Centrally Essential Rings

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[**MT18b**] Centrally essential rings which are not necessarily unital or associative (Russian). Diskretnaya Matematika. – 2018. – Vol. 30, no. 4. – P. 41-46.

[**MT19a**] Centrally essential rings. Discrete Math. Appl. – 2019. – Vol. 29, no. 3. – P. 189-194.

[**MT19b**] Rings essential over their centers. Comm. Algebra. – 2019. – https://doi.org/10.1080/00927872.2018.1513012.

[**MT19c**] Rings with Polynomial Identity and Centrally Essential Rings. Beiträge zur Algebra und Geometrie / Contributions to Algebra and Geometry. – 2019. – https://doi.org/10.1007/s13366-019-00447-w

[**MT19d**] Cayley-Dickson Process and Centrally Essential Rings. Journal of Algebra and Applications. – https://doi.org/10.1142/S0219498819502293

[**MT20a**] Uniserial Artinian Centrally Essential Rings, Beiträge zur Algebra und Geometrie / Contributions to Algebra and Geometry, (To appear).

[**MT20b**] Uniserial Noetherian Centrally Essential Rings, Communications in Algebra, (to appear).

[**MT20c**] Distributive Noetherian Centrally Essential Rings, Communications in Algebra, (to appear).

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We consider only associative rings with $1 \neq 0$, unless otherwise indicated.

A ring R with center C is said to be centrally essential if, for any non-zero element $a \in R$, there exist two non-zero elements $x, y \in C$ with $ax = y$, i.e. R_C is an essential extension of the module C_C.

In a (not necessarily unital) centrally essential ring R, all idempotents are central; in addition, if R is semiprime, then R is commutative.

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3.1. Let F be the field $\mathbb{Z}/3\mathbb{Z}$, V be a vector F-space with basis e_1, e_2, e_3, and let $\Lambda(V)$ be the exterior algebra of the space V.
Then $\Lambda(V)$ is a centrally essential, noncommutative, finite ring.

3.2. If F is the field of order 2 and $G = Q_8$ is the quaternion group, then the group algebra FG is a finite noncommutative centrally essential ring consisting of 256 elements.

3.3. Open question. What is the minimal number of elements of a centrally essential non-commutative finite ring?

3.4. There is a centrally essential ring R such that the ring $R/J(R)$ is not a PI ring. Thus, $R/N(R)$ is not a PI ring as well (in particular, the rings $R/J(R)$ and R are not commutative).

3.5. If R is a centrally essential ring and B is a proper ideal of R generated by some infinite set of central idempotents such that the factor ring R/B does not have non-trivial idempotents, then R/B is not necessarily a centrally essential ring.

3.6. If R is a centrally essential ring and $C = C(R)$ is a semiprime ring, then the ring R is commutative.

3.7. Any centrally essential ring R satisfies the following relations.

(1) $\forall n \in \mathbb{N}, x_1, \ldots, x_n, y_1, \ldots, y_n, r \in R,$

$$\left\{ \begin{array}{rcl} x_1y_1 + \ldots + x_ny_n & = & 1 \\ x_1ry_1 + \ldots + x_nry_n & = & 0 \end{array} \right. \Rightarrow r = 0.$$

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4.1. Let S be a subring of the ring R such that there exists a basis of the module R_S contained in $C(R)$. If S is a centrally essential ring, then R also is a centrally essential ring.

4.2. If R is a centrally essential ring, then for any commutative monoid G, the monoid ring RG is centrally essential. In particular, the rings $R[x]$ and $R[x, x^{-1}]$ are centrally essential.

4.3. Let F be a field of characteristic $p > 0$ and G a finite group.

4.3.1. The ring FG is centrally essential if and only if $G = P \times H$, where P is the unique Sylow p-subgroup of the group G, the group H is commutative, and the ring FP is centrally essential.

4.3.2. If G is a p-group with nilpotence class ≤ 2, then FG is centrally essential. There exists a group G' of order p^5 such that FG' is not centrally essential.

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5.1. For a ring R, the formal power series ring $R[[x]]$ is centrally essential $\Leftrightarrow$ the Laurent series ring $R((x))$ is centrally essential.

5.2. If R is a finite-dimensional algebra over a field, then
R is a centrally essential ring $\Leftrightarrow$
the formal power series ring $R[[x]]$ is centrally essential $\Leftrightarrow$
the Laurent series ring $R((x))$ is centrally essential.

