When the study of relatively maximal subgroups can be reduced to quotients?

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Topic

"A group" always means "a finite group".

The focus will be on the following types of tasks:

- Given a group G and a class $\mathscr{X}$ of groups (e. g. $\mathscr{X} = \mathscr{S}$ the class of solvable groups). Find (describe, list ect.) subgroups of G which belong to $\mathscr{X}$ (=are *$\mathscr{X}$-groups*).

Notation: $\mathscr{X}(G) = \{H \leqslant G \mid H \in \mathscr{X}\}$.

E. Galois, Mémoire sur les conditions de résolubilité des équations par radicaux, J. Math. Pures Appl. (Liouville) 11 (1846), 417–433.

C. M. Jordan, Commentaire sur le Mémoire de Galois. Comptes rendus 60 (1865), 770–774.

C. M. Jordan, Traité des substitutions et des équations algébriques, Paris: Gauthier-Villars, 1870.

$$\left[\begin{array}{c} f = \alpha_0 + \alpha_1 x \cdots + \alpha_n x^n \in \mathbb{Q}[x], \\ \alpha_n \neq 0 \end{array}\right] \to \left[\begin{array}{c} \text{the Galois group of } f \\ G_f \leqslant \mathrm{S}_n \end{array}\right]$$

$$\left[\begin{array}{c} f = 0 \\ \text{is solvable in radicals} \end{array}\right] \Leftrightarrow \left[\begin{array}{c} G_f \\ \text{is } \textit{solvable} \end{array}\right]$$

A group G is *solvable* if G has a series of normal subgroups

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \ldots \trianglerighteq G_n = 1$$

such that G_{i-1}/G_i is Abelian for all $i = 1, \ldots, n$.

- (C. Jordan): Find the solvable subgroups of S_n.

We always assume that $\mathscr{X}$ has "good" properties resembling those of the class $\mathscr{S}$ of solvable groups.
According to H.Wielandt, a non-empty class $\mathscr{X}$ is said to be *complete* if $\mathscr{X}$ is closed under

- subgroups ($G \in \mathscr{X}$ and $H \leqslant G \Rightarrow H \in \mathscr{X}$),
- homomorphic images ($G \in \mathscr{X}$, $\phi : G \to G^* \Rightarrow G^\phi \in \mathscr{X}$),
- and extensions ($N \trianglelefteq G$ and $N, G/N \in \mathscr{X} \Rightarrow G \in \mathscr{X}$).

Examples of complete classes:

- The classes $\mathscr{G}$ of all finite groups and $\mathscr{E}$ of groups of order 1.
- The class $\mathscr{S}$ of finite solvable groups.
- Take a subset π of the set $\mathbb{P}$ of all primes.
  A group G is a *π-group*, if $p \mid |G| \Rightarrow p \in \pi$ for all $p \in \mathbb{P}$.
  The classes $\mathscr{G}_\pi$ and $\mathscr{S}_\pi$ of finite π-groups and finite solvable π-groups, respectively, are complete.
  $$\mathscr{G}_{\{2,3\}} = \{G \mid |G| = 2^\alpha 3^\beta, \text{ where } \alpha, \beta \geqslant 0\}.$$
- For every $p \in \mathbb{P}$, the class of p-solvable groups is complete.

For complete $\mathscr{X}$, it is sufficient to find the *maximal $\mathscr{X}$-subgroup* (or *$\mathscr{X}$-maximal subgroups*) of G.
It is natural to search such subgroups up to conjugacy.

Definitions and notation :

- G is a group.
- $\mathscr{X}$ is a complete class.
- $\mathrm{m}_{\mathscr{X}}(G)$ is the set of $\mathscr{X}$-maximal subgroups of G.
- *A scheme over $\mathscr{X}$* or *an $\mathscr{X}$-scheme* of G is a set of representatives of all conjugacy classes of $\mathscr{X}$-maximal subgroups of G.
- $\mathrm{k}_{\mathscr{X}}(G)$ is the size of an $\mathscr{X}$-scheme (= the number of conjugacy classes of $\mathscr{X}$-maximal subgroups) of G.

Task: Given G and $\mathscr{X}$. Find an $\mathscr{X}$-scheme of G.

- N is a normal subgroup of G.

