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TANGENT BUNDLES AND GAUGE GROUPS

M. Rahula, V. Balan

Abstract

The differentials $T^k a$ $(k \geq 1)$ of a diffeomorphism a of a smooth manifold M induce in the fibers of the fiber bundles $T^k M$, i.e., in the corresponding tangent spaces, linear transformations, which embody the action of the gauge group $\mathcal{G}_k$. This action extends in a natural way to the osculating subbundles $\mathrm{Osc}^{k-1} M \subset T^k M$.

Key words: diffeomorphism of a smooth manifold, fiber bundles, action of the gauge group.

Introduction

The differential group $\mathcal{G}$ of a smooth manifold M induces in the tangent bundle $T^k M$ an action of the group of k-jets of transformations. More specifically, if a is a diffeomorphism of the manifold M, then its k-th differential $T^k a$ is a transformation of the level $T^k M$. Then the level $T^k M$ may be regarded as a homogeneous space J^k/H_k, where J^k is the group of k-jets of transformations and H_k is the stabilizer of an element $u_{(u)} \in T^k M$. The gauge group $\mathcal{G}_k$ is defined as a certain subgroup of the linear group $GL(2^k n, \mathbb{R})$, where $n = \dim M$ which is isomorphic to the stabilizer H_k. The action of the group $\mathcal{G}$ extends to the osculating subbundle $\mathrm{Osc}^{k-1} M \subset T^k M$.

The paper contains all the necessary definitions and founds all the previous considerations. Commented examples and groups of derived formulas are presented as exercises.

1. Tangent groups

1.1. Leibniz rule. We apply the tangent functor T to the Cartesian product of smooth manifolds M_1 and M_2:

$$T(M_1 \times M_2) = (TM_1 \times M_2) \oplus (M_1 \times TM_2),$$

and for the smooth mapping from $M_1 \times M_2$ to some smooth manifold M

$$\lambda : M_1 \times M_2 \longrightarrow M : (u, v) \mapsto w = u \cdot v,$$

we define the tangent mapping $T\lambda$. First, by fixing the points $u \in M_1$ and $v \in M_2$ we define two mappings λ_u and λ_v:

$$\lambda_u : M_2 \to M : v \mapsto u \cdot v, \qquad \lambda_v : M_1 \to M : u \mapsto u \cdot v.$$

Theorem 1. *To the pair of vectors $u_1 \in T_u M_1$ and $v_1 \in T_v M_2$ the mapping $T\lambda$ associates the vector $w_1 \in T_w M$, and we have*

$$\boxed{w = u \cdot v \quad \Rightarrow \quad w_1 = u_1 \cdot v + u \cdot v_1\,,} \tag{1}$$

where $u_1 \cdot v = T\lambda_v(u_1)$ and $u \cdot v_1 = T\lambda_u(v_1)$. In short, one can apply to the "product" $w = u \cdot v$ the Leibniz rule.

Proof. We specify that, by means of the tangent maps $T\lambda_v$ and $T\lambda_u$, two vectors $u_1 \in T_u M_1$ and $v_1 \in T_v M_2$ are transported from the points $u \in M_1$ and $v \in M_2$ to the point $w \in M$, where their sum defines the vector $w_1 \in T_w M$. Locally, this is confirmed by the formula:

$$w^\rho = \lambda^\rho(u^i, v^\alpha) \quad \Rightarrow \quad w_1^\rho = \frac{\partial \lambda^\rho}{\partial u^i} u_1^i + \frac{\partial \lambda^\rho}{\partial v^\alpha} v_1^\alpha,$$

where u^i, v^α, w^ρ are the coordinates of the points u, v, w and $u_1^i, v_1^\alpha, w_1^\rho$ are the components of the vectors u_1, v_1, w_1 on the neighborhoods $U_1 \subset M_1$, $U_2 \subset M_2$, $U \subset M$, $i = 1, \ldots, \dim M_1$, $\alpha = 1, \ldots, \dim M_2$, $\rho = 1, \ldots, \dim M$. □

Using the Leibniz rule we derive a set of important formulas in coordinate free form.

Exercise 1: action of Leibniz rule. Show that the Leibniz rule can be applied to the "product" of several factors, e.g.,

$$(u \cdot v \cdot w)_1 = u_1 \cdot v \cdot w + u \cdot v_1 \cdot w + u \cdot v \cdot w_1.$$

Exercise 2: prolongation of Leibniz rule. Prove that for the second tangent mapping $T^2\lambda$, the following formulas hold true:

$$\begin{aligned} w = u \cdot v, \quad w_1 &= u_1 \cdot v + u \cdot v_1, \\ w_2 &= u_2 \cdot v + u \cdot v_2, \\ w_{12} &= u_{12} \cdot v + u_2 \cdot v_1 + u_1 \cdot v_2 + u \cdot v_{12}. \end{aligned} \tag{2}$$

Exercise 3: functional equation.
Question: how can one solve the equation $(u \cdot v)_1 = u_1 \cdot v + u \cdot v_1$ with respect to u_1 for given v_1 and $(u \cdot v)_1$, or relative to v_1 for given u_1 and $(u \cdot v)_1$? This reminds *the method of integration by parts*:

$$d(uv) = u\,dv + v\,du \quad \rightsquigarrow \quad uv = \int u\,dv + \int v\,du, \quad \text{whence}$$

$$\text{either} \quad \int u\,dv = uv - \int v\,du, \quad \text{or} \quad \int v\,du = uv - \int u\,dv.$$

1.2. Coordinate-free story. The rule (1) is easy to use while building tangent groups and further, while studying representations of groups. If we have previously denoted the "product" of elements by a dot, as in (1) and (2), then while denoting the product of group elements, the dot will be omitted.

To a Lie group G with composition rule $\gamma : (a, b) \mapsto c = ab$, we associate the tangent group TG, having the composition law $T\gamma$:

$$\boxed{c = ab \quad \Rightarrow \quad c_1 = a_1 b + a b_1.} \tag{3}$$

The vectors $a_1 \in T_a G$ and $b_1 \in T_b G$ are transported by means of the *right shift* $r_b \doteq \gamma_b$ and of the *left shift* $l_a \doteq \gamma_a$, more exactly, by means of the tangent mappings Tr_b and Tl_a, from the points a and b to the point c, where the sum $a_1 b + a b_1 = Tr_b(a_1) + Tl_a(b_1)$ determines the vector $c_1 \in T_c G$. This is *the composition law on the tangent group* TG. The unity of the group TG is the null vector from $T_e G$. The inversion for the elements of TG is defined by the rule:

$$a_1 \in T_a G \quad \rightsquigarrow \quad a_1^{-1} = -a^{-1} a_1 a^{-1} \in T_{a^{-1}} G. \tag{4}$$

Exercise 4: unity and inverse elements. Using (3), confirm the assertion regarding the unity of the group TG and the inversion of the elements (4). The formula (4) is obtained by solving the equation $a_1 b + a b_1 = 0$ relative to b_1 for $b = a^{-1}$.

Exercise 5: matrix representation. Prove that the formulas (3) can be represented in matrix form as

$$\begin{pmatrix} a & 0 \\ a_1 & a \end{pmatrix} \cdot \begin{pmatrix} b & 0 \\ b_1 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} e & 0 \\ a^{-1} a_1 + b_1 b^{-1} & e \end{pmatrix} \cdot \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}. \tag{5}$$

The sum of vectors in $T_{ab}G$ reduces to the sum in TG_e :

$$a_1 b + a b_1 = T(l_a \circ r_b)(a^{-1}a_1 + b_1 b^{-1}).$$

Explain the meaning of the equality $a_1 b + a b_1 = (a_1 a^{-1})\, c + c\,(b^{-1} b_1)$.