5.3. Open Question. Is it true that any formal power series ring over a centrally essential ring is centrally essential?

5.4. If R is a centrally essential algebra and A is a commutative algebra, then $A \otimes R$ is a centrally essential algebra.

5.5. Open Question. Is it true that any tensor product of centrally essential algebras is centrally essential?

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6.1. Any finite, left uniserial, centrally essential ring is commutative.

6.2. There exists a non-commutative uniserial Artinian centrally essential ring.

6.3. There exists a right uniserial, right Noetherian, non-semiprimary PI ring $\widehat{R}$ with prime radical $\widehat{N}$ and Jacobson radical $\widehat{M}$ such that $\widehat{R}$ is not centrally essential, and $\widehat{R}$ is not left Noetherian or left uniserial, $\widehat{R}/\widehat{N}$ is a commutative discrete valuation domain, $\widehat{N}$ is a minimal right ideal, and $\widehat{N} = \widehat{M}\widehat{N} \neq \widehat{N}\widehat{M} = 0$.

6.4. A field F does not have a non-trivial derivation if and only if any left uniserial, left Artinian, centrally essential ring R with $R/J(R) \cong F$ is commutative.

6.5. Let R be a left Artinian, left uniserial ring with center C and Jacobson radical J and let n be the nilpotence index of the ideal J.
If $J^{[n/2]} \subseteq C$, then the ring R is centrally essential.

6.6. Is it true that the assertion, which is converse to the above asertion, holds?

6.7. Example. Let $F = GF(4)$, $F_0 = GF(2) \subseteq F$, and let $\sigma\colon x \mapsto x^2$ be the Frobenius automorphism of the field F. We consider the skew polynomial ring $S = F[X, \sigma]$ and its factor ring $R = S/(X^3)$. Then R is a left and right uniserial ring, left and right Artinian ring, $J(R)$ is a nilpotent ideal of nilpotence index 3, and $J(R)^{[3/2]+1} \subseteq C(R)$; however, the ring R is not centrally essential.

6.8. Open question. Is it true that every left Noetherian, left uniserial, centrally essential ring is right uniserial?

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7.1. A ring R is a right uniserial, right Noetherian, centrally essential ring if and only if R is a commutative discrete valuation domain or a uniserial Artinian ring.

7.2. A ring R is a right distributive, right Noetherian, centrally essential ring if and only if R is a direct product of finitely many commutative Dedekind domains and uniserial Artinian rings.

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8.1. Let R be a centrally essential semiperfect ring with center C, Jacobson radical $J(R)$ and prime radical $N(R)$. Then $R/J(R)$ is a commutative ring, R is a finite direct product of centrally essential local rings and $\operatorname{Soc} R_C \subseteq C$.

8.1.1. Is it true that $\operatorname{Soc} R_C = \operatorname{Soc} R_R$?

8.1.2. Is it true that $R = C + J(R)$?

8.2. If R is a right or left perfect ring with center C, then R is centrally essential if and only if $\operatorname{Soc} R_C \subseteq C$ and all idempotents of the ring R are central.

8.3. If R is a centrally essential ring and P is a semiprime nil-ideal of R such that the socle of the ring R/P is an essential ideal of R/P, then the ring R/P is commutative.

8.4. If R is a left or right semi-Artinian, centrally essential ring, then $R/J(R)$ is a commutative regular ring.

8.5. If R is a ring and I is an ideal of R such that I is contained in the center of R and R/I is a field, then R is a centrally essential ring.

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9.1. For a field F of characteristic 0 or $p \neq 2$ and a finite-dimensional vector F-space V, the exterior algebra $\Lambda(V)$ is a centrally essential ring if and only if $\dim V$ is an odd positive integer.

9.2. Let F be a finite field of odd characteristic and V a finite-dimensional vector F-space. If $\dim V$ is an odd positive integer > 1, then $\Lambda(V)$ is a centrally essential noncommutative finite ring.

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Let A be a not necessarily commutative ring with center C and n a positive integer. We define an A-algebra $\Lambda(A^n)$ of the finitely generated free module A^n of rank n. Namely, $\Lambda(A^n) = A \otimes_C \Lambda(C^n)$, where $\Lambda(C^n)$ is the exterior algebra of the free module C^n over the commutative ring C.