Question (H.Wielandt, 1963/64): When, instead of finding a $\mathscr{X}$-scheme for G, is it sufficient to find that for G/N?

H. Wielandt, Zusammengesetzte Gruppen endlicher Ordnung, Vorlesung an der Universität Tübingen im Wintersemester 1963/64.
In *Helmut Wielandt: Math. Works, Vol.* 1, 607–655.

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In the talk, we

- discuss why this question is important and
- give a comprehensive answer.

Main Theorem: $\mathrm{k}_{\mathscr{X}}(G) = \mathrm{k}_{\mathscr{X}}(G/N) \quad \Leftrightarrow \quad \mathrm{k}_{\mathscr{X}}(N) = 1.$

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What are the difficulties?

The task of finding a $\mathscr{X}$-scheme does not reduce well to the factors of a (sub)normal series.

If $G \trianglerighteq N$ (= G is an extension of N by G/N), then $H \leqslant G$ is an extension of $H \cap N$ by HN/N.

If $H \cap N \in \mathrm{m}_{\mathscr{X}}(N)$ and $HN/N \in \mathrm{m}_{\mathscr{X}}(G/N)$, then $H \in \mathrm{m}_{\mathscr{X}}(G)$.

The converse is not true:

- $H \in \mathrm{m}_{\mathscr{X}}(G) \not\Rightarrow H \cap N \in \mathrm{m}_{\mathscr{X}}(N)$,
- $H \in \mathrm{m}_{\mathscr{X}}(G) \not\Rightarrow HN/N \in \mathrm{m}_{\mathscr{X}}(G/N)$.
- $H \in \mathrm{m}_{\mathscr{X}}(G)$ and $\phi : G \to G^*$ a homomorphism $\not\Rightarrow H^\phi \in \mathrm{m}_{\mathscr{X}}(G^\phi)$.

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- $H \in \mathrm{m}_{\mathscr{X}}(G)$ and $\phi : G \to G^*$ a homomorphism
  $\not\Rightarrow \quad H^{\phi} \in \mathrm{m}_{\mathscr{X}}(G^{\phi})$.

Proposition (Wielandt, 1963).

Assume, there exists a group L having non-conjugate $\mathscr{X}$-maximal subgroups[1]*. Then every finite group G_0 is an image of an epimorphism $\phi : G \to G_0$ such that*

$$\{K^\phi \mid K \in \mathrm{m}_{\mathscr{X}}(G)\} = \mathscr{X}(G_0).$$

More exactly, under the natural epimorphism

$$\phi : G = L \wr G_0 \to G_0$$

from the regular wreath product $G = L \wr G_0$, every $H \in \mathscr{X}(G_0)$ coincides with K^ϕ for some $K \in \mathrm{m}_{\mathscr{X}}(G)$.

[1] The existence of a group L with non-conjugate $\mathscr{X}$-maximal subgroups is equivalent to the fact that $\mathscr{X}$ differs from $\mathscr{E}, \mathscr{G}, \mathscr{G}_p$ for all $p \in \mathbb{P}$.

Another difficulty for homomorphisms (quotients) is that the images of non-conjugate $\mathscr{X}$-maximal subgroups may be conjugate (the trivial homomorphism is an example).

In general, even if we know N and G/N, we can not calculate $\mathrm{k}_{\mathscr{X}}(G)$.

Example: In both $G_1 = \mathbb{Z}_2 \times \mathrm{PSL}_2(7)$ and $G_2 = \mathrm{PGL}_2(7)$, there is $N_i \trianglelefteq G_i$ such that $N_i \cong \mathrm{PSL}_2(7)$ and $G_i/N_i \cong \mathbb{Z}_2$ (i.e. every G_i is an extension of $\mathrm{PSL}_2(7)$ by $\mathbb{Z}_2$).
But $\mathrm{k}_{\mathscr{S}}(G_1) = 3$, while $\mathrm{k}_{\mathscr{S}}(G_2) = 4$.

Moreover, $\mathrm{k}_{\mathscr{X}}(G/N)$ is not defined by the isomorphism type of G and N only. It is depends on specifics of the embedding of N in G as a normal subgroup.