Exercise 6: second tangent group. Prove that in the second tangent group T^2G the product of elements is defined by the formulas

$$\begin{aligned} c = ab\,, \quad c_1 &= a_1 b + a b_1\,, \\ c_2 &= a_2 b + a b_2\,, \\ c_{12} &= a_{12} b + a_2 b_1 + a_1 b_2 + a b_{12}\,, \end{aligned} \tag{6}$$

and the inversion is performed by the rule

$$(a, a_1, a_2, a_{12})^{-1} \doteq (a^{-1}, a_1^{-1}, a_2^{-1}, a_{12}^{-1}), \quad \text{where}$$

$$\begin{aligned} a_1^{-1} &= -a^{-1} a_1 a^{-1}\,, \\ a_2^{-1} &= -a^{-1} a_2 a^{-1}\,, \\ a_{12}^{-1} &= -a^{-1} a_{12} a^{-1} + a^{-1} a_2 a^{-1} a_1 a^{-1} + a^{-1} a_1 a^{-1} a_2 a^{-1}\,. \end{aligned} \tag{7}$$

Exercise 7: classical formulas. Reduce the formulas (6) and (7) to the well known formulas from Analysis:

$$(uv)' = u'v + uv', \quad (uv)'' = u''v + 2u'v' + uv'',$$
$$\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}, \quad \left(\frac{1}{u}\right)'' = \frac{-uu'' + 2(u')^2}{u^3}.$$

Exercise 8: matrix relations. Using (6) and (7) prove the matrix relations:

$$\begin{pmatrix} c & 0 & 0 & 0 \\ c_1 & c & 0 & 0 \\ c_2 & 0 & c & 0 \\ c_{12} & c_2 & c_1 & c \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 \\ a_1 & a & 0 & 0 \\ a_2 & 0 & a & 0 \\ a_{12} & a_2 & a_1 & a \end{pmatrix} \cdot \begin{pmatrix} b & 0 & 0 & 0 \\ b_1 & b & 0 & 0 \\ b_2 & 0 & b & 0 \\ b_{12} & b_2 & b_1 & b \end{pmatrix},$$

$$\begin{pmatrix} a & 0 & 0 & 0 \\ a_1 & a & 0 & 0 \\ a_2 & 0 & a & 0 \\ a_{12} & a_2 & a_1 & a \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & 0 & 0 & 0 \\ a_1^{-1} & a^{-1} & 0 & 0 \\ a_2^{-1} & 0 & a^{-1} & 0 \\ a_{12}^{-1} & a_2^{-1} & a_1^{-1} & a^{-1} \end{pmatrix}.$$

Which endomorphism is involved here?

Exercise 9: logarithmic derivatives. Following the example of (5), represent the product of elements of the group T^2G in the form:

$$\begin{pmatrix} a & 0 & 0 & 0 \\ a_1 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & a_1 & a \end{pmatrix} \cdot \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ a^{-1}a_2 + b_2 b^{-1} & 0 & e & 0 \\ (a^{-1}a_2 + b_2 b^{-1})_1 & a^{-1}a_2 + b_2 b^{-1} & 0 & e \end{pmatrix} \cdot \begin{pmatrix} b & 0 & 0 & 0 \\ b_1 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & b_1 & b \end{pmatrix},$$

where $(a^{-1}a_2 + b_2 b^{-1})_1 = a^{-1}a_{12} - a^{-1}a_1 a^{-1} a_2 + b_{12} b^{-1} - b_2 b^{-1} b_1 b^{-1}$. In this generalization, both logarithmic derivatives $\big(\ln(uv)\big)'$ and $\big(\ln(uv)\big)''$, are present.

At the unity $e \in G$ we fix the tangent vector $e_1 \in T_eG$. This vector is displaced by left shifts l_a over the group G to produce the left-invariant vector field ae_1 and by right shifts r_a, to produce the right-invariant vector field $e_1 a$. If at the unit $e \in G$ we provide a frame, i.e., a basis of the space T_eG, then, in this way, two frame fields are defined on G (one right-invariant and left-invariant). The transition from one frame to another, at the point $a \in G$, is defined by some matrix $A(a)$, which is an element of the linear group $GL = GL(\dim G, \mathbb{R})$. By this way, we define a homomorphism of the group G into the linear group GL:

$$G \to GL: \; a \mapsto A(a). \tag{8}$$

Exercise 10: right/left shifts and inner automorphisms. Show that an 1-parametric subgroup a_t of the group G defines in the group G three flows corresponding to right shifts, left shifts and inner automorphisms

$$r_{a_t} = \exp tX, \quad l_{a_t} = \exp t\widetilde{X}, \quad A_{a_t} = l_{a_t} \circ r_{a_t}^{-1} = \exp t(\widetilde{X} - X)$$

and, accordingly, the left-invariant operator X, the right-invariant operator $\widetilde{X}$ and the adjoint representation operator $Y = \widetilde{X} - X$. Prove this, using the formulas

$$Xf = (f \circ r_{a_t})'_{t=0}, \quad \widetilde{X}f = (f \circ l_{a_t})'_{t=0}, \quad Yf = (f \circ A_{a_t})'_{t=0},$$

where f is an arbitrary smooth function on G, taking into consideration that left shifts commute with right shifts.

1.3. Elements of representation theory. We consider a differentiable manifold, which we shall call *representation space* for the group G, or, in the following, simply *space.* A smooth mapping

$$\lambda \ : \ M \times G \ \longrightarrow \ M \ : \ (u,a) \ \mapsto \ v = u \cdot a$$

defines an *action* of the group G on the space M, if all the mappings

$$\lambda_a \ : \ M \to M \ \ u \mapsto u \cdot a \ , \quad \forall a \in G \ ,$$

are transformations (diffeomorphisms) of the space M, and the mapping $a \mapsto \lambda_a$ is a homomorphism of the group G into the group of transformations of the space M. The homomorphism $a \mapsto \lambda_a$ is understood either in the sense of the equality $\lambda_{ab} = \lambda_a \circ \lambda_b$ or in the sense of the equality $\lambda_{ab} = \lambda_b \circ \lambda_a$. In the first case we say that the action of the group G on the space M is *left-sided*, and in the second case it is *right-sided.* By writing $v = a \cdot u$ we have in view a left action, while by $v = u \cdot a$ – a right action:

$$\begin{aligned} v = a \cdot u \quad &\rightsquigarrow \quad (ab) \cdot u = a \cdot (b \cdot u), \\ v = u \cdot a \quad &\rightsquigarrow \quad u \cdot (ab) = (u \cdot a) \cdot b. \end{aligned}$$

The next formulas correspond to the right-sided action of the group G on the space M.

The kernel of the homomorphism $a \mapsto \lambda_a$ is called the *stabilizer subgroup* of the group G. In the case of an effective action, the non-effectiveness kernel is trivial; it consists of the unity $e \in G$, and the mapping $a \mapsto \lambda_a$ is injective.

For a fixed point $u \in M$, the mapping

$$\lambda_u \ : \ G \to M \ : a \mapsto u \cdot a$$

defines in the space M the *orbit* of this point. The whole space M is fibered into orbits $\lambda_u(G)$. When $\lambda_u(G) = M$, i.e., when the space M is the only orbit of the group G, we say that the action of the Lie group G on the space M is *transitive.* In such a case, M is called *homogeneous group space.* If moreover, the dimensions of G and M are equal, then the action of G on M is *simply transitive* and such an action defines an *exact representation* of the group G.

Equation 11: action by right/left shifts and inner automorphisms. Show that the actions of the group G on itself, provided by left and right shifts, are simply transitive actions. Prove that the actions provided by inner automorphisms are non-transitive.