Let $\{e_1, \ldots, e_n\}$ be a basis of the module C^n. By identifying $1 \otimes x$ with x for every $x \in \Lambda(C^n)$, we obtain that the set

$$B_n = \{e_{i_1} \wedge \ldots \wedge e_{i_s} | 0 \leq s \leq n, \ \ 1 \leq i_1 < \ldots < i_s \leq n\}$$

is a basis of the A-module $\Lambda(A^n)$ (we assume that the product is equal to 1 for $s = 0$).

10.1. $\Lambda(A^n)$ is a centrally essential ring if and only if A is centrally essential and at least one of the following conditions holds:
a) the annihilator of 2 in A is an essential submodule in A_C;
b) n is an odd integer.

10.2. If the ring A is of finite characteristic s, then $\Lambda(A^n)$ is a centrally essential ring if and only if A is centrally essential and at least one of the following conditions holds:
a) $s = 2^m$ for some $m \in \mathbb{N}$;
b) n is an odd integer.

10.3. If A does not have zero-divisors, then the ring $\Lambda(A^n)$ is centrally essential if and only if A is centrally essential and at least one of the following conditions holds:
a) A is a ring of characteristic 2;
b) n is an odd integer.

11

In this section, we consider unital but not necessarily associative rings.

For a ring R, the *associator* of three elements $a, b, c \in R$ is the element $(a, b, c) = (ab)c - a(bc)$, the *associative center, commutative center, and center* of R are the sets

$$N(R) = \{x \in R : \forall a, b \in R,\ (x, a, b) = (a, x, b) = (a, b, x) = 0\},$$
$$K(R) = \{x \in R : \forall a \in R,\ [x, a] = 0\},$$
$$Z(R) = N(R) \cap K(R),$$

respectively. It is clear that $N(R)$ and $Z(R)$ are subrings in R and the ring R is a unitary (left and right) $N(R)$-module and $Z(R)$-module.

11.1. Centrally essential rings and N-essential rings.
A ring R is said to be *centrally essential* if $Z(R)r \cap Z(R) \neq 0$ for any non-zero element $r \in R$, i.e., $Z = Z(R)$ is an essential submodule of the module ${}_Z R$.
A ring R is said to be *left N-essential* if $N(R)r \cap N(R) \neq 0$ for any non-zero element $r \in R$, i.e., $N = N(R)$ is an essential submodule of the module ${}_N R$.

11.2. The Cayley-Dickson process. Let A be a ring with involution $*$ and α an invertible symmetrical element of the center of the ring A. We define a multiplication operation on the Abelian group $A \oplus A$ as follows:

$$(a_1, a_2)(a_3, a_4) = (a_1 a_3 + \alpha a_4 a_2^*\ ,\ a_1^* a_4 + a_3 a_2)$$

for any $a_1, \dots, a_4 \in A$. We denote the obtained ring by (A, α).
The elements of the ring (A, α) of the form $(a, 0)$, $a \in A$, form a subring in ring (A, α) which is isomorphic to the ring A; we will identify them with the corresponding elements of the ring A. We set $\nu = (0, 1) \in (A, \alpha)$. Then $a * \nu = (0, a) = \nu a$ for any $a \in A$ and $\nu^2 = \alpha$. Thus, $(A, \alpha) = A + A\nu$. In addition, $\nu^2 = \alpha$ and $\nu a = a^* \nu$ for any element $a \in A$, $(1, 0)$ is the identity element of the ring (A, α), the set $\{(a, 0) \,|\, a \in A\}$ is a subring of the ring (A, α) which is isomorphic to the ring A, and the mapping $(a, b) \mapsto (a^*, -b)$, $a, b \in A$, is an involution of the ring (A, α).

11.3. Associative Center of the Ring Obtained by Applying Cayley-Dickson Process In this subsection, we fix a ring A and an element α which satisfy the conditions of the Cayley-Dickson process; we also set $R = (A, \alpha)$.

11.4. N-Essentiality Criterion for the Ring Obtained by the Cayley-Dickson Process.

11.4.1. Let A be a ring with center $C = Z(A)$, $[A, A]$ the ideal of A generated by commutators of all its elements $I = \mathrm{Ann}_C([A, A])$, $R = (A, \alpha)$. The ring R is left (right) N-essential if and only if A is a centrally essential ring and I is an essential ideal of the ring C.

