New theories with massive gravitons and new Horndeski-type theories

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I. Massive spin-2 in curved space

C.Mazuet, M.S.V. JCAP 1807 (2018) 012

Massive fields in curved space

How to generalize wave equations to curved space ?

Spin-0: Klein-Gordon equation

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu - m^2)\Phi = 0$$

generalizes to curved space via simply via

$$\eta_{\mu\nu} \Rightarrow g_{\mu\nu}, \qquad \partial_\mu \Rightarrow \nabla_\mu$$

which yields

$$(g^{\mu\nu}\nabla_\mu\nabla_\nu - m^2)\Phi = 0$$

Similarly for spins $1/2, 1, 3/2$.

The procedure fails for massive spin-2

Fierz-Pauli equations /1939/

$$E_{\mu\nu} \equiv \partial^\sigma\partial_\mu h_{\sigma\nu} + \partial^\sigma\partial_\nu h_{\sigma\mu} - \partial^\sigma\partial_\sigma h_{\mu\nu} - \partial_\mu\partial_\nu h + \eta_{\mu\nu}\left(\partial^\sigma\partial_\sigma h - \partial^\alpha\partial^\beta h_{\alpha\beta}\right) + m^2(h_{\mu\nu} - h\,\eta_{\mu\nu}) = 0,$$

imply 5 constraints

$$\mathcal{C}_\nu \equiv \partial^\mu E_{\mu\nu} = m^2(\partial^\mu h_{\mu\nu} - \partial_\nu h) = 0,$$

$$\mathcal{C}_5 \equiv \left(\partial^\mu\partial^\nu + \frac{m^2}{2}\,\eta^{\mu\nu}\right) E_{\mu\nu} = -\frac{3}{2}\,m^4 h = 0.$$

hence

$$(\Box - m^2)h_{\mu\nu} = 0, \qquad \partial^\mu h_{\mu\nu} = 0, \qquad h = 0.$$

Replacing $\eta_{\mu\nu} \Rightarrow g_{\mu\nu}$ abd $\partial_\mu \Rightarrow \nabla_\mu$ one finds that

$$\left(\nabla^\mu\nabla^\nu + \frac{m^2}{2}\,g^{\mu\nu}\right) E_{\mu\nu}$$

is not a constraint anymore (contains second derivatives) $\Rightarrow$ there are 6 DoF, unless if $R_{\mu\nu} = \Lambda g_{\mu\nu}$. A long standing problem.

Massive integer spins are descried by symmetric tensors and one can write down equations of motion for them and find classical solutions only in special spacetimes (e.g. in Ricci flat spaces) ... a consistent classical action for the system of dynamical gravity and a higher massive field is still unknown and there are some indications that perhaps it does not exist at all.

/Buchbinder, Gitman, Krykhtin, Pershin 2000/

Solution

The dRGT massive gravity /2010/

$$G_{\mu\nu}(g) + \beta_0\, g_{\mu\nu} + \beta_1([\gamma]\, g_{\mu\nu} - \gamma_{\mu\nu}) + \beta_2\, |\gamma|\, ([\gamma]\, \gamma_{\mu\nu} - \gamma^{-2}_{\mu\nu}) + \beta_3\, |\gamma|\, \gamma_{\mu\nu} = 0 \qquad (1)$$

contains $g_{\mu\nu}$ and a reference metric $f_{\mu\nu}$ with $\gamma^{\mu}{}_{\nu} = \sqrt{g^{\mu\sigma} f_{\sigma\nu}}$. The idea is to represent

$$g_{\mu\nu} = \eta_{ab} e^{a}{}_{\mu} e^{b}{}_{\nu}$$

then linearize (1) with respect to tetrad perturbations

$$e^{a}{}_{\mu} \to e^{a}{}_{\mu} + \delta e^{a}{}_{\mu}$$

and use (1) to determine $f_{\mu\nu}$. This gives a linear theory for a non-symmetric tensor

$$X_{\mu\nu} = \eta_{ab} e^{a}{}_{\mu} \delta e^{b}{}_{\nu}$$

which propagates 5 DoF for any $g_{\mu\nu}$. Using $\delta g_{\mu\nu} = X_{\mu\nu} + X_{\nu\mu}$ gives very complicated expressions /Deffayet et al./

