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On Continuity of Functions With Generalized Derivatives

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Abstract—We study the question of the continuity of functions for which the generalized gradient belongs to the Lorentz spaces.

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We consider ball $B \subset \mathbb{R}^n$ and Sobolev space $W_p^1(B)$. An element of the Sobolev space, generally speaking, is a class of equivalent functions, so discussing properties of the Sobolev functions we will keep in mind that corresponding equivalence class contains which has the properties that we are interested in. According to the classical embedding theorem when $p > n$ Sobolev space $W_p^1(B)$ is embedded in the space of continuous functions and, moreover, the functions will satisfy the Hölder condition with exponent $1 - n/p$. When $p \leq n$, Sobolev space $W_p^1(B)$ contains functions which have unremovable discontinuities. Thus when $p = n$ there is a discrete shift from Hölder functions to discontinuous. It is evident that there should exist some conditions under which generalized differentiated function does not satisfy Hölder condition but is still continuous.

The condition of such type is found in paper [1]: If the function $u \in W_{1,\mathrm{loc}}^1(B)$ and its gradient ∇u belongs to the Lorentz space $L_{n,1}(B)$, then function u is n-absolutely continuous. As when all $p > n$ the embeddings

$$L_p(B) \subset L_{n,1}(B) \subset L_n(B),$$

are valid, we can use the Lorentz spaces to refine the classical Sobolev embedding theorem. We want to show that within Sobolev–Lorentz scale the result from [1] is a precise one.

While discussing the Lorentz spaces we will give only necessary information in a form which is relevant for us. The introduction to the theory of Lorentz spaces can be found in [2] (Chap. V, § 3).

The Lorentz space $L_{p,q}(B)$ when $1 \leq p < \infty$, $1 \leq q < \infty$ is defined as a set of all measurable functions h, for which the functional

$$\Lambda_{p,q}(h) = \left(\frac{q}{p} \int_0^{m_n(B)} [t^{1/p} h^*(t)]^q \frac{dt}{t} \right)^{1/q} = \left(q \int_0^\infty s^{q-1} (\omega(s))^{q/p} ds \right)^{1/q} < \infty,$$

is finite, where $\omega(s) = m_n(\{x \in B \mid |h(x)| > s\})$ is a distribution function and $h^*(t)$ is a nonincreasing permutation of function $h(x)$.

In the space $L_{p,q}(B)$ we can introduce a norm which satisfy the estimate

$$\Lambda_{p,q}(h) \leq \|h\|_{p,q} \leq \frac{p}{p-1} \Lambda_{p,q}(h).$$

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