The tangent map of the mapping λ, i.e., $T\lambda$, defines a representation of the tangent group TG on the first level TM,

$$T\lambda \ : \ TM \times TG \ \to \ TM \ : \ (u_1, a_1) \ \mapsto \ v_1 = u_1 \cdot a + u \cdot a_1 \, .$$

Formula (1) looks similarly:

$$\boxed{v = u \cdot a \quad \rightsquigarrow \quad v_1 = u_1 \cdot a + u \cdot a_1.} \tag{9}$$

We remark two particular cases:

for $a_1 = 0$ we define the action of the group G on the level TM,

$$a_1 = 0 \quad \Rightarrow \quad u_1 \ \mapsto \ v_1 = u_1 \cdot a;$$

for $u_1 = 0$ we define the action of the tangent group TG on the space M,

$$u_1 = 0 \quad \Rightarrow \quad e_1 = a^{-1}a_1 \ \mapsto \ v_1 = u \cdot a_1 = v \cdot a^{-1}a_1 = v \cdot e_1.$$

The formula

$$\boxed{v_1 = v \cdot a^{-1}a_1} \tag{10}$$

is *the fundamental formula* of the theory of Lie group representations.

In fact, to the vector $e_1 = a^{-1}a_1 \in T_eG$ (which is an element of the Lie algebra g) at the point $v \in M$ we associate some vector $v_1 \in T_vM$, and since v is an arbitrary point, it defines a vector field on the space M. This vector field is called the operator of the group G, or simply *group operator*[1]. Depending on the choice of the vector $e_1 \in T_eG$, in the space M we have an infinite set of group operators, and all of them, as vector fields, are tangent to the corresponding orbits.

For $v_1 = 0$ the equality (10) provides the equation $v \cdot a^{-1}a_1 = 0$, or $v \cdot e_1 = 0$, which determines in the space T_eG those directions e_1, along which the point $v \in M$ remains fixed. We define on the group G a Pfaff system, and its integral surface (solution), which contains the point $e \in G$, is a subgroup $H_v \subset G$ called *the stationary subgroup* or *the stabilizer* of the point v.

In coordinates (v^α) on the neighborhood $U \subset M$ of the point $v \in M$, Eq. (10) is written as a system $dv^\alpha = \xi^\alpha_i \omega^i$, where ω^i are the forms of the left-invariant coframe on the group G. There appears a matrix[2] $\xi = (\xi^\alpha_i)$, which determines a system of forms ϑ^α on the group G, and in the space M, a system of basic operators X_i:

$$\vartheta^\alpha = \xi^\alpha_i \omega^i, \quad X_i = \xi^\alpha_i \frac{\partial}{\partial v^\alpha}.$$

The number of operators X_i is equal to the dimension of G, and the number of forms ϑ^α is the dimension of M. Operators X_i and forms ϑ^α are not necessarily linearly independent. The Pfaff system $\xi^\alpha_i \omega^i = 0$ for a fixed point $v \in M$ is completely integrable and defines the stabilizer $H_v \subset G$.

Exercise 12: vision from the classical theory. Show that the system $\xi^\alpha_i \omega^i = 0$ is the coordinate form of the equation $v \cdot a^{-1}a_1 = 0$.

1.4. Adjoint representation. In the groups G and TG we define the action by left shifts:

$$\begin{aligned} l_a &: b \mapsto c = ab, \\ Tl_{a_1} &: b_1 \mapsto c_1 = (a_1a^{-1})\,c + ab_1, \\ &\boxed{c_1 = (a_1a^{-1})\,c}\,, \end{aligned} \tag{11}$$

[1] Since the time of S. Lie and frequently nowadays, group operators have been called *infinitesimal transformations* or *fundamental vector fields* of the group.

[2] The matrix ξ plays an essential role in the theory of Lie group representations (see, e.g., S. Lie Theorems).

right shifts:

$$
\begin{aligned}
r_a &: b \mapsto c = b\,a, \\
Tr_{a_1} &: b_1 \mapsto c_1 = b_1\,a + c\,(a^{-1}a_1), \\
&\boxed{c_1 = c\,(a^{-1}a_1)}\,,
\end{aligned}
\tag{12}
$$

and inner automorphisms:

$$
\begin{aligned}
A_a &: b \;\mapsto\; c = aba^{-1}, \\
TA_{a_1} &: b_1 \;\mapsto\; c_1 = (a_1a^{-1})\,c - c\,(a_1a^{-1}) + a\,b_1\,a^{-1}, \\
&\boxed{c_1 = (a_1a^{-1})\,c - c\,(a_1a^{-1})}\,.
\end{aligned}
\tag{13}
$$

The basic formula (10) is rewritten, for $b_1 = 0$, in the forms (11), (12) and (13), respectively.

Inner automorphisms are directly related to higher order movements.

Hence, when in the spaces A and B there take place the transformations a and b, the mapping $\varphi : A \to B$ is brought into the mapping $\widetilde{\varphi} : A \to B$. This is shown by the diagram:

$$
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
A & \xrightarrow{\widetilde{\varphi}} & B
\end{array}
\qquad
\boxed{\varphi \rightsquigarrow \widetilde{\varphi} = b\varphi a^{-1}}
$$

If we set here $A = B,\ a = b$ and if φ is a diffeomorphism, i.e., φ is a transformation of the space A, then this diagram describes the transformation of the mapping φ, subject to the influence of the transformation a:

$$
\begin{array}{ccc}
A & \xrightarrow{\varphi} & A \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\
A & \xrightarrow{\widetilde{\varphi}} & A
\end{array}
\qquad
\boxed{\varphi \rightsquigarrow \widetilde{\varphi} = a\varphi a^{-1}}
\tag{14}
$$

The transformation φ is subject to the inner automorphism.

Exercise 13: higher order transformations. The transformation of order 2 $\varphi \rightsquigarrow \widetilde{\varphi}$ is described by the 2-dimensional diagram (1.14). Show that the transformation of order 3, i.e., a transformation of transformation $\varphi \rightsquigarrow \widetilde{\varphi}$, is described by a 3-dimensional diagram and the transformation of order k is described by a corresponding k-dimensional diagram.

If the arrow a in diagram (14) is assumed to represent the 1-parametric group a_t of transformations of the space A, or in brief, the *flow* a_t, then we see how, to a change of the parameter t (of *time*), it corresponds to a change of the mapping $\varphi_t = a_t\varphi a_t^{-1}$. We can talk then about a 1-parametric family of mappings φ_t in the field a_t.

If the arrow φ in diagram (14) is regarded as a 1-parametric group of transformations b_τ of the space A (the flow b_τ), then we can see how this flow changes under the transformation a, i.e., $b_\tau \rightsquigarrow \widetilde{b}_\tau = ab_\tau a^{-1}$.

Exercise 14: transformation of the flow. Show that if b_τ is the flow of the vector field Y and $\widetilde{b}_\tau$ is the flow of the field $\widetilde{Y}$, then $\widetilde{Y} = TaY$, and the tangent mapping Ta acts on the field Y:

$$
b_\tau = \exp\tau Y \quad \rightsquigarrow \quad \widetilde{b}_\tau = ab_\tau a^{-1} = \exp\tau\widetilde{Y}, \qquad Y \rightsquigarrow \widetilde{Y} = TaY.
$$

Exercise 15: interaction of vector fields. Let X and Y be two vector fields. The flows of these fields $a_t = \exp tX$ and $b_\tau = \exp\tau Y$ interact according to the scheme:

$$
\begin{aligned}
b_\tau &\rightsquigarrow a_t b_\tau a_{-t}, \qquad & Y &\rightsquigarrow Ta_tY, \\
a_t &\rightsquigarrow b_\tau a_t b_{-\tau}, \qquad & X &\rightsquigarrow Tb_\tau X.
\end{aligned}
$$

Using derivatives of the function f

$$Xf = (f \circ a_t)'_{t=0} \quad \textit{and} \quad Yf = (f \circ b_\tau)'_{\tau=0},$$

perform the differentiation (the parameters t or τ from above the arrow mean differentiation relative to t for $t \to 0$ or to τ, for $\tau \to 0$):

$$f \circ (a_t b_\tau a_t^{-1}) \xrightarrow{t} (Xf) \circ b_\tau - X(f \circ b_\tau) \xrightarrow{\tau} (YX - XY)f,$$
$$f \circ (b_\tau a_t b_\tau^{-1}) \xrightarrow{\tau} (Yf) \circ a_t - Y(f \circ a_t) \xrightarrow{t} (XY - YX)f.$$

Check the validity of the relation $(Ta_tY)'_{t=0} = -(Tb_\tau X)'_{\tau=0}$ and establish a connection with the brackets $[X,Y] = XY - YX$.

