On the Hochschild cohomology of associative conformal algebras

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Origin of conformal algebras

• Axiomatic QFT (Wightman, 1956)
– d-Minkowski space & space of states V
– fields = End V-valued distributions on M
– Poincaré covariance of fields
– Locality principle (relativity):

$$\Phi(f)\Psi(g) = \Psi(g)\Phi(f)$$

if supports of test functions f and g are separated in M

Origin of conformal algebras

• $d = 2$ case, conformal covariance
(Belavin, Polyakov, Zamolodchikov, 1984): **chiral algebras**
– space of states V & translation $T : V \to V$
– fields: $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$, $a_n \in \operatorname{End} V$
–**T-invariance**:

$$[T, a(z)] = \frac{d}{dz} a(z)$$

– **Locality**:

$$[a(z), b(w)](z - w)^N = 0, \quad N = N(a, b) \in \mathbb{Z}_+,$$

Locality of formal distributions

A linear algebra (over $\mathbb{C}$)

$$\cdot : A \otimes A \to A \quad \text{(nonassociative)}$$

$a(z), b(z) \in A[[z, z^{-1}]]$ formal distributions over A

Kac, 1996:

$$a(w)b(z)(z-w)^N = 0 \iff a(w)b(z) = \sum_{n=0}^{N-1} c_n(z) \frac{1}{n!} \delta_z^{(n)}(z-w)$$

where

$$\delta(z-w) = \sum_{s\in\mathbb{Z}} z^s w^{-s-1}$$

$$c_n(z) = a(z)_{(n)} b(z)$$

Conformal algebras of formal distributions

$C \subset A[[z, z^{-1}]]$ subspace of pairwise local distributions closed w.r.t.

$$T = \frac{d}{dz}, \quad (\cdot_{(n)} \cdot),\ n \in \mathbb{Z}_+$$

Properties:

- $(a_{(n)} b) = 0$ for almost all n;
- $T(a_{(n)} b) = Ta_{(n)} b + a_{(n)} Tb$;
- $Ta_{(n)} b = -na_{(n-1)} b$.

— Axioms of **conformal algebra**

Conformal algebras of formal distributions

C conformal algebra $\Rightarrow A = \mathcal{A}(C)$ coefficient algebra:

$$C \subset A[[z, z^{-1}]]$$

Varieties of conformal algebras:
C ia a **Var-conformal** algebra (associative, Lie, etc.) $\iff \mathcal{A}(C)$ in Var

Examples

Trivial conformal algebra

$A = \mathbb{C}e,\ e^2 = 0$

$e(z) = e$: $Te = e\ _{(n)}\ e = 0$

Examples

Current conformal algebra

$\mathfrak{a}$ algebra over $\mathbb{C}$

$A = \mathfrak{a}[t, t^{-1}]$: $\quad t^n a \cdot t^m b = t^{n+m} ab$

$$a(z) = \sum t^n a z^{-n-1}, \quad a \in \mathfrak{a}$$

$$\boxed{a(w)b(z)(z-w) = 0} \quad a(z)b(w) = (ab)(z) \cdot \delta(z-w)$$

$$C = \mathbb{C}\left\{ \frac{d^n}{dz^n} a(z) \mid a \in \mathfrak{a},\ n \in \mathbb{Z}_+ \right\} = \operatorname{Cur}\mathfrak{a} \simeq \mathbb{C}[T] \otimes \mathfrak{a}$$

Examples

Conformal differential algebra

$\mathfrak{a}$ assoc. algebra with locally nilpotent derivation

$$(ab)' = a'b + ab'$$

$A = \mathfrak{a}[t, t^{-1}]$ differential Laurent polynomials

$$tb = bt + b', \quad b \in \mathfrak{a}$$

Then

$$a(z) = \sum_n a t^n z^{-n-1}, \quad a \in \mathfrak{a}$$

generate an associative conformal algebra $\simeq \mathbb{C}[T] \otimes \mathfrak{a}$

$$a_{(n)} b = ab^{(n)}$$

Examples

Conformal endomorphisms
$\mathfrak{a} = M_N(\mathbb{C}[x])$, $x' = 1$: locally nilpotent

