nu}, \quad x \in \gamma. \tag{6}$$

Let us calculate nontrivial solutions $u$ of problem (4)–(6) in the space of continuous and continuously differentiable in $\overline{\Omega}_i$ and $\overline{\Omega}_e$ and twice continuously differentiable in $\Omega_i$ and $\Omega_e$ functions, satisfying to condition (3). Denote this space of functions by $U$.

**Definition 1.** Let $\sigma > 0$ be a given number. A nonzero function $u \in U$ is called an eigenfunction of problem (4)–(6) satisfying a real eigenvalue $\chi$ if relationships (4)–(6) hold.

The next theorem follows from results of [4].

**Theorem 1.** *For any $\sigma > 0$ the eigenvalues $\chi$ of problem (4)–(6) can be only positive isolated numbers. Each number $\chi$ depends continuously on $\sigma$.*

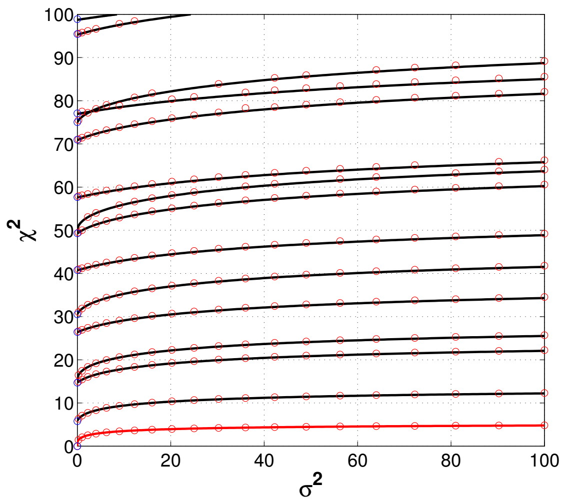


Figure 2: The dispersion curves for surface eigenmodes of a weakly guiding dielectric waveguide of the circular cross-section calculated by the spline-collocation method. The exact solution is plotted by solid lines. Our numerical results are marked by circles.

For waveguides of circular cross-section the analogous results about the localization of the surface modes spectrum and about the continuous dependence between the transverse wavenumbers $\sigma$ and $\chi$ were obtained in [3]. The results of [3] were obtained only for waveguides of circular cross-section by the method of separation of variables. Theorem 1 generalizes the results of [3] to the case of an arbitrary smooth boundary. The next theorem follows from results of [9] (see an illustration in Fig. 2).

**Theorem 2.** *The following statements hold:*

*1. For any $\sigma > 0$ there exist the denumerable set of positive eigenvalues $\chi_i(\sigma)$, where $i = 1, 2, ...,$ of a finite multiplicity with only cumulative point at infinity.*

*2. For any $\sigma > 0$ the smallest eigenvalue $\chi_1(\sigma)$ is positive and simple (its multiplicity is equal to one), corresponding eigenfunction $u_1$ can be chosen as the positive function in the domain $\Omega_i$.*

*3. $\chi_1(\sigma) \to 0$ as $\sigma \to 0$.*

For a given $\sigma > 0$ the smallest eigenvalue $\chi_1(\sigma)$ and corresponding eigenfunction $u_1$ define the eigenmode which is called the fundamental mode (see the bottom curve plotted by the red solid line in Fig. 2). Thus Theorem 2 states, particularly, that for any $\sigma > 0$ there exists exactly one fundamental mode.

## 3 The forward spectral problem

To compute eigenmodes we use the representation of eigenfunctions of problem (4)–(6) in the form of single-layer potentials [11]:

$$u(x) = \int_\gamma \Phi(\chi; x, y) f(y) dl(y), \quad x \in \Omega_i, \tag{7}$$

$$u(x) = \int_\gamma \Psi(\sigma; x, y) g(y) dl(y), \quad x \in \Omega_e. \tag{8}$$