If in one flow the points move along trajectories and under the influence of the other flow, this movement is transformed, and then the movement of the movement takes place, or a second-order movement. Under the influence of a third flow, the movement of second order changes its shape, then the movement of third order occurs, etc. In the infinitesimal approach this reduces to the iterations

$$a_t \rightsquigarrow Ta_t \rightsquigarrow T^2a_t \rightsquigarrow \dots$$

and to the corresponding vector fields on the levels

$$X \rightsquigarrow \overset{(1)}{X} \rightsquigarrow \overset{(2)}{X} \rightsquigarrow \dots, \qquad T^k a_t = \exp t\overset{(k)}{X}, \quad k = 0,1,2,\dots \tag{15}$$

In this way, the flow $T^k a_t$ induces *a movement of order* k.

1.5. Gauge groups. Let us fix on each level a point

$$u_{(k)} \in T^k M, \quad \text{such that} \quad \pi_k(u_{(k)}) = u_{(k-1)}, \;\; k = 0,1,2,\dots$$

In the neighborhood $T^kU \subset T^kM$ these points are defined by their coordinates:

$$\begin{aligned}
U &: u_{(0)} \rightsquigarrow (u^i), \\
TU &: u_{(1)} \rightsquigarrow (u^i, u^i_1), \\
T^2U &: u_{(2)} \rightsquigarrow (u^i, u^i_1, u^i_2, u^i_{12}), \\
T^3U &: u_{(3)} \rightsquigarrow (u^i, u^i_1, u^i_2, u^i_{12}, u^i_3, u^i_{13}, u^i_{23}, u^i_{123}), \\
&\quad \dots \qquad \dots
\end{aligned}$$

Transformation of coordinates in the neighborhood $U \subset M$

$$u^i \rightsquigarrow \widetilde{u}^i \circ a = a^i$$

induces a change of coordinates in each neighborhood T^kU:

$$(u^i,\, u^i_1,\, u^i_2,\, u^i_{12},\, \dots) \rightsquigarrow (\widetilde{u}^i,\, \widetilde{u}^i_1,\, \widetilde{u}^i_2,\, \widetilde{u}^i_{12},\, \dots) = (a^i,\, a^i_1,\, a^i_2,\, a^i_{12},\, \dots).$$

Namely, if these transformations of coordinates in the neighborhood TU are defined by the system

$$\begin{cases} \widetilde{u}^i = a^i, \\ \widetilde{u}^i_1 = a^i_1 \doteq a^i_j u^j_1, \end{cases}$$

with the Jacobian block-matrix

$$\begin{pmatrix} a^i_j & 0 \\ (a^i_j)_1 & a^i_j \end{pmatrix}, \quad \text{where} \quad a^i_j = \frac{\partial a^i}{\partial u^j}, \quad a^i_{jk} = \frac{\partial^2 a^i}{\partial u^j \partial u^k}, \quad (a^i_j)_1 \doteq a^i_{jk} u^k_1,$$

then the transformation of coordinates in the neighborhood T^2U are defined by the system

$$\begin{cases} \widetilde{u}^i = a^i, \\ \widetilde{u}^i_1 = a^i_1 \doteq a^i_j u^j_1, \\ \widetilde{u}^i_2 = a^i_2 \doteq a^i_j u^j_2, \\ \widetilde{u}^i_{12} = a^i_{12} \doteq a^i_{jk} u^j_1 u^k_2 + a^i_j u^j_{12}, \end{cases}$$

with the Jacobian block-matrix

$$\begin{pmatrix} a^i_j & 0 & 0 & 0 \\ (a^i_j)_1 & a^i_j & 0 & 0 \\ (a^i_j)_2 & 0 & a^i_j & 0 \\ (a^i_j)_{12} & (a^i_j)_2 & (a^i_j)_1 & a^i_j \end{pmatrix}, \quad \text{where} \quad \begin{cases} (a^i_j)_1 \doteq a^i_{jk} u^k_1, \\ (a^i_j)_2 \doteq a^i_{jk} u^k_2, \\ (a^i_j)_{12} \doteq a^i_{jkl} u^k_1 u^l_2 + a^i_{jk} u^k_{12}, \end{cases}$$

etc.

When performing a lift from one level to another, $U \rightsquigarrow TU \rightsquigarrow T^2U \rightsquigarrow \dots$, the Jacobian matrix is inductively built according to the scheme:

$$\mathfrak{a} \rightsquigarrow \begin{pmatrix} \mathfrak{a} & 0 \\ \mathfrak{a}_1 & \mathfrak{a} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \mathfrak{a} & 0 & 0 & 0 \\ \mathfrak{a}_1 & \mathfrak{a} & 0 & 0 \\ \mathfrak{a}_2 & 0 & \mathfrak{a} & 0 \\ \mathfrak{a}_{12} & \mathfrak{a}_2 & \mathfrak{a}_1 & \mathfrak{a} \end{pmatrix} \rightsquigarrow \dots, \tag{16}$$

with repeated, as shown above, n-dimensional blocks

$$\mathfrak{a} = (a^i_j), \;\; \mathfrak{a}_1 = (a^i_j)_1, \;\; \mathfrak{a}_2 = (a^i_j)_2, \;\; \mathfrak{a}_{12} = (a^i_j)_{12}, \;\; \dots$$

Therefore, there follows the general rule: *the Jacobian matrix of the transformation of coordinates on the neighborhood T^kU is of the form*

$$\begin{pmatrix} \mathcal{A} & 0 \\ \mathcal{A}_k & \mathcal{A} \end{pmatrix}, \tag{17}$$

where the block $\mathcal{A}$ is the Jacobian matrix on $T^{k-1}U$ and $\mathcal{A}_k \doteq d_k\mathcal{A}$, $k = 1, 2, \dots$

In other words, the Jacobian matrix on the neighborhood T^kU consists of four blocks, where the Jacobian matrix $\mathcal{A}$ of the neighborhood $T^{k-1}U$ is repeated on the diagonal, the upper-right block is zero, and the left-lower block is the differential of the block $\mathcal{A}$ taking into consideration the k-th level, i.e., $\mathcal{A}_k \doteq d_k\mathcal{A}$.

Formula (17) defines the sequence of matrices (16).

Exercise 16: inversion rule. Show that the inversion of the matrix (17) takes place according to the scheme:

$$\begin{pmatrix} \mathcal{A} & 0 \\ \mathcal{A}_k & \mathcal{A} \end{pmatrix} = \begin{pmatrix} \mathcal{E} & 0 \\ \mathcal{A}_k\mathcal{A}^{-1} & \mathcal{E} \end{pmatrix} \cdot \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{A} \end{pmatrix} \rightsquigarrow$$

$$\rightsquigarrow \begin{pmatrix} \mathcal{A} & 0 \\ \mathcal{A}_k & \mathcal{A} \end{pmatrix}^{-1} = \begin{pmatrix} \mathcal{A}^{-1} & 0 \\ 0 & \mathcal{A}^{-1} \end{pmatrix} \cdot \begin{pmatrix} \mathcal{E} & 0 \\ -\mathcal{A}_k\mathcal{A}^{-1} & \mathcal{E} \end{pmatrix},$$

where $\mathcal{E}$ is the identity block. See (4) and

$$a_1 = (a_1 a^{-1})a \;\rightsquigarrow\; a_1^{-1} = -a^{-1}(a_1 a^{-1}).$$

The matrix (17) depends on the point $u_{(k)} \in T^kU$. If this point is fixed, then a numeric matrix is defined, but still having the freedom to choose the function a^i (or the corresponding jet of the transformation).