Equations: $\Delta_{\mu\nu} + \mathcal{M}_{\mu\nu} = 0$

with the kinetic term

$$\begin{aligned}
\Delta_{\mu\nu} &= \frac{1}{2}\nabla^\sigma\nabla_\mu(X_{\sigma\nu} + X_{\nu\sigma}) + \frac{1}{2}\nabla^\sigma\nabla_\nu(X_{\sigma\mu} + X_{\mu\sigma}) \\
&- \frac{1}{2}\Box(X_{\mu\nu} + X_{\nu\mu}) - \nabla_\mu\nabla_\nu X - R^\sigma_\mu X_{\sigma\nu} - R^\sigma_\nu X_{\sigma\mu} \\
&+ g_{\mu\nu}\left(\Box X - \nabla^\alpha\nabla^\beta X_{\alpha\beta} + R^{\alpha\beta}X_{\alpha\beta}\right)
\end{aligned}$$

and the mass term

$$\begin{aligned}
\mathcal{M}_{\mu\nu} &= \beta_1\left(\gamma^\sigma{}_\mu X_{\sigma\nu} - g_{\mu\nu}\,\gamma^{\alpha\beta}\,X_{\alpha\beta}\right) \\
&+ \beta_2\,\{-\gamma^\alpha{}_\mu\gamma^\beta{}_\nu\,X_{\alpha\beta} - (\gamma^2)^\alpha{}_\mu\,X_{\alpha\nu} + \gamma_{\mu\nu}\,\gamma_{\alpha\beta}\,X^{\alpha\beta} \\
&\quad + [\gamma]\,\gamma^\alpha{}_\beta\,X_{\alpha\nu} + ((\gamma^2)_{\alpha\beta}\,X^{\alpha\beta} - [\gamma]\,\gamma_{\alpha\beta}\,X^{\alpha\beta})\,g_{\mu\nu}\} \\
&+ \beta_3\,|\gamma|\,\left(X_{\mu\sigma}(\gamma^{-1})^\sigma{}_\nu - [X](\gamma^{-1})_{\mu\nu}\right)
\end{aligned}$$

$\gamma_{\mu\nu}$ is algebraically related to the background $g_{\mu\nu}$ via

$$\begin{aligned}
G_{\mu\nu}(g) &+ \beta_0\,g_{\mu\nu} + \beta_1([\gamma]\,g_{\mu\nu} - \gamma_{\mu\nu}) \\
&+ \beta_2\,|\gamma|\,([\gamma]\,\gamma_{\mu\nu} - \gamma^{-2}_{\mu\nu}) + \beta_3\,|\gamma|\,\gamma_{\mu\nu} = 0
\end{aligned}$$

Constraints

There are 16 equations

$$E_{\mu\nu} \equiv \Delta_{\mu\nu} + \mathcal{M}_{\mu\nu} = 0$$

for 16 components of $X_{\mu\nu}$. The imply 11 conditions:

$$\Delta_{[\mu\nu]} = 0 \quad \Rightarrow \quad \mathcal{M}_{[\mu\nu]} = 0 \quad \Rightarrow \quad \text{6 algebraic constraints}$$

$$\mathcal{C}_\nu = \nabla^\mu E_{\mu\nu} = 0 \quad \Rightarrow \quad \text{4 vector constraints}$$

$$\begin{aligned} \mathcal{C}_5 &= \nabla_\mu((\gamma^{-1})^{\mu\nu}\mathcal{C}_\nu) + \frac{\beta_1}{2} E^\alpha_{\ \alpha} + \beta_2 \gamma^{\mu\nu} E_{\mu\nu} \\ &+ \beta_3 \frac{|\gamma|}{g^{00}} \Big((\gamma^{-1})^{0\alpha}(\gamma^{-1})^{0\beta} - (\gamma^{-1})^{00}(\gamma^{-1})^{\alpha\beta} \\ &\times \Big(E_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}(E^\sigma_{\ \sigma} - \frac{1}{g^{00}} E^{00}) \Big) \Big) = 0 \;\Rightarrow\; \text{scalar constraint} \end{aligned}$$

The number of DoF is $16 - 6 - 4 - 1 = 5$.

The two special models

If $R_{\mu\nu} = \Lambda g_{\mu\nu}$ then $X_{\mu\nu} = X_{\nu\mu}$ and the theory reproduces the standard description of massive gravitons.
The mass term is a non-linear function of the background $R_{\mu\nu}$.

$$\mathcal{M}_{\mu\nu} = B_0\, g_{\mu\nu} + B_1\, R_{\mu\nu} + B_2\, (R^2)_{\mu\nu} + B_3\, (R^3)_{\mu\nu}$$

where y_k, B_m are functions of scalar invariants of $R^{\mu}{}_{\nu}$ and of β_A.
$\mathcal{M}_{\mu\nu}$ is linear in Ricci if either $\beta_2 = \beta_3 = 0$ or $\beta_1 = \beta_2 = 0 \Rightarrow$ two special models (with $m^2 = \beta_0$):

$$\text{model I:} \qquad \mathcal{M}_{\mu\nu} = \gamma_{\mu\alpha} X^{\alpha}_{\ \nu} - g_{\mu\nu}\, \gamma_{\alpha\beta} X^{\alpha\beta}$$

$$\gamma_{\mu\nu} = R_{\mu\nu} + \left(m^2 - \frac{R}{6} \right) g_{\mu\nu}$$

$$\text{model II:} \qquad \mathcal{M}_{\mu\nu} = -X_{\mu}^{\ \alpha} \gamma_{\alpha\nu} + X \gamma_{\mu\nu}$$