Here

$$\Phi(\chi; x, y) = \frac{i}{4} H_0^{(1)}(\chi |x - y|), \tag{9}$$

$$\Psi(\sigma; x, y) = \frac{1}{2\pi} K_0(\sigma |x - y|) \tag{10}$$

are the fundamental solutions of Helmholtz equations (4) and (5) correspondingly, unknown densities $f$ and $g$ belong to the Holder space $C^{0,\alpha}(\gamma)$. Then $f$ and $g$ satisfy the following system of equations:

$$\mathcal{A}_{11}(\chi) f + \mathcal{A}_{12}(\sigma) g = 0, \quad x \in \gamma, \tag{11}$$

$$\mathcal{A}_{21}(\chi) f + \mathcal{A}_{22}(\sigma) g = 0, \quad x \in \gamma, \tag{12}$$

where

$$(\mathcal{A}_{11}(\chi) f)(x) = \int_\gamma \Phi(\chi; x, y) f(y) dl(y),$$

$$(\mathcal{A}_{12}(\sigma) g)(x) = -\int_\gamma \Psi(\sigma; x, y) g(y) dl(y),$$

$$(\mathcal{A}_{21}(\chi) f)(x) = \frac{1}{2} f(x) + \int_\gamma \frac{\partial \Phi(\chi; x, y)}{\partial \nu(x)} f(y) dl(y),$$

$$(\mathcal{A}_{22}(\sigma) g)(x) = \frac{1}{2} g(x) - \int_\gamma \frac{\partial \Psi(\sigma; x, y)}{\partial \nu(x)} g(y) dl(y).$$

System (11), (12) is reduced as in [4] to a nonlinear spectral problem for the integral operator-valued function:

$$A(\chi, \sigma) w = (I + B(\chi, \sigma)) w = 0, \tag{13}$$

where $B$ is a compact operator and $I$ is the identical operator acting in an appropriate Banach space. By analogy with [4] we prove that problem (4)–(6) is spectrally equivalent to (13). On the base of (13) we formulate the forward and the inverse spectral problems for weakly guiding step-index waveguides.

Let us formulate the forward spectral problem by the following way. Suppose that the boundary $\gamma$ of the waveguide's cross-section and the number $\sigma > 0$ are given. It is necessary to calculate all characteristic values $\chi$ of the operator-valued function $A(\chi)$ in the given interval.

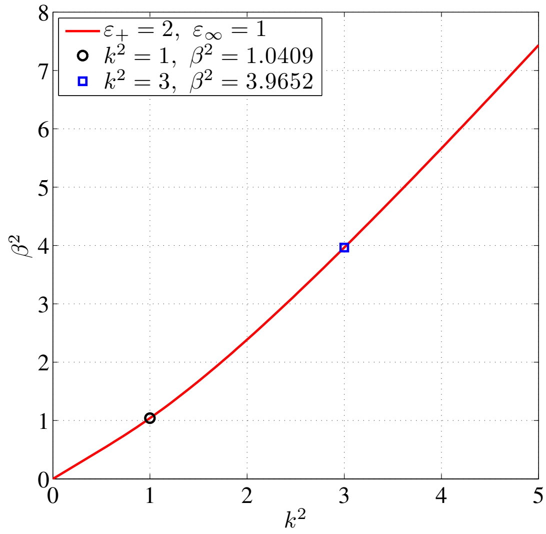


Figure 3: The red solid line is the plot of function $\beta^2 = \beta^2(k^2)$ corresponding to the fundamental mode of the circular waveguide.

Clearly, if the numbers $\chi$, $\sigma$, and the permittivities $\varepsilon_+$, $\varepsilon_\infty$ are known, then the longitudinal wavenumber $k$ and the propagation constant $\beta$ are calculated by the following explicit formulas:

$$k^2 = \frac{\sigma^2 + \chi^2}{\varepsilon_+ - \varepsilon_\infty}, \quad \beta^2 = \frac{\varepsilon_+ \sigma^2 + \varepsilon_\infty \chi^2}{\varepsilon_+ - \varepsilon_\infty}.$$

For each given $\varepsilon_+$, $\varepsilon_\infty$, and $i$ the function $\chi_i(\sigma)$ generates a function $\beta^2$ of the variable $k^2$. As an example in Fig. 3 we present the plot of function $\beta^2 = \beta^2(k^2)$ corresponding to the fundamental mode of the circular waveguide. Here $\varepsilon_+ = 2$, $\varepsilon_\infty = 1$, and $i = 1$. Two points marked by circle and by square we will use in the next section as test points for inverse eigenvalue problems.

A spline-collocation method was proposed in [6] for the numerical solution of problem (13). The original problem (13) was reduced to a nonlinear finite-dimensional eigenvalue problem.

We solved numerically the algebraic nonlinear eigenvalue problem by the residual inverse iteration method. The results of some numerical experiments are presented in Fig. 2 for the dielectric waveguide of the circular cross section.