Example: $G = \mathrm{PGL}_2(7) \times \mathrm{PSL}_2(7)$ has two normal subgroups N_1 and N_2 such that $N_1 \cong N_2 \cong \mathrm{PSL}_2(7)$ and $G/N_i \cong G_i$, $i = 1, 2$, where G_1 and G_2 are defined above. Therefore, $\mathrm{k}_{\mathscr{S}}(G/N_1) \neq \mathrm{k}_{\mathscr{S}}(G/N_2)$.

At the same time, there are pairs (G, N), where $N \trianglelefteq G$, such that not only $HN/N \in \mathrm{m}_{\mathscr{X}}(G/N)$ for all $H \in \mathrm{m}_{\mathscr{X}}(G)$, but also the map $H \mapsto HN/N$ induces a bijection between the conjugacy classes of $\mathscr{X}$-maximal subgroups in G and G/N.

Proposition:

Let $^{-} : G \to G/N$ *be the canonical epimorphism. Then*

- $\mathscr{X}(\overline{G}) = \{\overline{H} \mid H \in \mathscr{X}(G)\}$;
- $\mathrm{m}_{\mathscr{X}}(\overline{G}) \subseteq \{\overline{H} \mid H \in \mathrm{m}_{\mathscr{X}}(G)\}$;
- $\mathrm{k}_{\mathscr{X}}(G) \geqslant \mathrm{k}_{\mathscr{X}}(\overline{G})$;
- $\mathrm{k}_{\mathscr{X}}(G) = \mathrm{k}_{\mathscr{X}}(\overline{G}) \Rightarrow \mathrm{m}_{\mathscr{X}}(\overline{G}) = \{\overline{H} \mid H \in \mathrm{m}_{\mathscr{X}}(G)\}$.

Question (H.Wielandt): Find all pairs (G, N), $N \trianglelefteq G$, such that

$$\mathrm{k}_{\mathscr{X}}(G) = \mathrm{k}_{\mathscr{X}}(G/N).$$

Definition: We say that, *the reduction $\mathscr{X}$-theorem holds for a pair (G, N)*, where $N \trianglelefteq G$, if

$$\mathrm{k}_{\mathscr{X}}(G) = \mathrm{k}_{\mathscr{X}}(G/N).$$

The reduction $\mathscr{X}$-theorem holds for a group A if it holds for every pair (G, N) such that $N \cong A$.

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The reduction $\mathscr{X}$-theorem holds for a group A if it holds for every pair (G, N) such that $N \cong A$.

Theorem (Chunikhin, Wielandt):
The reduction $\mathscr{X}$-theorem holds for a group A:

- *if A is an $\mathscr{X}$-group,*
- *if A is an $\mathscr{X}'$-group,*
- *if A is an $\mathscr{X}$-separable group.*

A group G is called an *$\mathscr{X}'$-group* ($G \in \mathscr{X}'$) if G has no non-trivial $\mathscr{X}$-subgroups.

A group G is called *$\mathscr{X}$-separable*, if G has a (sub)normal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \ldots \trianglerighteq G_{n-1} \trianglerighteq G_n = 1$$

such that $G^i \in \mathscr{X} \cup \mathscr{X}'$ for every section $G^i = G_{i-1}/G_i$.
A solvable group is $\mathscr{X}$-separable for every $\mathscr{X}$.

The reduction $\mathscr{X}$-theorem holds for a group A if $\mathrm{k}_{\mathscr{X}}(G) = \mathrm{k}_{\mathscr{X}}(G/N)$ for every pair (G, N) such that $G \trianglerighteq N \cong A$.

Observation. If the reduction $\mathscr{X}$-theorem holds for a group A then $\mathrm{k}_{\mathscr{X}}(A) = \mathrm{k}_{\mathscr{X}}(A/A) = 1$, i. e. the $\mathscr{X}$-maximal subgroups are conjugate in A.

$$\left[\begin{array}{c}\text{the}\\ \text{reduction}\\ \mathscr{X}\text{-theorem}\end{array}\right] \Rightarrow \left[\begin{array}{c}\text{the conjugacy of}\\ \mathscr{X}\text{-maximal}\\ \text{subgroups}\end{array}\right]$$

Problem. Does the conjugacy of $\mathscr{X}$-maximal subgroups imply the reduction $\mathscr{X}$-theorem?

$$\left[\begin{array}{c}\text{the}\\ \text{reduction}\\ \mathscr{X}\text{-theorem}\end{array}\right] \overset{?}{\Leftarrow} \left[\begin{array}{c}\text{the conjugacy of}\\ \text{the } \mathscr{X}\text{-maximal}\\ \text{subgroups}\end{array}\right]$$

Wielandt's strategy

Problem. Does the conjugacy of $\mathscr{X}$-maximal subgroups imply the reduction $\mathscr{X}$-theorem?