$$\gamma_{\mu\nu} = R_{\mu\nu} - \left(m^2 + \frac{R}{2} \right) g_{\mu\nu}$$

Massive spin-2 in the expanding universe

The background geometry

$$g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) d\mathbf{x}^2$$

where $a(t)$ fulfills the Einstein equations

$$3\,\frac{\dot{a}^2}{a^2} = \frac{\boldsymbol{\rho}}{M_{\rm Pl}^2} \equiv \rho, \qquad 2\,\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -\frac{\boldsymbol{p}}{M_{\rm Pl}^2} \equiv -p,$$

where $\boldsymbol{\rho}, \boldsymbol{p}$ are the energy density and pressure of the background matter. We wish to solve the equations

$$\Delta_{\mu\nu} + \mathcal{M}_{\mu\nu} = 0.$$

for the two special models.

Fourier decomposition

$$X_{\mu\nu}(t,\mathbf{x}) = a^2(t)\sum_{\mathbf{k}} X_{\mu\nu}(t,\mathbf{k})e^{i\mathbf{kx}}$$

where the Fourier amplitude splits into the sum of the tensor, vector, and scalar harmonics,

$$X_{\mu\nu}(t,\mathbf{k}) = X^{(2)}_{\mu\nu} + X^{(1)}_{\mu\nu} + X^{(0)}_{\mu\nu}$$

The spatial part of the background Ricci tensor $R_{ik} \sim \delta_{ik}$ hence

$$X_{ik} = X_{ki}$$

$\Rightarrow X_{\mu\nu}$ has only 13 independent components:

Tensor, vector, scalar harmonics

$$X^{(2)}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{D}_+ & \mathrm{D}_- & 0 \\ 0 & \mathrm{D}_- & -\mathrm{D}_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; X^{(1)}_{\mu\nu} = \begin{bmatrix} 0 & W^+_+ & W^+_- & 0 \\ W^-_+ & 0 & 0 & i\mathrm{k}\mathrm{V}_+ \\ W^-_- & 0 & 0 & i\mathrm{k}\mathrm{V}_- \\ 0 & i\mathrm{k}\mathrm{V}_+ & i\mathrm{k}\mathrm{V}_- & 0 \end{bmatrix},$$

$$X^{(0)}_{\mu\nu} = \begin{bmatrix} S^+_+ & 0 & 0 & i\mathrm{k}S^+_- \\ 0 & S^-_- & 0 & 0 \\ 0 & 0 & S^-_- & 0 \\ i\mathrm{k}S^-_+ & 0 & 0 & S^-_- - \mathrm{k}^2\mathrm{S} \end{bmatrix},$$

where $\mathrm{D}_\pm$, $\mathrm{V}_\pm$, S, $W^\pm_\pm$, $S^\pm_\pm$ are 13 functions of time. The equations split into three independent groups – for the tensor modes $X^{(2)}_{\mu\nu}$, for vector modes $X^{(1)}_{\mu\nu}$, and for scalar modes $X^{(0)}_{\mu\nu}$.

Two tensor modes

The effective action is

$$I_{(2)} = \int (K\dot{\mathrm{D}}_{\pm}^2 - U\mathrm{D}_{\pm}^2)\, a^3 dt$$

with

$$K = 1, \qquad U = M_{\mathrm{eff}}^2 + \mathrm{k}^2/a^2$$

where

$$\begin{aligned} \mathrm{I}: \quad M_{\mathrm{eff}}^2 &= M^2 + \frac{1}{3}\,\rho, \qquad m_{\mathsf{h}}^2 = M_{\mathrm{eff}}^2, \\ \mathrm{II}: \quad M_{\mathrm{eff}}^2 &= M^2 - p, \qquad m_{\mathsf{h}}^2 = M^2 + \rho \end{aligned}$$

m_{h} reduces to the Higuchi mass if $R_{\mu\nu} = \Lambda g_{\mu\nu}$.

Two vector modes

The effective action

$$I_{(1)} = \int (K\dot{\mathrm{V}}_{\pm}^2 - U\mathrm{V}_{\pm}^2)\, a^3 dt$$

with $(\epsilon = \rho + p)$

$$\begin{aligned} K &= \frac{\mathrm{k}^2 m_{\mathrm{H}}^4}{m_{\mathrm{H}}^4 + (\mathrm{k}^2/a^2)(m_{\mathrm{H}}^2 - \epsilon/2)}, \\ U &= M_{\mathrm{eff}}^2\, \mathrm{k}^2 \end{aligned}$$

If $R_{\mu\nu} = \Lambda g_{\mu\nu}$ then $m_{\mathrm{h}} = M_{\mathrm{H}}$ (Higuchi mass), vector modes do not propagate if $M_{\mathrm{H}} = 0$ (massless limit).