Conformal algebra in the previous example:

$$\mathbb{C}[T] \otimes M_N(\mathbb{C}[x]) = \mathrm{Cend}_N$$

Conformal subalgebras:

$$\mathrm{Cend}_{Q,N} = Q(x) M_N(\mathbb{C}[T, x])$$

Examples

Commutator conformal algebra

C associative conformal algebra

Define $C^{(-)}$:

$$[a_{(n)} b] = (a_{(n)} b) - \sum_{s\geq 0}(-1)^{n+s}\frac{1}{s!}T^s(b_{(n+s)} a), \quad a, b \in C$$

is a Lie conformal algebra

$\mathrm{Cend}_N^{(-)} = \mathrm{gc}_N$

Examples

Quadratic conformal algebra

$\mathfrak{a}$ commutative algebra with a derivation
$A = \mathfrak{a}[t, t^{-1}]$: Lie algebra w.r.t.

$$[t^n a, t^m b] = t^{n+m-1}(nab' - ma'b)$$

Examples

Quadratic conformal algebra

$\mathfrak{a}$ commutative algebra with a derivation

$A = \mathfrak{a}[t, t^{-1}]$: Lie algebra w.r.t.

$$[t^n a, t^m b] = t^{n+m-1}(nab' - ma'b)$$

$$a(z) = \sum t^n a z^{-n-1}, \quad a \in \mathfrak{a}$$

$$[a(w), b(z)](z-w)^2 = 0 : \quad [a(w), b(z)] = \frac{d}{dz}(a'b)(z)\cdot\delta + (ab)'(z)\cdot\delta'_z$$

$$C \simeq \mathbb{C}[T] \otimes \mathfrak{a}$$

Examples

Virasoro conformal algebra

$\mathfrak{a} = \mathbb{C}[x]$, $x' = 1$

$$x(w)x(z) = Tx(z) \cdot \delta + 2x(z)\delta'_z$$

$$\mathrm{Vir} = \mathbb{C}\left\{ \frac{d^n}{dz^n} x(z) \mid n \in \mathbb{Z}_+ \right\} \simeq \mathbb{C}[T]x$$

with

$$x_{(0)} x = Tx, \quad x_{(1)} x = 2x$$

— simple conformal algebra

Examples

Virasoro/central extensions

$\mathcal{A}(\mathrm{Vir}) = W_1$, where $W_1 = \mathrm{Der}\,\mathbb{C}[t, t^{-1}]$ is the Witt algebra

The well-known nontrivial central extension ($c \in \mathbb{C}$):

$$0 \to \mathbb{C}e \to \widehat{W}_{1,c} \to W_1 \to 0$$

$$[t^n x, t^m x] = (n-m)t^{n+m-1}x + \delta_{n+m,-1}\, c\,\frac{n(n-1)(n+1)}{12} e$$

$$\Downarrow$$

$$0 \to \mathbb{C}e \to \mathrm{Vir}_c \to \mathrm{Vir} \to 0$$

— extended Virasoro conformal algebra with central charge c

$$[x(w), x(z)](w-z)^4 = 0$$

Examples

Current/central extensions

$\mathfrak{g}$ semisimple f-dim Lie algebra

$$0 \to \mathbb{C}e \to \mathfrak{g}\widehat{[t, t^{-1}]} \to \mathfrak{g}[t, t^{-1}] \to 0$$

$$[t^n a, t^m b] = t^{n+m}[a, b] + n\delta_{m+n,0}(a, b)e,$$

$$(\cdot, \cdot) \sim \text{Killing form}$$

$$\Downarrow$$

$$0 \to \mathbb{C}e \to \widehat{\mathrm{Cur}\,\mathfrak{g}} \to \mathrm{Cur}\,\mathfrak{g} \to 0$$

— Kac–Moody conformal algebra

$$[a(w), b(z)](w - z)^2 = 0$$

Cohomologies of conformal algebras

Bakalov, Kac, Voronov, 1999:
– Standard (Eilenberg) complex for Lie conformal algebras
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$$(d\gamma)_{\lambda_1,\dots,\lambda_{n+1}}(a_1,\dots,a_{n+1}) =$$
$$= \sum_{i=1}^{n+1}(-1)^{i+1} a_{i\,\lambda_i} \gamma_{\lambda_1,\dots,\hat{\lambda}_i,\dots,\lambda_{n+1}}(a_1,\dots,\hat{a}_i,\dots,a_{n+1}) +$$
$$+ \sum_{\substack{i,j=1\\ i<j}}^{n+1}(-1)^{i+j} \gamma_{\lambda_i+\lambda_j,\lambda_1,\dots,\hat{\lambda}_i,\dots,\hat{\lambda}_j,\dots,\lambda_{n+1}}([a_{i\,\lambda_i} a_j], a_1,\dots,\hat{a}_i,\dots,\hat{a}_j,\dots,a_{n+1}),$$

Algebras in pseudo-tensor categories

Beilinson, Drinfeld (2005):
Category theory in vertex algebras $\Rightarrow$ **pseudo-tensor category**

Bakalov, D'Andrea, Kac (2001):
Conformal algebras $\Leftarrow$ pseudo-tensor category $\mathcal{M}^*(H)$, $H = \mathbb{C}[T]$
"Spaces": H-modules
"Multi-linear" maps: $V_1 \otimes \cdots \otimes V_n \to H^{\otimes n} \otimes_H V$
with a well-defined composition rule

Categorical approach $\Rightarrow$ Basic algebraic features:

- associativity/Lie, etc.;
- morphism/ideal;
- module/representation;
- extension/cohomology.