Exercise 17: gauge group. Show that all matrices of the form (17), with the point $u_{(k)} \in T^kU$ fixed, determine a subgroup of the linear group of order $2^k n$,

$$\mathcal{G}_k \subset GL(2^k n, \mathbb{R}).$$

Prove the existence of the groups $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and extend to $\mathcal{G}_k$.

We call the group $\mathcal{G}_k$ of matrices (17) with fixed point $u_{(k)} \in T^kU$ *the gauge group* of order k on the manifold M. By setting $k = 0, 1, 2, 3, \ldots$, we obtain an infinite sequence of gauge groups

$$\mathcal{G} \rightsquigarrow \mathcal{G}_1 \rightsquigarrow \mathcal{G}_2 \rightsquigarrow \mathcal{G}_3 \rightsquigarrow \ldots \tag{18}$$

Theorem 2. *The gauge group of order k is isomorphic to the k-th tangent group of the linear group $GL(n, \mathbb{R})$, which, in its turn, is embedded in the linear group $GL(2^k n, \mathbb{R})$:*

$$\mathcal{G}_k \approx T^k\big(GL(n, \mathbb{R})\big) \subset GL(2^k n, \mathbb{R}), \quad k = 0, 1, 2, \ldots \tag{19}$$

In this case

$$\dim GL(2^k n, \mathbb{R}) = (2^k n)^2 \quad \textit{and} \quad \dim \mathcal{G}_k = 2^k n^2.$$

Proof. We fix the element $u_{(k)} \in T^kM$ of the k-th level. Matrices (17) generate a subgroup $\mathcal{G}_k$ of the linear group $GL(2^k n, \mathbb{R})$ (see Exercise 17). The fixing of the point $u_{(k)}$ does not limit the freedom of choice for the element (17) in the group $\mathcal{G}_k$. Hence, the group $\mathcal{G}_k$ is uniquely defined regardless of the point $u_{(k)} \in T^kM$. On the other side, the tangent group $T^k\big(GL(n, \mathbb{R})\big)$ coincides up to an isomorphism, with the matrix group (17), or $\mathcal{G}_k$. This follows from the formulas (3)–(7) and Exercises 12–17, if we assume $G = GL(n, \mathbb{R})$. □

Further, in the matrix (17), besides the point $u_{(k)} \in T^kU$, there exists the k-jet of coordinate transformations $(a^j_i, a^j_{i_1 i_2}, \ldots, a^j_{i_1 i_2 \ldots i_k})$. We shall denote as J_k the group of such jets at the point u (see Exercise 5). The homomorphism is defined:

$$\chi_k : J_k \to \mathcal{G}_k. \tag{20}$$

Exercise 18: jets and gauge group. Show that for $k = 2$, the mapping χ_2 is homomorphic, i.e., to a composition of 2-jets (a^i_k, a^j_{kl}) and (b^k_j, a^k_{jl}) there corresponds the product of matrices $\mathcal{A}_2$,

$$\begin{pmatrix} a^i_k & 0 \\ a^i_{kl}u^l_1 & a^i_k \end{pmatrix} \cdot \begin{pmatrix} b^k_j & 0 \\ b^k_{jl}u^l_1 & b^k_j \end{pmatrix} = \begin{pmatrix} a^i_k b^k_j & 0 \\ (a^i_k b^k_j)_l u^l_1 & a^i_k b^k_j \end{pmatrix},$$

and to the inverse 2-jet $(a^i_j, a^i_{jl})^{-1} \doteq (\bar{a}^i_j, -\bar{a}^i_s a^s_{kl} \bar{a}^k_j)$, there corresponds the inverse matrix $\mathcal{A}_2^{-1}$

$$\begin{pmatrix} a^i_k & 0 \\ a^i_{kl}u^l_1 & a^i_k \end{pmatrix}^{-1} = \begin{pmatrix} \bar{a}^i_j & 0 \\ -\bar{a}^i_s a^s_{kl} \bar{a}^k_j u^l_1 & \bar{a}^i_j \end{pmatrix}.$$

Generalize this to the general case k.

Exercise 19: homogeneity of tangent space. Show that the kernel of the homomorphism χ_k is the stabilizer $H_{u_{(k)}}$ of the element $u_{(k)} \in T^kM$ in the group J_k. The tangent space $T^k_{u_{(k)}}M$ is identified with the homogeneous space $J_k/H_{u_{(k)}}$.

Let us consider once again the gauge groups of the sequence (18). The first group $\mathcal{G}$ is the linear group $GL(n, \mathbb{R})$,

$$\mathcal{G} = GL(n, \mathbb{R}).$$

The second group $\mathcal{G}_1$ is isomorphic to the tangent group $T\big(GL(n,\mathbb{R})\big)$. Its elements are block matrices of the form

$$\begin{pmatrix} \mathfrak{a} & 0 \\ \mathfrak{a}_1 & \mathfrak{a} \end{pmatrix}, \quad \text{where} \quad \mathfrak{a} \in GL(n,\mathbb{R}) \quad \text{and} \quad \mathfrak{a}_1 \in gl(n,\mathbb{R}).$$

The correspondence $\mathcal{G}_1 \leftrightsquigarrow T\big(GL(n,\mathbb{R})\big)$ is one-to-one. The product of elements in the group $\mathcal{G}_1$,

$$\begin{pmatrix} \mathfrak{a} & 0 \\ \mathfrak{a}_1 & \mathfrak{a} \end{pmatrix} \cdot \begin{pmatrix} \mathfrak{b} & 0 \\ \mathfrak{b}_1 & \mathfrak{b} \end{pmatrix} = \begin{pmatrix} \mathfrak{ab} & 0 \\ (\mathfrak{ab})_1 & \mathfrak{ab} \end{pmatrix},$$

reduces to the Leibniz rule in the tangent group $T\big(GL(n,\mathbb{R})\big)$,

$$(\mathfrak{ab})_1 = \mathfrak{a}_1\mathfrak{b} + \mathfrak{a}\mathfrak{b}_1,$$

and the inversion of elements in $\mathcal{G}_1$,

$$\begin{pmatrix} \mathfrak{a} & 0 \\ \mathfrak{a}_1 & \mathfrak{a} \end{pmatrix}^{-1} = \begin{pmatrix} \mathfrak{a}^{-1} & 0 \\ -\mathfrak{a}^{-1}\mathfrak{a}_1\mathfrak{a}^{-1} & \mathfrak{a}^{-1} \end{pmatrix},$$

reduces to the rule

$$\mathfrak{a}_1^{-1} = -\mathfrak{a}^{-1}\mathfrak{a}_1\mathfrak{a}^{-1}.$$

This speaks about an isomorphism between the groups $\mathcal{G}_1$ and $T\big(GL(n,\mathbb{R})\big)$. An inner authomorphism in $\mathcal{G}_1$ is generated as allows:

$$\begin{pmatrix} \mathfrak{a} & 0 \\ \mathfrak{a}_1 & \mathfrak{a} \end{pmatrix} \cdot \begin{pmatrix} \mathfrak{b} & 0 \\ \mathfrak{b}_1 & \mathfrak{b} \end{pmatrix} \cdot \begin{pmatrix} \mathfrak{a} & 0 \\ \mathfrak{a}_1 & \mathfrak{a} \end{pmatrix}^{-1} = \begin{pmatrix} \mathfrak{aba}^{-1} & 0 \\ (\mathfrak{aba}^{-1})_1 & \mathfrak{aba}^{-1} \end{pmatrix},$$

with the block $(\mathfrak{aba}^{-1})_1 = \mathfrak{ab}_1\mathfrak{a}^{-1} + \mathfrak{a}_1\mathfrak{a}^{-1}(\mathfrak{aba}^{-1}) - (\mathfrak{aba}^{-1})\mathfrak{a}_1\mathfrak{a}^{-1}$, etc.