Scalar sector

$$I_{(0)} = \int (K\dot{\mathrm{S}}^2 - U\mathrm{S}^2)\, a^3 dt$$

where the kinetic term

$$K = \frac{3\mathrm{k}^4 m_{\mathrm{H}}^4 (m_{\mathrm{H}}^2 - 2H^2)}{(m_{\mathrm{H}}^2 - 2H^2)[9m_{\mathrm{H}}^4 + 6(\mathrm{k}^2/a^2)(2m_{\mathrm{H}}^2 - \epsilon)] + 4(\mathrm{k}^4/a^4)(m_{\mathrm{H}}^2 - \epsilon)}$$

and the potential

$$\begin{aligned} U/K &\rightarrow M_{\mathrm{eff}}^2 \quad \text{as} \quad k \rightarrow 0 \\ U/K &\rightarrow c^2\,(\mathrm{k}^2/a^2) \quad \text{as} \quad k \rightarrow \infty \end{aligned}$$

where c is the sound speed. Only one DoF in the scalar sector (!!!)
If $R_{\mu\nu} = \Lambda g_{\mu\nu}$ then $m_{\mathrm{h}} = M_{\mathrm{H}}$ and the scalar mode does not propagate if either $M_{\mathrm{H}} = 0$ (massless limit) or if $M_{\mathrm{H}}^2 = 2H^2$ (partially massless limit).

Stability

The kinetic term and the sound speed squared should be positive

$$K > 0, \qquad c^2 > 0$$

(no ghosts and tachyons). These conditions are fulfilled

- all the time after the inflation if $M \geq 10^{13}$ GeV
- at present if $M \geq 10^{-33}$ eV
- Assuming that $X_{\mu\nu}$ couples only to gravity and hence massive spin-2 particles do not have other decay channels, it follows that they could be a part of Dark Matter (DM) at present

Summary of Part I

- A consistent theory of a free massive spin-2 field propagating 5 DoF in arbitrary spacetimes is constructed.

- This allows for the first time to consistently consider a model of Dark Matter made of massive spin-2 particles.

II. Massive gravity with a healthy Boulware-Deser mode

S. Mukohyama, M.S.V., JCAP 1810 (2018) 037

Theory of Ogievetsy and Polubarinov (1965)

The OP were looking for a non-linear completion of the theory of free massive gravitons

$$
\begin{aligned}
(\Box - m^2)h_{\mu\nu} &= (\text{terms non-linear in } h_{\mu\nu}) \\
\partial^\mu h_{\mu\nu} &= 0 \\
h^\sigma{}_\sigma &= 0
\end{aligned}
$$

trying to construct the non-linear terms without using differential geometry. They constructed

$$
\begin{aligned}
(\Box - m^2)h_{\mu\nu} &= (\text{terms non-linear in } h_{\mu\nu}) \\
\partial^\mu h_{\mu\nu} + q\, h^\sigma{}_\sigma &= 0 \qquad (2)
\end{aligned}
$$

by requiring that the liner condition (2) follows from the non-linear field equations. They derived the Einstein-Hilbert kinetic term with only the field-theory methods for the first time (before Deser).

Theory of Ogievetsy and Polubarinov (1965)

$$S = M_{\rm Pl}^2 \int \left(\frac{1}{2} R(g) - m^2 U + 2\Lambda \right) \sqrt{-g}\, d^4x \,,$$

Introducing $\hat{S} = \hat{g}^{-1}\hat{f} \equiv \hat{1} + \hat{\chi}$ its arbitrary real power is

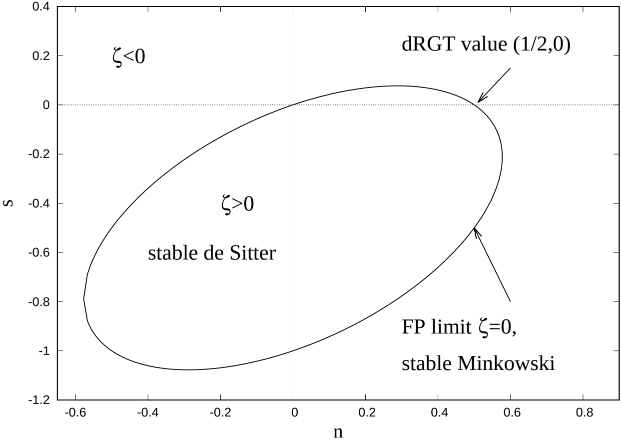
$$\hat{S}^n = \hat{1} + n\hat{\chi} + \frac{n(n-1)}{2}\,\hat{\chi}^2 + \ldots$$

The OP mass term depends on two real parameters n, s:

$$U = \frac{1}{4n^2}\left(\det(\hat{S})\right)^{-s/2}[\hat{S}^n] + \frac{n-2s-2}{2n^2} \equiv \mathcal{U} + \lambda_g .$$

For $s = 0$, $n = 1/2$ this theory coincides with the dRGT theory for $\beta_0 = -3$, $\beta_1 = 1$, $\beta_2 = \beta_3 = 0 \Rightarrow$ OP obtained first "ghost-free" theory already in 1965. For generic n, s there are 6 DoF, even around flat space $\Rightarrow$ the Boulware-Deser ghost.