Wielandt posed this problem in an extended form using the concept of an $\mathscr{X}$-submaximal subgroup which generalizes that of $\mathscr{X}$-maximal subgroup.

Proposition (Wielandt, 1963). *If the $\mathscr{X}$-submaximal subgroups of A are conjugate then the reduction $\mathscr{X}$-theorem hold A.*

$$\begin{bmatrix} \text{the conjugacy of} \\ \mathscr{X}\text{-submaximal} \\ \text{subgroups} \end{bmatrix} \Rightarrow \begin{bmatrix} \text{the} \\ \text{reduction} \\ \mathscr{X}\text{-theorem} \end{bmatrix} \Rightarrow \begin{bmatrix} \text{the conjugacy of} \\ \mathscr{X}\text{-maximal} \\ \text{subgroups} \end{bmatrix}$$

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Theorem 1:

For a finite group A, the following statements are equivalent:

(i) *the $\mathscr{X}$-maximal subgroups of A are conjugate;*

(ii) *the $\mathscr{X}$-submaximal subgroups of A are conjugate;*

(iii) *the reduction $\mathscr{X}$-theorem holds for A;*

(iv) *in every composition factor of A, the $\mathscr{X}$-maximal subgroups are conjugate;*

(v) *every composition factor of A is isomorphic to a group from a known list $\mathcal{L}(\mathscr{X})$.*

W.Guo, R., E.Vdovin, The reduction theorem for relatively maximal subgroups, Bull. Math. Sci., 11 (2021), N2, 2150001, 47 pp., doi: 10.1142/S1664360721500016.

Theorem 2: $\mathrm{k}_{\mathscr{X}}(G) = \mathrm{k}_{\mathscr{X}}(G/N) \quad \Leftrightarrow \quad \mathrm{k}_{\mathscr{X}}(N) = 1.$

Recall that there is G with $N_1, N_2 \trianglelefteq G$ such that $N_1 \cong N_2$ while $\mathrm{k}_{\mathscr{X}}(G/N_1) \neq \mathrm{k}_{\mathscr{X}}(G/N_2)$.

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Recall that there is G with $N_1, N_2 \trianglelefteq G$ such that $N_1 \cong N_2$ while $\mathrm{k}_{\mathscr{X}}(G/N_1) \neq \mathrm{k}_{\mathscr{X}}(G/N_2)$.

Question (H.Wielandt): When, instead of finding a $\mathscr{X}$-scheme for G, is it sufficient to find that for G/N?

Answer: It is if and only if $\mathrm{k}_{\mathscr{X}}(N) = 1$. Equivalently, if and only if $\mathrm{k}_{\mathscr{X}}(S) = 1$ for every composition factor S of N.
All simple groups S with this property are known:
either $S \in \mathscr{X}$ or the natural arithmetic parameters of S satisfy some conditions under

$$\pi(\mathscr{X}) = \{p \in \mathbb{P} \mid p \text{ divides } |X| \text{ for some } X \in \mathscr{X}\}.$$

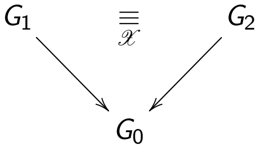
Example: $S = \mathrm{PSL}_n(q)$ where q is a power of $p \in \pi(\mathscr{X})$.
Then $\mathrm{k}_{\mathscr{X}}(S) = 1 \Leftrightarrow$ either $S \in \mathscr{X}$
or for every $r \in \pi(\mathscr{X})$ if r divides $|S|$ then r divides $q(q-1)$ and $r > n$.

Corollary A: *Let* $\mathscr{X}$ *be a complete class and let* N *be a normal subgroup of a finite group* G*. Then* $\mathrm{k}_{\mathscr{X}}(G) > \mathrm{k}_{\mathscr{X}}(G/N)$ *if* $\mathrm{k}_{\mathscr{X}}(N) > 1$.