Cohomologies of conformal algebras

Bakalov, Kac, Voronov, 1999:
$H^n(C, M)$ for semisimple finite Lie conformal algebras / irreducible modules

Dolguntseva, 2007: $H^2(\mathrm{Cend}_1, M)$

Su, Yue, 2008: $H^2(\mathrm{gc}_N, H^N)$, $H^3(\mathrm{gc}_1, H)$ $\qquad H = \mathbb{C}[T]$

Dolguntseva, 2009: $H^2(\mathrm{Cend}_N, M)$

Kozlov, 2017: $H^2(\mathrm{Cend}_{x,1}, M)$ Weyl conformal algebra

K., Kozlov, 2019: $H^2(C, M)$ for all simple assoc. conf. algebras of linear growth

Universal associative enveloping conformal algebras

$C \mapsto C^{(-)}$
Assoc. conformal $\to$ Lie conformal
(!) No left adjoint functor $U(\cdot)$

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Roitman, 2000:
L Lie conformal algebra
$B \subset L$ set of generators
$N : B \times B \to \mathbb{Z}_+$ function
$\Rightarrow$ $U(L; B, N)$ universal enveloping assoc. conf. algebra with

$$(a \,{}_{(n)}\, b) = 0, \quad n \geq N(a, b),\ a, b \in B$$

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Representation of $U(L; B, N) \Rightarrow$ Representation of L
(!) No $\Leftarrow$ (only for finite representations)

Universal associative enveloping conformal algebras

Example

$L = \text{Vir}$ Virasoro conformal algebra
$B = \{x\}$
$N(x, x) = 2$
Then

$$U(\text{Vir}; B, N) = U(2) \simeq \text{Cend}_{x,1}$$

— the Weyl conformal algebra

Universal associative enveloping conformal algebras

Example

$L = \operatorname{Cur}\mathfrak{g}$ current conformal algebra / semisimple f.-dim Lie algebra $\mathfrak{g}$
$B =$ basis of $\mathfrak{g}$
$N(a,b) = 1$
Then

$$U(\operatorname{Cur}\mathfrak{g}; B, N) \simeq \operatorname{Cur} U(\mathfrak{g})$$

Cohomologies of universal conformal envelopes

Lie cohomology vs Hochschild cohomology

$$H^n(\mathfrak{g}, M) = H^n(U(\mathfrak{g}), M)$$

for ordinary algebras

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For conformal algebras, observe that

$$\dim H^n(\mathrm{Vir}, \mathbb{C}e) = \begin{cases} 1, & n = 2, 3 \\ 0, & n \neq 2, 3 \end{cases}$$

but $H^n(U(2), \mathbb{C}e) = 0$

Cohomologies of universal conformal envelopes

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$$H^n(\mathrm{Cur}\,\mathfrak{g}, \mathbb{C}e) = H^n(\mathfrak{g}, \mathbb{C}e) + H^{n+1}(\mathfrak{g}, \mathbb{C}e)$$

but $H^n(\mathrm{Cur}\,U(\mathfrak{g}), \mathbb{C}e) = H^n(U(\mathfrak{g}), \mathbb{C}e) = H^n(\mathfrak{g}, \mathbb{C}e)$

Cohomologies of universal conformal envelopes

$U(3) = U(\mathrm{Vir}; x, 3)$
— associative conformal algebra of quadratic growth, its structure was found in [K., 2020]

Theorem

$\dim H^2(U(3), \mathbb{C}e) = 1$, as for Vir
but $H^3 = 0$.

Problem

For $U = U(\mathrm{Cur}\,\mathfrak{g}; B, N)$, $N = 2, 3$, calculate $H^2(U, \mathbb{C}e)$.

The structure of $U(\widehat{\mathrm{Cur}\,\mathfrak{g}}; B, N)$, $N = 2, 3$ [K., Kozlov, 2021] implies the answer is $\neq 0$