## 4 The inverse spectral problem

### *4.1 Core's permittivity reconstructions*

We solved three variants of the inverse spectral problem. The first variant is the problem on the reconstruction of the permittivity of the core. It is formulated as follows. Suppose that the boundary $\gamma$ of the waveguide's cross-section and the permittivity $\varepsilon_\infty$ of the cladding are given. Suppose that the propagation constant $\beta$ of the fundamental mode is measured for a given wavenumber $k$. The measuring can be done by experimental methods. It is necessary to calculate the permittivity $\varepsilon_+$ of the waveguide's core.

The mathematical analysis of existence of the solution of the forward spectral problem is presented in Theorems 1 and 2. An illustration of the theoretical results is shown in Fig. 2. Analyzing Fig. 2 we observe that the fundamental mode (red solid curve in Fig. 2) exists for each wavenumber $k > 0$. The fundamental mode is unique, its dispersion curve does not intersect with any others curves and well separated from them. Therefore, the inverse spectral problem's solution exists and unique for each wavenumber $k > 0$, this solution depends continuously on given data. In other words the inverse spectral problem is well-posed by Hadamard.

An example of this continues dependence we can see in Fig. 4. The red solid line is the plot of the function $\varepsilon_+$ of $\beta^2$ for the fundamental mode.

If a numerical solution of the forward problem was obtained, then the solution of the inverse spectral problem is calculated by the following way. First, we compute the number

$$\sigma = \sqrt{\beta^2 - k^2\varepsilon_\infty},$$

which is calculated for given values of $\beta$, $k$, and $\varepsilon_\infty$. Then the transverse wavenumber $\chi(\sigma)$ is calculated by the spline-collocation method for the obtained $\sigma$ of the fundamental mode. This number is calculated by an interpolation of the function $\chi_1(\sigma)$ with respect to the points obtained when the forward problem was numerically solved (see the bottom curve plotted by the red solid line in Fig. 2). Finally, the permittivity of the waveguide's core is calculated by the following explicit formula:

$$\varepsilon_+ = \left(\chi^2 + \beta^2\right)/k^2.$$

In our computations by analogy with [12] we have introduced a random noise in the propagation constant as

$$\widetilde{\beta} = \beta(1 + p\alpha),$$

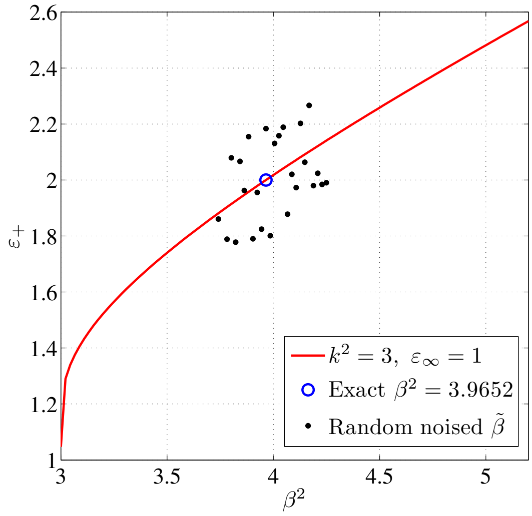


Figure 4: The red solid line is the plot of function $\varepsilon_+ = \varepsilon_+(\beta^2)$ for the fundamental mode. The approximate solution obtained by the spline-collocation method is marked by the blue circle for the exact $\beta$ and by black points for the random noised $\widetilde{\beta}$.

where $\beta = 3.98518$ is the exact measured propagation constant, $\alpha \in (-1, 1)$ are randomly distributed numbers, and $p$ is the noise level. In our computations we have used $p = 0.05$ and thus, the noise level was 5%.

Some numerical results of reconstruction of $\varepsilon_+$ are presented in Fig. 4. The approximated value of $\varepsilon_+$ for the noise-free data marked in Fig. 4 by the blue circle. Approximated values of $\varepsilon_+$ for randomly distributed noise $\widetilde{\beta}$ with the 5% noise level marked at Fig. 4 by the black points. Using this figure we observe that the approximate solutions even for the randomly noised $\widetilde{\beta}$ were stable.

We see that for the unique and stable reconstruction of the constant waveguide permittivity $\varepsilon_+$ it is enough to measure the propagation constant $\beta$ of the fundamental mode for only one wavenumber $k$.

### *4.2 Cladding's permittivity reconstructions*

The second variant of the problem is the reconstruction of the permittivity of the cladding. It is formulated as follows. Suppose that the permittivity of the core is given and that the propagation constant $\beta$ of the fundamental mode is measured for a given wavenumber $k$. It is necessary to calculate the permittivity $\varepsilon_\infty$ of the waveguide's cladding.