Corollary B: *Let* $\mathscr{X}$ *be a complete class. In every finite group* G*, there exists the largest normal subgroup* N *satisfying* $\mathrm{k}_{\mathscr{X}}(G) = \mathrm{k}_{\mathscr{X}}(G/N)$.

An epimorphism $\phi : G \to G^*$ is called an *$\mathscr{X}$-isoschematism* if ϕ maps some (=every) $\mathscr{X}$-scheme of G to an $\mathscr{X}$-scheme of G^*.

Two groups are called *isoschemic over $\mathscr{X}$* if they admits $\mathscr{X}$-isoschematisms on one and the same group: $G_1 \underset{\mathscr{X}}{\equiv} G_2$



A group G is said to be *schematically reduced over $\mathscr{X}$* if each $\mathscr{X}$-isoschematism with the domain G is an $\mathscr{X}$ isomorphism.

Corollary C: *The relation $\underset{\mathscr{X}}{\equiv}$ is an equivalence. Every equivalency class contains a unique schematically reduced over $\mathscr{X}$ group (an universal attracting object).*

Sketch of the proof of Theorems 1 and 2

$$\pi = \pi(\mathscr{X}) = \{p \in \mathbb{P} \mid p \text{ divides } |X| \text{ for some } X \in \mathscr{X}\}.$$

$$\mathscr{S}_\pi \subseteq \mathscr{X} \subseteq \mathscr{G}_\pi.$$

An $\mathscr{X}$-subgroup H of G is called *a $\mathscr{X}$-Hall subgroup* ($H \in \mathrm{Hall}_{\mathscr{X}}(G)$) if $|G : H|$ is divisible by no prime in π.

$\mathrm{h}_{\mathscr{X}}(G)$ is the number of conjugacy classes of $\mathscr{X}$-Hall subgroup of G.

- $\mathrm{Hall}_{\mathscr{X}}(G) \subseteq \mathrm{m}_{\mathscr{X}}(G)$;
- $\mathrm{h}_{\mathscr{X}}(G) \leqslant \mathrm{k}_{\mathscr{X}}(G)$;
- $\mathrm{k}_{\mathscr{X}}(G) = 1 \quad \Rightarrow \quad \mathrm{m}_{\mathscr{X}}(G) = \mathrm{Hall}_{\mathscr{X}}(G)$;
- $\mathrm{k}_{\mathscr{X}}(G) = \mathrm{k}_{\mathscr{X}}(G/N) \quad \Rightarrow \quad \mathrm{m}_{\mathscr{X}}(N) = \mathrm{Hall}_{\mathscr{X}}(N)$.

- $H \in \mathrm{Hall}_{\mathscr{X}}(G) \quad \Rightarrow$
  $H \cap N \in \mathrm{Hall}_{\mathscr{X}}(N)$ and $HN/N \in \mathrm{Hall}_{\mathscr{X}}(G/N)$.

Theorem 1: $\mathrm{k}_{\mathscr{X}}(G) = \mathrm{k}_{\mathscr{X}}(G/N) \quad \Leftarrow \quad \mathrm{k}_{\mathscr{X}}(N) = 1.$

It is sufficient to consider the case $\mathrm{k}_{\mathscr{X}}(G/N) = 1$.
If G is a minimal counterexample then N is nonabelian simple, $N \notin \mathscr{X}$, and $G \leqslant \mathrm{Aut}(N)$.

If $\mathscr{X} = \mathscr{G}_\pi$ then the statement is true.

E. P. Vdovin, R., Theorems of Sylow type, Russian Math. Surveys, 66:5 (2011), 829–870.

If 2 or $3 \notin \pi$ then π-Hall (and $\mathscr{X}$-Hall) subgroups of N are solvable and the statement for $\mathscr{X}$ is equivalent the statement for $\mathscr{G}_\pi$.

If $2, 3 \in \pi$ then, case by case, we prove that $\mathrm{k}_{\mathscr{X}}(N) > 1$.