OP parameter space



Stability of de Sitter

$$\zeta = \frac{1 - (n - 2s - 1)^2 - 3n^2}{4n^2}$$

The analysis of perturbations shows that

- The tensor and vector sectors contain 4 DoF which are always healthy. The sound speeds approaches unity in the UV limit, for large momenta.
- The scalar sector contains two modes whose kinetic energy is always positive if

  $$\zeta > 0, \qquad 6H^2 > \zeta M^2$$

  hence the BD mode is benign. The sound speed approaches unity in the UV limit.
- The same analysis around flat space shows that one of the two eigenvalues of the kinetic matrix in the scalar sector is always negative $\Rightarrow$ the BD mode is ghost.

Stability of Milne

- The Milne universe (sector of Minkowski space) is UV stable, hence its instable modes can only be in the IR.
- This presumably means that the ghost instability of flat space is similar to the classical Jeans instability.

Summary of Part II

The Boulware-Deser mode can be benign

III. Screening Horndeski cosmologies and the seed of GW

A.Starobinsky, S.Sushkov, M.S.V., JCAP 1606 (2016) 007

Horndeski /1974/

$$L_{\rm H} = \sqrt{-g}\,(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5),$$

$$\begin{aligned}
\mathcal{L}_2 &= G_2(X,\Phi)\,, \\
\mathcal{L}_3 &= G_3(X,\Phi)\,\Box\Phi\,, \\
\mathcal{L}_4 &= G_4(X,\Phi)\,R + \partial_X G_4(X,\Phi)\,\delta^{\mu\nu}_{\alpha\beta}\,\nabla^\alpha_\mu\Phi\nabla^\beta_\nu\Phi\,, \\
\mathcal{L}_5 &= G_5(X,\Phi)\,G_{\mu\nu}\nabla^{\mu\nu}\Phi - \frac{1}{6}\,\partial_X G_5(X,\Phi)\,\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\,\nabla^\alpha_\mu\Phi\nabla^\beta_\nu\Phi\nabla^\gamma_\rho\Phi\,,
\end{aligned}$$

with $X \equiv -\frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi$, $\quad \delta^{\lambda\rho}_{\nu\alpha} = 2!\,\delta^\lambda_{[\nu}\delta^\rho_{\alpha]}$, $\quad \delta^{\lambda\rho\sigma}_{\nu\alpha\beta} = 3!\,\delta^\lambda_{[\nu}\delta^\rho_\alpha\delta^\sigma_{\beta]}$.

The most general theory with second order field equations.

The GW170817 event shows that the speed of GW is one $\Rightarrow$ one has to have $\partial_X G_4 = G_5 = 0$ for redshifts $z < 0.3$. No restrictions for larger redshifts.

Fab Four (F4)

$$L_{\mathrm{F4}} = \sqrt{-g}\,(\mathcal{L}_J + \mathcal{L}_P + \mathcal{L}_G + \mathcal{L}_R - 2\Lambda)$$

$$\begin{aligned}
\mathcal{L}_J &= V_J(\Phi)\, G_{\mu\nu} \nabla^\mu \Phi \nabla^\nu \Phi\,, \\
\mathcal{L}_P &= V_P(\Phi)\, P_{\mu\nu\rho\sigma} \nabla^\mu \Phi \nabla^\rho \Phi \nabla^{\nu\sigma} \Phi\,, \\
\mathcal{L}_G &= V_G(\Phi)\, R\,, \\
\mathcal{L}_R &= V_R(\Phi)\,(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2),
\end{aligned}$$

the dual of Riemann $P^{\mu\nu}{}_{\alpha\beta} = -\frac{1}{4}\,\delta^{\mu\nu\gamma\delta}_{\sigma\lambda\alpha\beta}\, R^{\sigma\lambda}{}_{\gamma\delta}, \quad P^{\mu\alpha}{}_{\nu\alpha} = G^\mu_\nu$

The most general Horndeski subset where flat space is a solution, despite $\Lambda \neq 0 \Rightarrow$ screening of the cosmological constant.

The model

$$S = \int \left(M_{\mathrm{Pl}}^2 \, R - (\alpha \, G_{\mu\nu} + \varepsilon \, g_{\mu\nu}) \nabla^\mu \Phi \nabla^\nu \Phi - 2\Lambda \right) \sqrt{-g} \, d^4x + S_{\mathrm{mat}}$$

seems to be incompatible with the GW170817 observation.

However ...

FLRW

$$ds^2 = -dt^2 + \mathrm{a}^2(t)\left[dx^2 + dy^2 + dz^2\right],$$

With $H = \dot{\mathrm{a}}/\mathrm{a}$, $\psi = \dot{\Phi}$ the equations reduce to

$$3M_{\mathrm{Pl}}^2 H^2 = \frac{1}{2}\,\varepsilon\,\psi^2 - \frac{9}{2}\,\alpha\,\psi^2 H^2 + \Lambda + \rho,$$

and to

$$\mathrm{a}^3\left(3\alpha\,H^2 - \varepsilon\right)\psi = C$$

where C=const is a “scalar charge”.