The following group $\mathcal{G}_2$ is isomorphic to the tangent group $T^2\big(GL(n,\mathbb{R})\big)$. The stair-like structure appears again:

$$\begin{pmatrix} \mathfrak{a} & 0 & 0 & 0 \\ \mathfrak{a}_1 & \mathfrak{a} & 0 & 0 \\ \mathfrak{a}_2 & 0 & \mathfrak{a} & 0 \\ \mathfrak{a}_{12} & \mathfrak{a}_2 & \mathfrak{a}_1 & \mathfrak{a} \end{pmatrix} \cdot \begin{pmatrix} \mathfrak{b} & 0 & 0 & 0 \\ \mathfrak{b}_1 & \mathfrak{b} & 0 & 0 \\ \mathfrak{b}_2 & 0 & \mathfrak{b} & 0 \\ \mathfrak{b}_{12} & \mathfrak{b}_2 & \mathfrak{b}_1 & \mathfrak{b} \end{pmatrix} = \begin{pmatrix} \mathfrak{ab} & 0 & 0 & 0 \\ (\mathfrak{ab})_1 & \mathfrak{ab} & 0 & 0 \\ (\mathfrak{ab})_2 & 0 & \mathfrak{ab} & 0 \\ (\mathfrak{ab})_{12} & (\mathfrak{ab})_2 & (\mathfrak{ab})_1 & \mathfrak{ab} \end{pmatrix},$$

where

$$\begin{aligned} (\mathfrak{ab})_1 &= \mathfrak{a}_1\mathfrak{b} + \mathfrak{a}\mathfrak{b}_1, \\ (\mathfrak{ab})_2 &= \mathfrak{a}_2\mathfrak{b} + \mathfrak{a}\mathfrak{b}_2, \\ (\mathfrak{ab})_{12} &= \mathfrak{a}_{12}\mathfrak{b} + \mathfrak{a}_2\mathfrak{b}_1 + \mathfrak{a}_1\mathfrak{b}_2 + \mathfrak{a}\mathfrak{b}_{12}. \end{aligned}$$

Exercise 20: logarithmic rule for gauge group. Show that while forming the blocks

$$\mathfrak{a} \rightsquigarrow \mathfrak{a}^{-1}\mathfrak{a}_1 \rightsquigarrow (\mathfrak{a}^{-1}\mathfrak{a}_1)_2 = \mathfrak{a}^{-1}\mathfrak{a}_{12} - \mathfrak{a}^{-1}\mathfrak{a}_2\mathfrak{a}^{-1}\mathfrak{a}_1 \rightsquigarrow \dots$$

there appears the following property of the logarithmic function:

$$\ln u \rightsquigarrow \frac{u'}{u} \rightsquigarrow \frac{u''}{u} - \frac{(u')^2}{u^2} \rightsquigarrow \dots$$

We shall further denote the Lie algebra of the group $\mathcal{G}_k$ by $\overline{\mathcal{G}}_k$.

The general scheme is the following. An element of the group $\mathcal{G}_k$ is generated according to the principle:

$$\begin{pmatrix} \mathcal{A} & 0 \\ \mathcal{A}_k & \mathcal{A} \end{pmatrix}, \quad \text{where} \quad \mathcal{A} \in \mathcal{G}_{k-1},\ \mathcal{A}_k \in \overline{\mathcal{G}}_{k-1}.$$

The product and the inversion of elements,

$$\begin{pmatrix} \mathcal{A} & 0 \\ \mathcal{A}_k & \mathcal{A} \end{pmatrix} \cdot \begin{pmatrix} \mathcal{B} & 0 \\ \mathcal{B}_k & \mathcal{B} \end{pmatrix} = \begin{pmatrix} \mathcal{AB} & 0 \\ (\mathcal{AB})_k & \mathcal{AB} \end{pmatrix},$$

$$\begin{pmatrix} \mathcal{A} & 0 \\ \mathcal{A}_k & \mathcal{A} \end{pmatrix}^{-1} = \begin{pmatrix} \mathcal{A}^{-1} & 0 \\ \mathcal{A}_k^{-1} & \mathcal{A}^{-1} \end{pmatrix},$$

reduce to the rules:

$$(\mathcal{AB})_k = \mathcal{A}_k\mathcal{B} + \mathcal{A}\mathcal{B}_k, \quad \mathcal{A}_k^{-1} = -\mathcal{A}^{-1}\mathcal{A}_k\mathcal{A}^{-1}.$$

The Lie algebra $\overline{\mathcal{G}}_{k-1}$ is identified with the additive subgroup of the matrix group $\mathcal{G}_k$, whose matrices have the form:

$$\begin{pmatrix} \mathcal{E} & 0 \\ \mathcal{A}_k & \mathcal{E} \end{pmatrix}, \tag{21}$$

where $\mathcal{E}$ is the unit block, i.e., the unity of the group $\mathcal{G}_{k-1}$. The product and the inversion of such matrices are performed in the following way:

$$\begin{pmatrix} \mathcal{E} & 0 \\ \mathcal{A}_k & \mathcal{E} \end{pmatrix} \cdot \begin{pmatrix} \mathcal{E} & 0 \\ \mathcal{B}_k & \mathcal{E} \end{pmatrix} = \begin{pmatrix} \mathcal{E} & 0 \\ \mathcal{A}_k + \mathcal{B}_k & \mathcal{E} \end{pmatrix},$$

$$\begin{pmatrix} \mathcal{E} & 0 \\ \mathcal{A}_k & \mathcal{E} \end{pmatrix}^{-1} = \begin{pmatrix} \mathcal{E} & 0 \\ -\mathcal{A}_k & \mathcal{E} \end{pmatrix}.$$

All these matrices generate within the group $\mathcal{G}_k$ a normal divisor,

$$\begin{pmatrix} \mathcal{A} & 0 \\ \mathcal{A}_k & \mathcal{A} \end{pmatrix} \cdot \begin{pmatrix} \mathcal{E} & 0 \\ \mathcal{B}_k & \mathcal{E} \end{pmatrix} \cdot \begin{pmatrix} \mathcal{A} & 0 \\ \mathcal{A}_k & \mathcal{A} \end{pmatrix}^{-1} = \begin{pmatrix} \mathcal{E} & 0 \\ \mathcal{A}\mathcal{B}_k\mathcal{A}^{-1} & \mathcal{E} \end{pmatrix}.$$

An inner automorphism of the group $\mathcal{G}_k$ leads to the transformation of the block

$$\mathcal{B}_k \quad \rightsquigarrow \quad \widetilde{\mathcal{B}}_k = \quad \mathcal{A}\mathcal{B}_k\mathcal{A}^{-1}.$$

Under such a transformation, the spectrum of the matrix $\mathcal{B}_k$ is preserved. The invariants will be the eigenvalues of this matrix and the corresponding symmetric polynomials, which are coefficients in the Hamilton – Cayley formula.

Exercise 21: Lie algebra of the Lie group. Show that the Lie algebra of an arbitrary Lie group G may be regarded as an additive subgroup and a normal divisor of the tangent group TG. Describe the cosets of this normal divisor and the corresponding quotient group of the group TG.