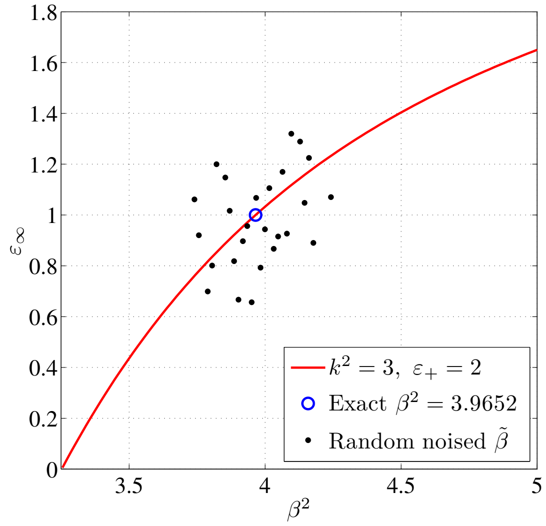


Figure 5: The red solid line is the plot of function $\varepsilon_\infty = \varepsilon_\infty(\beta^2)$ for the fundamental mode. The approximate solution obtained by the spline-collocation method is marked by the blue circle for the exact $\beta$ and by black points for the random noised $\widetilde{\beta}$.

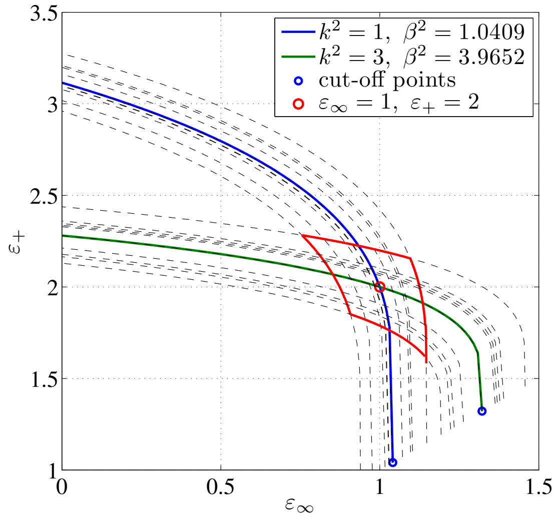


Figure 6: The solid lines are plots of function $\varepsilon_+ = \varepsilon_+(\varepsilon_\infty)$ for given pairs of $k$ and $\beta$ for the fundamental mode. The approximate solution obtained by the spline-collocation method is marked by the red circle for the exact $\beta$ and by the red rhomb for the random noised $\widetilde{\beta}$.

The solution of this problem is calculated by the following way. Firstly, we compute the number

$$\chi = \sqrt{k^2\varepsilon_+ - \beta^2}$$

which is calculated for given values of $\beta$, $k$ and $\varepsilon_+$. Secondly, the transverse wavenumber $\sigma$ is calculated by the spline-collocation method for the obtained $\chi$ for the fundamental mode. Thirdly, the permittivity of the waveguide's cladding is calculated by the following explicit formula:

$$\varepsilon_\infty = \left(\beta^2 - \sigma^2\right)/k^2.$$

As in the previous figure, the red solid line is the plot of continuous function $\varepsilon_\infty$ of squared $\beta$ for the fundamental mode. The approximate solution obtained by the spline-collocation method is marked by the blue circle for the exact $\beta$ and by black points for the random noised $\widetilde{\beta}$. Using this figure we observe that the approximate solutions for the randomly noised $\widetilde{\beta}$ were stable in this case too.

For the unique and stable reconstruction of the constant permittivity $\varepsilon_\infty$ it is enough to measure the propagation constant $\beta$ of the fundamental mode for only one wavenumber $k$.

### *4.3 Full permittivity reconstructions*

The full variant of our problem is the reconstruction of both permittivities of the core and of the cladding. The solution for the fundamental mode of the forward spectral problem gives us an implicit function $\varepsilon_+$ of the variable $\varepsilon_\infty$ for each fixed wavenumber and propagation constant. For example in Fig. 6 the blue and green solid lines are plots of this function for given pairs of $k$ and $\beta$.