$C = 0$

GR branch

$$H^2 = \frac{\Lambda + \rho}{3M_{\rm Pl}^2}, \qquad \psi = 0$$

Screening branch:

$$H^2 = \frac{\varepsilon}{3\alpha}, \qquad \psi^2 = \frac{\alpha\,(\Lambda + \rho) - \varepsilon\, M_{\rm Pl}^2}{\alpha\,\varepsilon}$$

cosmological term is $\varepsilon/3\alpha$ while the Λ, ρ are screened.

$C \neq 0$

Scalar field

$$\psi = \frac{C}{\mathrm{a}^3\,\left[3\alpha\, H^2 - \varepsilon\right]},$$

the Friedmann equation reduces to algebraic equation for $H(a)$

$$3M_{\rm Pl}^2\, H^2 = \frac{C^2\,\left[\varepsilon - 9\alpha H^2\right]}{2a^6\,\left[\varepsilon - 3\alpha\, H^2\right]^2} + \Lambda + \rho.$$

Dimensionless master equation

$$H^2 = H_0^2\, y, \qquad \mathrm{a} = \mathrm{a}_0\, a\,, \qquad \rho_{\rm cr} = 3 M_{\rm Pl}^2 H_0^2\,,$$

$$\Omega_0 = \frac{\Lambda}{\rho_{\rm cr}}, \quad \Omega_6 = \frac{C^2}{6\alpha\, \mathrm{a}_0^6\, H_0^2\, \rho_{\rm cr}}, \quad \zeta = \frac{\varepsilon}{3\alpha\, H_0^2},$$

and assuming

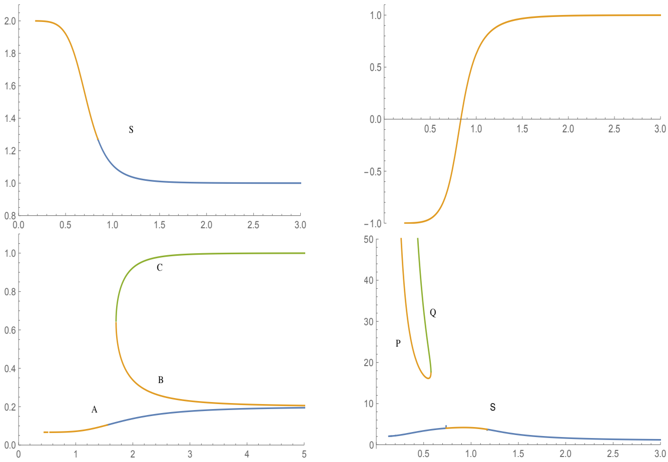
$$\rho = \rho_{\rm cr} \left(\frac{\Omega_4}{a^4} + \frac{\Omega_3}{a^3} \right) \qquad = \ \text{radiation+dust}$$

yields the cubic algebraic equation for $y(a)$,

$$y = \Omega_0 + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{\Omega_6\,[\zeta - 3y]}{a^6\,[\zeta - y]^2}$$

There are many different solutions. Most of them show ghosts and tachyons.

$y = (H/H_0)^2$ against a.



Exceptional solution

No ghosts. Stable for $a \to \infty$. Passes the GW test for large a.

Exists with or without matter, contains two inflationary stages with hierarchy of Hubble scales

$$\frac{\zeta}{3} \leftarrow \left(\frac{H}{H_0}\right)^2 \to \Omega_0$$

Late times – standard dynamic dominated by the $\Lambda +$ matter.
Early times matter and Λ are screened.

$$-\frac{3C}{2\varepsilon \mathrm{a}^3} \leftarrow \psi = \frac{C}{\mathrm{a}^3\,[3\alpha\, H^2 - \varepsilon]} \to \frac{C\, M_{\mathrm{Pl}}^2}{\alpha\Lambda - \varepsilon M_{\mathrm{Pl}}^2}\frac{1}{\mathrm{a}^3}$$

Speed of GW

The tensor modes are governed by the effective action

$$I = \frac{1}{2}\int \mathrm{K}\left(\dot{\mathrm{D}}_{\pm}^2 - c_s^2\frac{p^2}{a^2}\,\mathrm{D}_{\pm}^2\right) a^3 dt$$

with

$$\mathrm{K} = M_{\mathrm{Pl}}^2 + \alpha\psi^2, \qquad c_s^2 = \frac{1-\alpha\psi^2/(2M_{\mathrm{Pl}}^2)}{1+\alpha\psi^2/(2M_{\mathrm{Pl}}^2)}$$

Since $\psi \to 0$ at late times, one has

$$c_s^2 - 1 = \frac{\alpha}{2M_{\mathrm{Pl}}^2}\psi^2 = \frac{\alpha C^2 M_{\mathrm{Pl}}^2}{(\alpha\Lambda - \varepsilon M_{\mathrm{Pl}}^2)^2}\frac{1}{\mathrm{a}^6} \propto \frac{1}{\mathrm{a}^6} \qquad (!)$$

$\Rightarrow$ perfectly agrees with GW170817 $\Rightarrow$ not all G_4, G_5 Horndeski models are eliminated.