Exercise 22: structure constants iterated. The structure constants of a Lie group G have three indices and can be placed into a spacial matrix a). Prove that the structure constants of the tangent groups TG, T^2G and T^3G can be similarly put into a spacial matrices of type b), c) and d), respectively.

2. Tangent bundles and osculators

2.1. Levels and sector-forms. The tangent functor T iterated k times associates to a smooth manifold M its k-fold tangent bundle T^kM (the k-th *level* of M) and associates to a smooth map $\varphi : M_1 \to M_2$ the graded morphism $T^k\varphi : T^kM_1 \to T^kM_2$, the k-th *derivative* of φ. The level T^kM has a multiple vector bundle structure with k projections onto $T^{k-1}M$:

$$\rho_s \doteq T^{k-s}\pi_s : T^kM \to T^{k-1}M, \quad s = 1, 2, \ldots, k,$$

where π_s is the natural projection $T^sM \to T^{s-1}M$.

Local coordinates in neighborhoods

$$T^sU \subset T^sM,\ s = 1, 2, \ldots, k, \quad \text{where} \quad T^{s-1}U = \pi_s(T^sU),$$

are determined automatically by those in the neighborhood $U \subset M$, the quantities (u^i) being regarded either as coordinate functions on U or as the coordinate components of the point $u \in U$:

U: (u^i), $i = 1, 2, \ldots, n = \dim M$,
TU: (u^i, u^i_1), with $u^i \doteq u^i \circ \pi_1$, $u^i_1 \doteq du^i$,
T^2U: $(u^i, u^i_1, u^i_2, u^i_{12})$,
with $u^i \doteq u^i \circ \pi_1\pi_2$, $u^i_1 \doteq du^i \circ \pi_2$, $u^i_2 \doteq d(u^i \circ \pi_1)$, $u^i_{12} \doteq d(du^i)$, etc.

We set up the following convention: *to introduce coordinates on T^kU, we take the coordinates on $T^{k-1}U$ and repeat them with an additional index k*, so that a tangent vector is preceded by its point of origin. This indexing is convenient since at present the symbols with index s become fiber coordinates for the projection ρ_s, $s = 1, 2, \ldots, k$.

Thus, for example, under the projections $\rho_s : T^3U \to T^2U$, $s = 1, 2, 3$, the coordinates with indices 1, 2 and 3 are each suppressed in turn:

$$\begin{array}{ccc} & (u^i\, u^i_1\, u^i_2\, u^i_{12}\, u^i_3\, u^i_{13}\, u^i_{23}\, u^i_{123}) & \\ \rho_1 \swarrow & \rho_2 \downarrow & \searrow \rho_3 \\ (u^i\, u^i_2\, u^i_3\, u^i_{23}) & (u^i\, u^i_1\, u^i_3\, u^i_{13}) & (u^i\, u^i_1\, u^i_2\, u^i_{12}). \end{array}$$

The level T^kM is a smooth manifold of dimension 2^kn and admits an important subspace of dimension $(k+1)n$ called the *osculating bundle* of M (briefly – *osculator*) of order $k-1$ and denoted by $\operatorname{Osc}^{k-1}M$. The bundle $\operatorname{Osc}^{k-1}M$ is determined by the equality of the projections

$$\rho_1 = \rho_2 = \ldots = \rho_k,$$

meaning that an element of T^kM belongs to the bundle $\operatorname{Osc}^{k-1}M$ precisely when all its k projections into $T^{k-1}M$ coincide. In this case all coordinates with the same number of lower indices coincide. For example, the first bundle $\operatorname{Osc}M$ is determined in $T^2U \subset T^2M$ by the equation $u^i_1 = u^i_2$, and the second bundle Osc^2M is determined in $T^3U \subset T^3M$ by $u^i_1 = u^i_2 = u^i_3$, $u^i_{12} = u^i_{13} = u^i_{23}$, etc. The coordinates in $\operatorname{Osc}^{k-1}M$ will be denoted by the derivatives of the coordinate functions on U, that is $(u^i, du^i, d^2u^i, \ldots, d^ku^i)$.

The immersion $\zeta : \operatorname{Osc}M \hookrightarrow T^2M$ and its derivative $T\zeta$ are determined in coordinates by matrix formulas:

$$\begin{pmatrix} u^i \\ u^i_1 \\ u^i_2 \\ u^i_{12} \end{pmatrix} \circ \zeta = \begin{pmatrix} u^i \\ du^i \\ du^i \\ d^2u^i \end{pmatrix}, \qquad \begin{pmatrix} u^i_3 \\ u^i_{13} \\ u^i_{23} \\ u^i_{123} \end{pmatrix} \circ T\zeta = \begin{pmatrix} du^i \\ d^2u^i \\ d^2u^i \\ d^3u^i \end{pmatrix},$$

$$T\zeta\left(\frac{\partial}{\partial u^i},\ \frac{\partial}{\partial(du^i)},\ \frac{\partial}{\partial(d^2u^i)}\right)=\left(\frac{\partial}{\partial u^i},\ \frac{\partial}{\partial u^i_1}+\frac{\partial}{\partial u^i_2},\ \frac{\partial}{\partial u^i_{12}}\right).$$

The fibres of the bundle $\operatorname{Osc} M$ are the integral manifolds of the distribution

$$\langle\,\partial^1_i+\partial^2_i,\partial^{12}_i\rangle,\quad \text{with}\quad \partial^1_i+\partial^2_i\doteq\frac{\partial}{\partial u^i_1}+\frac{\partial}{\partial u^i_2},\quad \partial^{12}_i\doteq\frac{\partial}{\partial u^i_{12}}.$$

The functions $(u^i_1-u^i_2)$ vanish on $\operatorname{Osc} M$.

Historically, osculating bundles were introduced under various names long before the bundles T^kM. The systematic study which was initiated 60 years ago by works of V. Vagner [2] has been culminated in recent times in Miron – Atanasiu theory [3]. Meanwhile, the theme of levels T^kM remained unjustly neglected for the obvious reason that the multiple fibre bundle structure demands a whole new understanding and new approach (see [1, 4–6]). Attempts such as [7] and the so-called synthetic formulation of T^kM [8] made progress in that direction.

While an infinitesimal displacement of the point $u\in M$ is determined by a tangent vector u_1 to M, an infinitesimal displacement of the element $(u,u_1)\in TM$ is determined by the quantities (u_2,u_{12}), representing a tangent vector to TM, etc. This interpretation of the elements of T^kM allows us to develop the theory of higher order motion. Clearly, the future belongs to these bundles.

White considers on the level T^kM or on a k-multiple vector bundle certain *sector-forms* which are functions simultaneously linear on the fibres of all k projections (see [7]). In particular, the sector-forms on T^2U and T^3U can be written as

$$\begin{aligned}\Phi&=\varphi_{ij}u^i_1u^j_2+\varphi_iu^i_{12},\\ \Psi&=\psi_{ijk}u^i_1u^j_2u^k_3+\psi^1_{ij}u^i_1u^j_{23}+\psi^2_{ij}u^i_2u^j_{13}+\psi^3_{ij}u^i_3u^j_{12}+\psi_iu^i_{123},\end{aligned}$$

with coefficients in U. For example, in each term of Ψ the index 1 (or 2 or 3 respectively) appears exactly once. This means that the function Ψ is linear on the fibres of ρ_1 (and ρ_2 and ρ_3).

Any scalar function can be lifted from the level $T^{k-1}M$ to the level T^kM by k different projections $\rho_s : T^kM\to T^{k-1}M$. For example, for the sector-form Φ (see above) there are three possibilities of lifting to T^3M:

$$\Phi\circ\rho_1=\varphi_{ij}u^i_2u^j_3+\varphi_iu^i_{23},\quad \Phi\circ\rho_2=\varphi_{ij}u^i_1u^j_3+\varphi_iu^i_{13},\quad \Phi\circ\rho_3=\varphi_{ij}u^i_1u^j_2+\varphi_iu^i_{12}.$$

Proposition 1. *Every exterior k-form can be regarded as a sector-form in the sense of White, a scalar function on T^kM that is constant on the fibres of* $\operatorname{Osc}^{k-1}M$.