The intersection of this lines is the unique exact solution of our problem. Therefore the permittivities $\varepsilon_+$ and $\varepsilon_\infty$ we calculate as the solution of the following nonlinear system of two equations:

$$\begin{cases} \chi^2(\beta_1^2 - k_1^2\varepsilon_\infty) = k_1^2\varepsilon_+ - \beta_1^2, \\ \chi^2(\beta_2^2 - k_2^2\varepsilon_\infty) = k_2^2\varepsilon_+ - \beta_2^2. \end{cases}$$

Here $k_1$ and $k_2$ are some given longitudinal wavenumbers, $\beta_1$ and $\beta_2$ are corresponding measured propagation constants. The approximate solution obtained by the spline-collocation method is marked at Fig. 6 by the red circle for the exact propagation constant. As we described above we introduced the random noise in the propagation constant. The approximate solutions for the noised $\widetilde{\beta}$ are intersections of dashed lines. We see that the approximate solutions for the randomly noised $\beta$ belong to the red rhomb and are stable.

## 5 Conclusion

In this work we showed that our inverse spectral problem is well-posed. It is important to note that any information on specific values of eigenfunctions is not required. For solution of the inverse problem it is enough to know that the fundamental mode is excited, and then to measure its propagation constant for one ore for two frequencies.

This approach satisfies the practice of physical experiments because usually the fundamental mode is excited for practical purposes. Moreover, the fundamental mode can be excited only for the enough wide interval of frequencies.

For the approximate solution of the inverse problems we propose first to solve the forward spectral problem in order to compute the dispersion curve for the fundamental mode. These calculations are done accurately by the spline-collocation method. Next, we uniquely and stably reconstruct the permittivities in our inverse algorithms.

## Acknowledgements

This work was funded by the subsidy allocated to Kazan Federal University for the state assignment in the sphere of scientific activities, and also supported by RFBR and by Government of Republic Tatarstan, grant 12-01-97012-r_povolzh'e_a.

The authors would like to thank the following individuals for helpful discussions concerning the topics covered in this work: Larisa Beilina, Michael Havrilla, George Hanson, Anna Repina.

## References

[1] Chu, M. T., Golub, G. H., 2005, Inverse Eigenvalue Problems: Theory, Algorithms, and Applications. Oxford University Press.

[2] Janezic, M. D., Jargon, J. A., 1999, Complex permittivity determination from propagation constant measurements, *IEEE Microwave and Guided Wave Letters*. Vol. **9**, No. 2, pp. 76–78.

[3] Snyder, A. W., Love, J. D., 1983, Optical Waveguide Theory. Chapman and Hall, London.

[4] Karchevskii, E. M., 1999, Analysis of the eigenmode spectra of dielectric waveguides, *Computational Mathematics and Mathematical Physics*. Vol. **39**, No. 9, pp. 1493–1498.

[5] Karchevskii, E. M., 2000, The fundamental wave problem for cylindrical dielectric waveguides. *Differential Equations*. Vol. **36**, No. 7, pp. 1109–1111.

[6] Spiridonov, A. O., Karchevskiy, E. M., 2013, Projection methods for computation of spectral characteristics of weakly guiding optical waveguides. *Proceedings of the International Conference Days on Diffraction 2013, DD 2013*, pp. 131–135.

[7] Karchevskii, E. M., Solov'ev, S. I., 2000: Investigation of a spectral problem for the Helmholtz operator on the plane. *Differential Equations*. Vol. **36**, No. 4, pp. 631–634.

[8] Kartchevski, E. M., Nosich, A. I., Hanson, G. W., 2005, Mathematical analysis of the generalized natural modes of an inhomogeneous optical fiber. *SIAM J. Appl. Math.* Vol. **65**, No. 6, pp. 2033–2048.

[9] Frolov, A., Kartchevskiy, E., 2013, Integral equation methods in optical waveguide theory. *Springer Proceedings in Mathematics and Statistics*. Vol. **52**, pp. 119–133.

[10] Karchevskiy, E., Shestopalov, Y., 2013, Mathematical and numerical analysis of dielectric waveguides by the integral equation method. *Progress in Electromagnetics Research Symposium, PIERS 2013 Stockholm, Proceedings*, pp. 388–393.

[11] Colton, D., Kress, R., 1983, Integral Equation Methods in Scattering Theory. Jhon Wiley and Sons, New York.

[12] Beilina, L., Klibanov, M. V., 2012, A new approximate mathematical model for global convergence for a coefficient inverse problem with backscattering data. *Inverse and Ill-Posed Problems*, Vol. **20**, pp. 513–565.