One has $\psi \propto 1/a^3$ also for small a, hence the solution is unstable near singularity $\Rightarrow$ presumably should be replaced by something else (inhomogeneous phase ?); further analysis is needed.

Summary of part III

The GW170817 results eliminate not the G_4, G_5 Horndeski models but rather some of their solutions.

IV. New Horndeski-type theories in the Palatini approach

in preparation

Horndeski à là Palatini ?

$$L_{\rm H} = \sqrt{-g}\,(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5),$$

$$\begin{aligned}
\mathcal{L}_2 &= G_2(X,\Phi)\,, \\
\mathcal{L}_3 &= G_3(X,\Phi)\,\Box\Phi\,, \\
\mathcal{L}_4 &= G_4(X,\Phi)\,R + \partial_X G_4(X,\Phi)\,\delta^{\mu\nu}_{\alpha\beta}\,\nabla^\alpha_\mu\Phi\nabla^\beta_\nu\Phi\,, \\
\mathcal{L}_5 &= G_5(X,\Phi)\,G_{\mu\nu}\nabla^{\mu\nu}\Phi - \frac{1}{6}\,\partial_X G_5(X,\Phi)\,\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\,\nabla^\alpha_\mu\Phi\nabla^\beta_\nu\Phi\nabla^\gamma_\rho\Phi
\end{aligned}$$

What happens if we vary this Lagrangian à là Palatini ?

The answer depends on whether one varies the black part or the red part.

The KGB part of Horndeski

$$S = \int \left(G_4(\phi) \overset{(\Gamma)}{R} + K(\phi, X) + G_3(\phi, X) \overset{(\Gamma)}{\Box} \phi \right) \sqrt{-g}\, d^4x$$

$$\begin{aligned}
\overset{(\Gamma)}{R} &= g^{\mu\nu} \overset{(\Gamma)}{R}_{\mu\nu}, \\
\overset{(\Gamma)}{R}_{\mu\nu} &= \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\sigma\alpha} \Gamma^\sigma_{\mu\nu} - \Gamma^\alpha_{\mu\sigma} \Gamma^\sigma_{\nu\alpha} \\
\overset{(\Gamma)}{\Box} \phi &= g^{\mu\nu} \overset{(\Gamma)}{\nabla}_\mu \overset{(\Gamma)}{\nabla}_\nu \phi \\
X &= \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \equiv g^{\mu\nu} X_{\mu\nu}
\end{aligned}$$

If $\Gamma^\alpha_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\}$ then this is the standard Horndeski theory.
Let us vary independently with respect to $\Gamma^\alpha_{\mu\nu}$, $g_{\mu\nu}$, ϕ.

Varying with respect to $\Gamma^{\alpha}_{\mu\nu}$

yields the covariant derivative of the metric,

$$\overset{(\Gamma)}{\nabla}_{\alpha} g^{\mu\nu} = g^{\mu\nu}\partial_{\alpha}\omega + \frac{2}{3}\gamma\left(g^{\mu\nu}\partial_{\alpha}\phi + \delta^{(\mu}_{\alpha}\phi^{\nu)}\right)$$

where

$$G_4 = e^{\omega}, \qquad G_3 = \gamma G_4, \qquad K = \kappa G_4.$$

This can be resolved to obtain

$$\begin{aligned}\Gamma^{\alpha}_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\} \quad &+ \quad \frac{1}{2}\left(\delta^{\alpha}_{\mu}\partial_{\nu}\omega + \delta^{\alpha}_{\nu}\partial_{\mu}\omega - g_{\mu\nu}\partial^{\alpha}\omega\right) \\ &+ \quad \frac{1}{3}\gamma\left(\delta^{\alpha}_{\mu}\partial_{\nu}\phi + \delta^{\alpha}_{\nu}\partial_{\mu}\phi\right)\end{aligned}$$

$\Rightarrow$ not a metric connection. This yields

$$\overset{(\Gamma)}{R}_{\mu\nu} = R_{\mu\nu} - \nabla_{\mu}\nabla_{\nu}\,\omega - \gamma\nabla_{\mu}\nabla_{\nu}\phi + \frac{1}{2}\,\partial_{\mu}\omega\partial_{\nu}\omega + \ldots$$

