CENTRALLY ESSENTIAL RINGS

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All the results of this report were obtained jointly with V.T. Markov.

We consider only associative rings with $1 \neq 0$.

A ring R with center C is said to be centrally essential if, for any non-zero element $a \in R$, there exist two non-zero elements $x, y \in C$ with $ax = y$, i.e. R_C is an essential extension of the module C_C.

In a centrally essential ring R, all idempotents are central; in addition, if R is semiprime or right (left) nonsingular, then R is commutative.

1

Let F be the field $\mathbb{Z}/3\mathbb{Z}$, V be a vector F-space with basis e_1, e_2, e_3, and let $\Lambda(V)$ be the exterior algebra of the space V. Then $\Lambda(V)$ is a centrally essential noncommutative finite ring.

There is a centrally essential ring R such that the ring $R/J(R)$ is not a PI ring. Thus, $R/N(R)$ also is not a PI ring (in particular, the rings $R/J(R)$ and R are not commutative).

2

If R is a centrally essential ring, then for any commutative monoid G, the monoid ring RG is centrally essential. In particular, the rings $R[x]$ and $R[x, x^{-1}]$ are centrally essential.

Let F be a field of characteristic $p > 0$ and G a finite group.

1. The ring FG is centrally essential if and only if $G = P \times H$, where P is the unique Sylow p-subgroup of the group G, the group H is commutative, and the ring FP is centrally essential.

2. If G is a p-group with nilpotence class ≤ 2, then FG is centrally essential. There exists a group G' of order p^5 such that FG' is not centrally essential.

3

If R is a finite-dimensional centrally essential algebra, then R is a centrally essential ring $\Leftrightarrow$ $\Leftrightarrow$ the ring $R[[x]]$ is centrally essential $\Leftrightarrow$ the ring $R((x))$ is centrally essential.

Open Question. Is it true that any formal power series ring over a centrally essential ring is centrally essential?

If R is a centrally essential algebra and A is a commutative algebra, then $A \otimes R$ is a centrally essential algebra.

Open Question. Is it true that any tensor product of centrally essential algebras is centrally essential?

4

The exterior algebra $\Lambda(V)$ of a finite-dimensional vector space V over a field F of characteristic 0 or $p \neq 2$ is a centrally essential ring if and only if $\dim V$ is an odd positive integer. In particular, if F is a finite field of odd characteristic and $\dim V$ is an odd positive integer exceeding 1, then $\Lambda(V)$ is a centrally essential noncommutative finite ring.

5

A ring R is a right distributive, right Noetherian, centrally essential ring if and only if R is a direct product of finitely many commutative Dedekind domains and uniserial Artinian rings.

Let R be a left Artinian, left uniserial ring with center C and Jacobson radical J and let n be the nilpotence index of the ideal J. If $J^{[\frac{n}{2}} \subseteq C$, then the ring R is centrally essential. **Open question:** Is the converse assertion true?

If a field F has a non-trivial derivation δ, then there is a non-commutative Artinian uniserial centrally essential ring R with $R/J(R) \cong F$.

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