11.4.2. The Criterion of Central Essentiality for the Ring Obtained by the Cayley-Dickson Process.
We fix a ring A with center $C = Z(A)$ and an element α which satisfy the Cayley-Dickson process. Let $R = (A, \alpha)$, $I = \mathrm{Ann}_C([A, A])$, $B = \{a \in C : \ a = a^*\}$, $J = \mathrm{Ann}_B(\{a - a^* \,|\, a \in A\})$. We note that the sets B and J are invariant with respect to the involution and are closed with respect to the multiplication by α.

11.4.3. $Z(R) = \{(x, y) \,|\, x \in B, \ y \in I \cap J\}$.

11.4.4. The ring $R = (A, \alpha)$ is centrally essential if and only if B is an essential B-submodule of the ring R and $J' = J \cap I$ is an essential ideal of the ring B.

11.5. Generalized Quaternion Algebra and Octonion Algebra over a Commutative Ring.

Let K be a commutative associative ring with the identity involution and a an invertible element of the ring R. We consider the ring $A_1 = (K, a)$. It can be verified that A_1 is a commutative associative ring. It is natural to write elements of the ring A_1 in the form $x+yi$, where x, y are elements of the ring K, $i = (0, 1)$. On the ring A_1, an involution is defined by the relation $(x + yi)^* = x - yi$ for any $x, y \in K$. We choose an invertible element $b \in K$. Then b is an invertible symmetrical element of the center of the ring A_1 and we can construct the ring $A_2 = (A_1, b)$. We consider the K-basis of the algebra A_2 which is formed by the elements $1 = (1, 0)$, $i = (i, 0)$, $j = (0, 1)$ and $k = (0, -i)$. The relations $i^2 = a$, $j^2 = b$, $ij = -ji = k$, $ik = -ki = aj$, $kj = -jk = bi$ are directly verified. Consequently, the obtained ring is the generalized quaternion algebra (a, b, K). It is well known that the ring A_2 is associative. The center of the ring A_2 is of the form $K + Ni + Nj + Nk$, where $N = \mathrm{Ann}_K(2)$. For $A = A_2$, let B, I, J be defined as in 11.4.2. It is easy to verify that $B = C = Z(A_2)$, $I = J = N + Ni + Nj + Nk$.

11.5.1. The quaternion algebra $((K,a),b)$ is a non-commutative centrally essential ring if and only if $\mathrm{Ann}\,_K(2)$ is a proper essential ideal of the ring K.

11.5.2. We consider an arbitrary invertible element $c \in K$. The octonion algebra $(((K,a),b),c)$ is a non-associative centrally essential ring if and only if $\mathrm{Ann}_K\,(2)$ is a proper essential ideal of the ring K.

11.5.3. Example. Let $K = \mathbb{Z}_4$. We prove that $R = (((K,1),1),1)$ is a non-associative non-commutative centrally essential ring.
Indeed, $\mathrm{Ann}_K\,(2) = 2K$ is an essential proper ideal in K. Therefore, the non-commutativity of the ring $((K,1),1)$ (and the non-commutativity of the ring R containing $((K,1),1)$) follows from Proposition 5.2 and the non-associativity of the ring R follows from Proposition 5.3..
We note that the ring $R = (((K,1),1),1)$ is an alternative ring and the ring $(R,1)$ is not even a right-alternative ring, i.e., $(R,1)$ does not satisfy the identity $(x,y,y) = 0$. Thus, there exist alternative non-associative finite centrally essential rings and non-alternative finite centrally essential rings. $\square$

11.6. Open Questions.

11.6.1. Is it true that there exist left N-essential rings which are not right N-essential?

11.6.2. Is it true that there exist commutative N-essential (equivalently, centrally essential) non-associative rings?

11.6.3. Is it true that there exist right-alternative centrally essential or N-essential non-alternative rings?

11.6.4. How can we generalize the obtained results to the case of non-unital rings and the case, where the element α in definition 11.2 is not supposed to be invertible?

11.6.5. Since the Cayley-Dickson process gives non-associative division algebras, it seems natural to state the following question: What can be said about N-essentiality of these division rings?

11.6.6. Note that centrally essential semiprime rings are commutative, but we do not know if N-essential semiprime ring are associative.

Thank you!