Varying with respect to $\phi, g_{\mu\nu}$

$$
\begin{aligned}
0 \;=\;& \partial_\phi G_4 \overset{(\Gamma)}{R} + \partial_\phi K + \partial_\phi G_3 \overset{(\Gamma)}{\Box}\phi \\
& -\nabla_\mu \left(\partial_X K \,\nabla^\mu \phi\right) - \nabla_\mu \left(\partial_X G_3 \overset{(\Gamma)}{\Box}\phi\, \nabla^\mu \phi\right) \\
& + \frac{1}{\sqrt{-g}} \overset{(\Gamma)}{\nabla}_\mu \overset{(\Gamma)}{\nabla}_\nu \left(\sqrt{-g}\, G_3\, g^{\mu\nu}\right)
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\;& \overset{(\Gamma)}{R}_{\mu\nu} - \frac{1}{2} \overset{(\Gamma)}{R} g_{\mu\nu} + \gamma \overset{(\Gamma)}{\nabla}_\mu \overset{(\Gamma)}{\nabla}_\nu \phi \\
& + \left(\kappa_X + \gamma_X \overset{(\Gamma)}{\Box}\phi\right) X_{\mu\nu} - \frac{1}{2}\left(\kappa + \gamma \overset{(\Gamma)}{\Box}\phi\right) g_{\mu\nu}
\end{aligned}
$$

Equations

$$G_{\mu\nu}(g) + T_{\mu\nu} = 0, \qquad \nabla_\mu J^\mu = \Sigma \quad \text{where}$$

$$\begin{aligned} T_{\mu\nu} &= -\omega' \partial_\mu \partial_\nu \phi - \gamma\, \partial_{(\mu}\phi \partial_{\nu)} X \\ &+ \left(\kappa_X + \gamma \Box \phi + 2\omega' X \gamma - 2\omega'' + \omega'^2 - 2\gamma' - \frac{2}{3}\, \partial_X(\gamma^2 X) \right) X_{\mu\nu} \\ &+ \left(\frac{1}{2} \langle \partial\phi \partial X \rangle \gamma - \frac{1}{2} \kappa + \omega' \Box \phi + (2\omega'' + \frac{1}{2}\omega'^2 + \gamma' + \frac{1}{3}\gamma^2) X \right) g_{\mu\nu} \end{aligned}$$

and

$$\begin{aligned} J^\mu &= \left\{ \partial_X K + (\omega' - \frac{2}{3}\gamma')(2X\partial_X + 1) G_3 - \partial_\phi G_3 \right\} \partial^\mu \phi \\ &\quad + \partial_X G_3 \left\{ \Box\phi\, \partial^\mu \phi - \partial^\mu X \right\}, \\ \Sigma &= \partial_\phi K + \partial_\phi G_4 \overset{(\Gamma)}{R} + \partial_\phi G_3 \overset{(\Gamma)}{\Box} \phi\,. \end{aligned}$$

These are different from Horndeski equations $\Rightarrow$ new theory.

Propagating modes

Assuming

$$ds^2 = -dt^2 + a(t)^2 \left(dx^2 + dy^2 + dz^2\right), \qquad \phi = \phi(t), \qquad \Phi \equiv \dot{\phi},$$

and considering small perturbations

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \qquad \phi \to \phi + \delta\phi,$$

one obtains in the tensor sector

$$I_{\mathrm{T}} = \frac{1}{2} \int G_4(\phi) \left(\dot{D}_{\pm}^2 - \frac{p^2}{a^2} D_{\pm}^2 \right) a^3\, d^4x,$$

$\Rightarrow$ the kinetic term $G_4 = e^{\omega}$ is always positive while the sound speed is equal to one $\Rightarrow$ the GW propagate with the speed of light.

The result in the scalar sector is more complex – both the kinetic term and the sound speed are background-dependent.

Example of red theory: $G_4 = X$

$$
\begin{aligned}
\mathcal{L} &= \left(\overset{(\Gamma)}{R} + \alpha G_{\mu\nu} \partial^\mu \Phi \partial^\nu \Phi \right) \sqrt{-g} \\
&= \overset{(\Gamma)}{R}_{\mu\nu} \left(\left[1 - \frac{\alpha}{2} \partial_\sigma \Phi \partial^\sigma \Phi \right] g^{\mu\nu} + \alpha\, \partial^\mu \Phi \partial^\nu \Phi \right) \sqrt{-g} \\
&= \overset{(\Gamma)}{R}_{\mu\nu}\, \tilde{g}^{\mu\nu} \sqrt{-\tilde{g}}
\end{aligned}
$$

Varying this à là Palatini yields third order equations for $g_{\mu\nu}$, however, they are equivalent to the vacuum GR

$$R_{\mu\nu}(\tilde{g}) = 0$$

Rest of Horndeski – also higher order equations for $g_{\mu\nu}$, but it unclear if one cay make sense out of this.

Summary of part IV

- Varying à là Palatini the black (allowed) part of the Horndeski Lagrangian produces second order equations which are different from Horndeski equations $\Rightarrow$ new theory. Some exact solutions can be obtained (wormholes).
- The GW speed is one.
- Varying the red (forbidden) part produces higher order equations. In some cases the higher derivatives can be absorbed by disformal transformations.