Proof. The sector-form Φ is constant on $\operatorname{Osc} M$ if and only if its derivatives vanish on $\operatorname{Osc} M$. Thus

$$\begin{aligned}&\Phi=\varphi_{ij}u^i_1u^j_2+\varphi_iu^i_{12}\quad\Rightarrow\\ &\qquad(\partial^1_i+\partial^2_i)\Phi=\varphi_{ij}u^j_2+\varphi_{ji}u^j_1=(\varphi_{ij}+\varphi_{ji})u^j_1-\varphi_{ij}(u^j_1-u^j_2),\\ &\qquad\partial^{12}_i\Phi=\varphi_i\quad\Rightarrow\quad\varphi_{(ij)}=0,\quad\varphi_i=0\,.\end{aligned}$$

If Φ is an antisymmetric bilinear form then it can be expressed in the coordinates (u^i,du^i) as a 2-form $\Phi=\varphi_{[ij]}du^i\wedge du^j$. Thus the sector-form Φ is constant on $\operatorname{Osc} M$ if and only if it is a Cartan 2-form.

If $k=3$ the fibres Osc^2M of dimension $3n$ are the integral manifolds of the distribution

$$\langle\,\partial^1_i+\partial^2_i+\partial^3_i,\ \partial^{23}_i+\partial^{13}_i+\partial^{12}_i,\ \partial^{123}_i\rangle.$$

For the sector-form Ψ (see above) we have

$$\begin{aligned}
&\Psi = \psi_{ijk}u_1^iu_2^ju_3^k + \psi^1_{ij}u_1^iu_{23}^j + \psi^2_{ij}u_2^iu_{13}^j + \psi^3_{ij}u_3^iu_{12}^j + \psi_i u^i_{123} \Rightarrow \\
&(\partial_i^1 + \partial_i^2 + \partial_i^3)\Psi = \psi_{ijk}u_2^ju_3^k + \psi_{jik}u_1^ju_3^k + \psi_{jki}u_1^ju_2^k + \psi^1_{ij}u^j_{23} + \psi^2_{ij}u^j_{13} + \psi^3_{ij}u^j_{12}\,, \\
&(\partial_i^{23} + \partial_i^{13} + \partial_i^{12})\Psi = \psi^1_{ji}u_1^j + \psi^2_{ji}u_2^j + \psi^3_{ji}u_3^j, \\
&\partial_i^{123}\Psi = \psi_i.
\end{aligned}$$

The derivatives vanish on the fibres $\mathrm{Osc}^2 M$ when the following conditions hold:

$$\varphi_{(ijk)} = 0, \quad \psi^1_{ij} + \psi^2_{ij} + \psi^3_{ij} = 0, \quad \psi_i = 0.$$

These conditions are necessary and sufficient for the sector-form Ψ to be constant on $\mathrm{Osc}^2 M$, but not for Ψ to be a Cartan 3-form. However, every 3-form $\widetilde{\Psi} = \varphi_{ijk}\, du^i \wedge du^j \wedge du^k$ can be regarded as a homogeneous sector-form that is constant on $\mathrm{Osc}^2 M$.

The argument extends likewise to the cases when $k > 3$. □

White's theory of sector-forms is much more extensive than that of Cartan exterior forms. In particular, exterior differentiation is an operation on the set of sector-forms that are constant on the osculating bundles.

2.2. Gauge groups on osculating spaces. The action of the gauge group $\mathcal{G}_k$ on the k-th level $T^k M$ extends in a natural way to the osculating bundle $\mathrm{Osc}^{k-1} M$. The diagram from below shows how the block-matrix 4×4 reduces, for $u_1 = u_2$, to a 3×3 block-matrix:

$$u_1 = u_2 \quad \Rightarrow \quad \begin{pmatrix} a & 0 & 0 & 0 \\ a_1 & a & 0 & 0 \\ a_2 & 0 & a & 0 \\ a_{12} & a_2 & a_1 & a \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} a & 0 & 0 \\ da & a & 0 \\ d^2 a & da & a \end{pmatrix}.$$

The blocks of the matrix from the right side are generated in the following way:

$$\begin{aligned}
& a \sim a^i_j, \qquad \left.\begin{aligned} a_1 &\sim a^i_{jk}u_1^k \\ a_2 &\sim a^i_{jk}u_2^k \end{aligned}\right\} \rightsquigarrow \; da \sim da^i_j = a^i_{jk}du^k, \\
& a_{12} \sim a^i_{jkl}u_1^ku_2^l + a^i_{jk}u^k_{12} \rightsquigarrow d^2 a \sim a^i_{jkl}\, du^k du^l + a^i_{jk}d^2u^k.
\end{aligned}$$

The action of the gauge group $\mathcal{G}_2$ on the level $T^2 M$ is obviously transported to the subbundle $\mathrm{Osc}\, M \subset T^2 M$. While one passes from $T^2 M$ to $\mathrm{Osc}\, M$ by considering

$$(a_1 = a_2, a_{12}) \rightsquigarrow (da\,, d^2 a), \quad (\partial^1 + \partial^2\,, \partial^{12}) \rightsquigarrow \Big(\frac{\partial}{\partial(du)}, \frac{\partial}{\partial(d^2u)}\Big),$$

the transformation of the natural basis on $T^2 M$ is transported to the transformation of the natural basis on $\mathrm{Osc}\, M$:

$$(\partial \; \partial^1 \; \partial^2 \; \partial^{12}) \cdot \begin{pmatrix} a & 0 & 0 & 0 \\ a_1 & a & 0 & 0 \\ a_2 & 0 & a & 0 \\ a_{12} & a_2 & a_1 & a \end{pmatrix} \rightsquigarrow \Big(\frac{\partial}{\partial u} \; \frac{\partial}{\partial(du)} \; \frac{\partial}{\partial(d^2u)}\Big) \cdot \begin{pmatrix} a & 0 & 0 \\ da & a & 0 \\ d^2 a & da & a \end{pmatrix}.$$

In the general case, the action of the group $\mathcal{G}_k$ on the level $T^k M$ extends in a similar way to the subbundle $\mathrm{Osc}^{k-1} M$.

Резюме

М. Рахула, В. Балан. Касательные расслоения и калибровочные группы.

Дифференциалы $T^k a$ $(k \geq 1)$ диффеоморфизма a гладкого многообразия M индуцируют в слоях расслоений $T^k M$, то есть в соответствующих касательных пространствах, линейные преобразования, заключающие в себе действие калибровочной группы $\mathcal{G}_k$. Это действие естественным образом распространяется на соприкасающиеся подрасслоения $\mathrm{Osc}^{k-1} M \subset T^k M$.

Ключевые слова: диффеоморфизм гладкого многообразия, пространство расслоения, действие калибровочной группы.

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Rahula, Maido – Doctor of Physics and Mathematics, Professor Emeritus, Faculty of Mathematics and Computer Science, University of Tartu, Tartu, Estonia.

Рахула, Майдо – доктор физико-математических наук, почетный профессор факультета математики и информатики Тартуского университета, г. Тарту, Эстония.

E-mail: *rahula@ut.ee*

Balan, Vladimir – Doctor of Mathematics, Professor, Faculty of Applied Sciences, University Politehnica of Bucharest, Bucharest, Romania.

Балан, Владимир – доктор математических наук, профессор факультета прикладных наук Бухарестского политехнического университета, г. Бухарест, Румыния.

E-mail: *vladimir.balan@upb.ro*