Theorem 2: $\mathrm{k}_{\mathscr{X}}(G) = \mathrm{k}_{\mathscr{X}}(G/N) \quad \Rightarrow \quad \mathrm{k}_{\mathscr{X}}(N) = 1.$

This statement was unknown even in the case where $\mathscr{X} = \mathscr{G}_\pi$

- Suppose, G is a minimal counterexample. Then we may assume that N is minimal normal:

  $$N = S_1 \times \cdots \times S_n, \text{ where all } S_i \text{ are conjugate simple.}$$

  $S := S_1$.
  S acts on $\mathrm{Hall}_{\mathscr{X}}(S)$ via conjugation.
  Δ is the set of orbits and $\mathrm{h}_{\mathscr{X}}(S) = |\Delta|$.
- $\mathrm{h}_{\mathscr{X}}(S) \in \{0, 1, 2, 3, 4, 9\}$ (Vdovin, R., 2010).
- $\mathrm{m}_{\mathscr{X}}(S) = \mathrm{Hall}_{\mathscr{X}}(S)$. In particular, $\mathrm{h}_{\mathscr{X}}(S) = \mathrm{k}_{\mathscr{X}}(S) > 0$.
  It is sufficient to show that $\mathrm{h}_{\mathscr{X}}(S) = 1$.
- We exclude the case $\mathrm{h}_{\mathscr{X}}(S) = 9$.
  In this case, $S \cong \mathrm{PSp}_{2m}(q)$, q is odd (Vdovin, R., 2010), and one can show that $\mathrm{m}_{\mathscr{X}}(S) \neq \mathrm{Hall}_{\mathscr{X}}(S)$.

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- Notation: if $H \leqslant G$ then
$$\mathrm{Aut}_H(S) := N_H(S)/C_H(S) \leqslant \mathrm{Aut}(S).$$
$$\mathrm{Aut}_H(S) = N_H(S)/C_H(S) \cong (H \cap N_G(S))C_G(S)/C_G(S) \leqslant$$
$$\leqslant N_G(S)/C_G(S) = \mathrm{Aut}_G(S).$$
$\mathrm{Aut}_H(S)$ acts on Δ.
- Every $\mathscr{X}$-subgroup of $\mathrm{Aut}_G(S)$ stabilizes an element of Δ.
- If $H \in \mathrm{m}_{\mathscr{X}}(G)$ then $\mathrm{Aut}_H(S)$ stabilizes exactly one element of Δ. In particular, $\mathrm{h}_{\mathscr{X}}(S) = |\Delta| \neq 2$.
- Assume, that $\mathrm{h}_{\mathscr{X}}(S) \neq 1$, i. e. $\mathrm{h}_{\mathscr{X}}(S) \in \{3, 4\}$. Then $2, 3 \in \pi$ (Vdovin, R., 2010).
- Take $K \in \mathrm{Hall}_{\mathscr{X}}(S)$ and $H \in \mathrm{m}_{\mathscr{X}}(G)$ such that $K \leqslant H$. Then $\mathrm{Aut}_H(S)$ leaves invariant K^S and acts without fix points on the remaining 2 or 3 elements of Δ. This means that both $\mathrm{Aut}_H(S)$ and the stabilizer of K^S in $\mathrm{Aut}_G(S)$ act transitively on these elements. Therefore,
- $\mathrm{Aut}_G(S)$ acts 2-transitively on Δ and its image A in $\mathrm{Sym}(\Delta) \cong \mathrm{S}_3$ or S_4 is a transitive $\{2, 3\}$-group. In particular, A is an $\mathscr{X}$-group and is the image of an $\mathscr{X}$-subgroup U of $\mathrm{Aut}_G(S)$ which acts transitively on Δ.

Main Theorem: $\mathrm{k}_{\mathscr{X}}(G) = \mathrm{k}_{\mathscr{X}}(G/N) \quad \Leftrightarrow \quad \mathrm{k}_{\mathscr{X}}(N) = 1.$

The completeness of $\mathscr{X}$ is essential:
for $\mathscr{X} = \mathscr{N}$ (the class of nilpotent groups) the the theorem fails.

Example: $G = \mathrm{S}_3$, $N = \mathrm{A}_3$,
$\mathrm{k}_{\mathscr{N}}(G) = 2$, $\mathrm{k}_{\mathscr{N}}(G/N) = 1$, while $\mathrm{k}_{\mathscr{N}}(N) = 1$.

Thank you for your attention!